ON A NEW SUBFAMILY OF MALTIVALENT FUNCTIONS WITH NEGATIVE COFFICIENTS

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Abstract: In the present paper, we establish a new subfamily of multivalent functions with negative coefficients. Sharp results concerning coefficients, distortion theorem and the radius of convexity for the class $WH_p(\alpha,\beta,\varepsilon)$ are obtained. Furthermore it is shown that the class $WH_p(\alpha,\beta,\varepsilon)$is closed under convex linear combinations. The arithmetic mean is also obtained.

2000 Mathematics Subject Classification: Primary 30C45.

Key Words: Multivalent Function, Distortion Theorem, Radius of Convexity, Convex Linear Combination, Arithmetic Mean.

1. Introduction :

Let W_p (p a fixed integer greater than 1) denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad , \quad p, n \in IN = \{1,2,3,...\} \qquad (1.1)$$

which are analytic and multivalent functions in the open unit disk $U = \{z \in C : |z| < 1\}$. Also let H_p denote the subclass of W_p consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad , \; a_{n+p} \geq 0, n, p \in IN. \qquad (1.2)$$

A function $f \in H_p$ is said to be in the class $WH_p(\alpha, \beta, \varepsilon)$ if and only if

$$\left| \frac{\left(f''(z)z^{2-p} - p(p-1)\right) + (f'(z)z^{1-p} - p)}{2\varepsilon\left(f''(z)z^{2-p} - \alpha\right) - \left(f''(z)z^{2-p} - p(p-1)\right)} \right| < \beta, \qquad (1.3)$$

$z \in U$, for $0 \leq \alpha < \frac{\text{p}}{2\varepsilon}, 0 < \beta \leq 1, \frac{1}{2} < \varepsilon \leq 1.$

Such type of study and study another different classes of univalent and multivalent functions was carried out by Aouf [1] caplinger [5], Gupte – Jain [6], Juneja – Mogra [7], Kulkarni [8], Atshan [2] and Atshan – Kulkarni [3,4].

In the present paper, sharp results concerning coefficients, distortion theorem and the radius of convexity for the class $WH_p(\alpha, \beta, \varepsilon)$ are obtained. Finally, we prove that the class $WH_p(\alpha, \beta, \varepsilon)$ is closed under the arithmetic mean and convex linear combinations.

2. <u>Coefficient Theorem :</u>

<u>Theorem 1:</u> A function $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$ is in the class $WH_p(\alpha, \beta, \varepsilon)$ if and only if

$$\sum_{n=1}^{\infty} (n+p)\left[(n+p) + (n+p-1)(2\varepsilon - 1)\beta\right] a_{n+p} \leq 2\varepsilon\beta\left(p(p-1) - \alpha\right). \qquad (2.1)$$

The result (2.1) is sharp, the extermal function being

$$f(z) = z^p - \frac{2\varepsilon\beta\left(p(p-1) - \alpha\right)}{(n+p)\left[(n+p) + (n+p-1)(2\varepsilon - 1)\beta\right] a_{n+p}} z^{n+p}. \qquad (2.2)$$

Proof: Let $|z|=1$. Then

$$\left|\left(f''(z)z^{2-p}-p(p-1)\right)+(f'(z)z^{1-p}-p)\right|-\beta\left|2\varepsilon\left(f''(z)z^{2-p}-\alpha\right)-\left(f''(z)z^{2-p}-p(p-1)\right)\right|$$

$$=\left|-\sum_{n=1}^{\infty}(n+p)^2 a_{n+p}z^n\right|-\beta\left|2\varepsilon\left(p(p-1)-\alpha\right)-(2\varepsilon-1)\sum_{n=1}^{\infty}(n+p)(n+p-1)a_{n+p}z^n\right|$$

$$\leq\sum_{n=1}^{\infty}(n+p)\left[(n+p)+(n+p-1)(2\varepsilon-1)\beta\right]a_{n+p}-2\varepsilon\beta(p(p-1)-\alpha)\leq 0,$$

by hypothesis. Hence, by the maximum modulus theorem $f\in WH_p(\alpha,\beta,\varepsilon)$.

Conversely, suppose that

$$\left|\frac{\left(f''(z)z^{2-p}-p(p-1)\right)+(f'(z)z^{1-p}-p)}{2\varepsilon\left(f''(z)z^{2-p}-\alpha\right)-\left(f''(z)z^{2-p}-p(p-1)\right)}\right|$$

$$=\left|\frac{-\sum_{n=1}^{\infty}(n+p)^2 a_{n+p}z^n}{2\varepsilon(p(p-1)-\alpha)-(2\varepsilon-1)\sum_{n=1}^{\infty}(n+p)(n+p-1)a_{n+p}z^n}\right|<\beta.$$

Since $|\mathrm{Re}(z)|\leq|z|$ for all *z*, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}(n+p)^2 a_{n+p}z^n}{2\varepsilon(p(p-1)-\alpha)-(2\varepsilon-1)\sum_{n=1}^{\infty}(n+p)(n+p-1)a_{n+p}z^n}\right\}<\beta.$$

We select the values of *z* on the real axis so that $f''(z)z^{2-p}$, $f'(z)z^{1-p}$ are real. Simplifying the denominator in the in the above expression and letting $z\to 1$ through real values, we obtain

$$\sum_{n=1}^{\infty}(n+p)^2 a_{n+p} \le 2\varepsilon\beta(p(p-1)-\alpha)-(2\varepsilon-1)\beta\sum_{n=1}^{\infty}(n+p)(n+p-1)a_{n+p},$$

and it results in the required condition.
The result is sharp for the function (2.2) .

3. <u>Distortion Theorem:</u>

<u>Theorem2</u>: Let $f \in WH_p(\alpha,\beta,\varepsilon)$. Then for $|z|=r$,

$$r^p - \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)\big[(p+1)+p(2\varepsilon-1)\beta\big]}r^{p+1} \le |f(z)| \le r^p + \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)\big[(p+1)+p(2\varepsilon-1)\beta\big]}r^{p+1}, \quad (3.1)$$

and

$$pr^{p-1} - \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)+p(2\varepsilon-1)\beta}r^p \le |f'(z)| \le pr^{p-1} + \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)+p(2\varepsilon-1)\beta}r^p, \quad (3.2)$$

<u>Proof:</u> In view of Theorem 1 , we have

$$\sum_{n=1}^{\infty} a_{n+p} \le \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)\big[(p+1)+p(2\varepsilon-1)\beta\big]}$$

Hence $|f(z)| \le r^p + \sum_{n=1}^{\infty} a_{n+p} r^{n+p} \le r^p + \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)\big[(p+1)+p(2\varepsilon-1)\beta\big]}r^{p+1},$

and $|f(z)| \ge r^p - \sum_{n=1}^{\infty} a_{n+p} r^{n+p} \ge r^p - \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)\big[(p+1)+p(2\varepsilon-1)\beta\big]}r^{p+1}.$

In the same way, we have

$$|f'(z)| \le pr^{p-1} + \sum_{n=1}^{\infty}(n+p)a_{n+p}r^{n+p-1} \le pr^{p-1} + \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)+p(2\varepsilon-1)\beta}r^p,$$

and

$$|f'(z)| \ge pr^{p-1} - \sum_{n=1}^{\infty}(n+p)a_{n+p}r^{n+p-1} \ge pr^{p-1} - \frac{2\varepsilon\beta\big(p(p-1)-\alpha\big)}{(p+1)+p(2\varepsilon-1)\beta}r^p.$$

This complete the proof of the theorem.

The above bounds are sharp. Equalities are attended for the following function

$$f(z) = z^p - \frac{2\varepsilon\beta\left(p(p-1)-\alpha\right)}{(p+1)\left[(p+1)+p(2\varepsilon-1)\beta\right]} z^{p+1}, z = \pm 1. \qquad (3.3)$$

4. **Radius of Convexity :**

Theorem 3: Let $f \in WH_p(\alpha,\beta,\varepsilon)$. Then f is convex in the disk $|z| < r = r(p,\alpha,\beta,\varepsilon)$, where

$$r(p,\alpha,\beta,\varepsilon) = \inf_{n\in IN}\left\{\frac{p^2(n+p)\left[(n+p)+(n+p-1)(2\varepsilon-1)\beta\right]}{(n+p)^2 2\varepsilon\beta(p(p-1)-\alpha)}\right\}^{\frac{1}{n}}.$$

The result is sharp, the extermal function being of the form (2.2).

<u>Proof:</u> It is enough to show that.

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| \le p \quad \text{for } |z|<1.$$

First, we note that

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| = \left|\frac{zf''(z)+(1-p)f'(z)}{f'(z)}\right| \le \frac{\sum_{n=1}^{\infty} n(n+p)a_{n+p}|z|^n}{p-\sum_{n=1}^{\infty}(n+p)a_{n+p}|z|^n}.$$

Thus, the result follows if

$$\sum_{n=1}^{\infty} n(n+p)a_{n+p}|z|^n \le p\left\{p-\sum_{n=1}^{\infty}(n+p)a_{n+p}|z|^n\right\},$$

or, equivalently,

$$\sum_{n=1}^{\infty}(\frac{n+p}{p})^2 a_{n+p}|z|^n \le 1.$$

But, in view of Theorem 1, we have

$$\sum_{n=1}^{\infty}(n+p)[(n+p)+(n+p-1)(2\varepsilon-1)\beta]a_{n+p} \le 2\varepsilon\beta(p(p-1)-\alpha).$$

Thus *f* is convex if

$$(\frac{n+p}{p})^2|z|^n \le \frac{(n+p)[(n+p)+(n+p-1)(2\varepsilon-1)\beta]}{2\varepsilon\beta(p(p-1)-\alpha)}, n=1,2,3,...,$$

hence

$$|z| = \left\{\frac{p^2(n+p)[(n+p)+(n+p-1)(2\varepsilon-1)\beta]}{(n+p)^2 2\varepsilon\beta(p(p-1)-\alpha)}\right\}^{\frac{1}{n}}, n=1,2,3,...,$$

which complete the proof.

5. **Closure Theorem:**

Next, two results respectively show that the family $WH_p(\alpha,\beta,\varepsilon)$ is closed under taking "arithmetic mean" and "convex linear combination".

Theorem4: Let $f(z)=z^p-\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$ and $g(z)=z^p-\sum_{n=1}^{\infty}b_{n+p}z^{n+p}$ are in the class $WH_p(\alpha,\beta,\varepsilon)$.Then

$h(z)=z^p-\frac{1}{2}\sum_{n=1}^{\infty}(a_{n+p}+b_{n+p})z^{n+p}$ is also in the class $WH_p(\alpha,\beta,\varepsilon)$.

Proof: *f* and *g* both being members of $WH_p(\alpha,\beta,\varepsilon)$, we have in accordance with Theorem 1,

$$\sum_{n=1}^{\infty}(n+p)[(n+p)+(n+p-1)(2\varepsilon-1)\beta]a_{n+p} \le 2\varepsilon\beta(p(p-1)-\alpha) \quad (5.1)$$

and

$$\sum_{n=1}^{\infty}(n+p)[(n+p)+(n+p-1)(2\varepsilon-1)\beta]\, b_{n+p} \le 2\varepsilon\beta(p(p-1)-\alpha).(5.2)$$

To show that h is member of $WH_p(\alpha,\beta,\varepsilon)$ it is enough to show

$$\frac{1}{2}\sum_{n=1}^{\infty}(n+p)[(n+p)+(n+p-1)(2\varepsilon-1)\beta](a_{n+p}+b_{n+p}) \le 2\varepsilon\beta(p(p-1)-\alpha).$$

This is exactly an immediate consequence of (5.1) and (5.2).

Let the function $f_j(z) \quad (j=1,2,\ldots,\ell)$ be defined by

$$f_j(z) = z^p - \sum_{n=1}^{\infty} a_{n+p,j} z^{n+p}, \left(a_{n+p,j} \ge 0, n \in IN, n \ge 1\right). \quad (5.3)$$

Theorem 5: $WH_p(\alpha,\beta,\varepsilon)$ is closed under convex linear combination.

Proof: Let the function $f_j(z)(j=1,2)$ defined by (5.3) be in the class $WH_p(\alpha,\beta,\varepsilon)$. It is sufficient to show that the function *h(z)* defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z), \quad (0 \le \lambda \le 1)$$

is in the class $WH_p(\alpha,\beta,\varepsilon)$.Since, for $0 \le \lambda \le 1$,

$$h(z) = z^p - \sum_{n=1}^{\infty}\left[\lambda a_{n+p,1} + (1-\lambda) a_{n+p,2}\right] z^{n+p}$$

by applying Theorem 1, we have

$$\sum_{n=1}^{\infty}\frac{(n+p)\left[(n+p)+(n+p-1)(2\varepsilon-1)\beta\right]}{2\varepsilon\beta(p(p-1)-\alpha)}\left[\lambda a_{n+p,1}+(1-\lambda)a_{n+p,2}\right]$$

$$=\lambda\sum_{n=1}^{\infty}\frac{(n+p)\left[(n+p)+(n+p-1)(2\varepsilon-1)\beta\right]}{2\varepsilon\beta(p(p-1)-\alpha)}a_{n+p,1}+$$

$$(1-\lambda)\sum_{n=1}^{\infty}\frac{(n+p)\left[(n+p)+(n+p-1)(2\varepsilon-1)\beta\right]}{2\varepsilon\beta(p(p-1)-\alpha)}a_{n+p,2}\leq 1,$$

which implies that *h(z)* is in the class $WH_p(\alpha,\beta,\varepsilon)$ and this completes the proof.

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