ON JORDAN- CENTRALIZERS ON GAMMA RINGS WITH INVOLUTION*

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الخلاصة العربية: قدمنا في هذا البحث دراسة حول تطبيق جوردان المركزي على بعض الحلقات من نوع كاما

ABSTRACT

Let M be a 2-torsion free Γ-ring with involution satisfies the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$.an additive mapping *: $M\rightarrow M$is called Involution if and only if $(a\alpha b)^*=b^*\alpha a^*$and $(a^*)^*=a$. In section one of this paper ,we prove if M be a completely primeΓ-ring and $T:M\rightarrow M$ an additive mapping such that $T(a\alpha a)=T(a)\alpha a^*$ (resp., $T(a\alpha a)=a^*\alpha T(a)$)holds for all $a\in M,\alpha\in\Gamma$.Then T is an anti- left *centralizer or M is commutative (res.,an anti-right* centralizer or M is commutative) and so every Jordan* centralizer on completely primeΓ-ring M is an anti- *centralizer or M is commutative. In section two we prove that every Jordan* left centralizer (resp., every Jordan* right centralizer) onΓ-ring has a commutator right non-zero divisor(resp., onΓ-ring has a commutator left non-zero divisor)is an anti- left *centralizer(resp., is an anti-right *centralizer) and so we prove that every Jordan* centralizer on Γ-ring has a commutator non –zero divisor is an anti-* centralizer .

<u>Key wards</u> : Γ-ring,involution, primeΓ-ring,semi-primeΓ-ring , left centralizer, Left* centralizer ,Right centralizer, Right* centralizer, centralizer, Jordan *centralizer.

1-INTRODUCTION

Throughout this paper,M will represent Γ-ring with center Z .In [8] B.Zalar proved that anyJordan left (resp.,right)centralizer on a 2-torsion free semi-prime ring is a left (resp.,right)Centralizer . In [3] authors proved that anyJordan left (resp.,right) σ- centralizer on a 2-torsion free R has a commutator right (resp., left) non- zero divisor is a left (resp.,right) σ-Centralizer . In [7] Vukman proved that if R is2-torsion free semi-prime ring and $T:R\rightarrow R$ be an additive mapping such that $2T(x^2)=T(x)x+xT(x)$ holds for all $x,y\in R$ .Then T is left and right centralizer.In [6]Rajaa C.Shaheen defined Jordan centralizer onΓ-ring and showed that the existence of a non-zero Jordan centralizer Ton a 2-torsion free completely primeΓ-ring M which satisfies the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$ implies either T is centralizer or M is commutative Γ-ring.We should mentioned the reader that the concept of Γ-ring was introduced by Nobusawa[5] and generalized by Barnes[1],as follows

Let M and Γ be additive abelian groups, M is called a Γ-ring if for any x,y,z $\in M$ and α, $\beta\in\Gamma$,the following conditions are satisfied

(1)$x\alpha y\in M$

(2)$(x+y)\alpha z=x\alpha z+y\alpha z$

$x(\alpha+\beta)z=x\alpha z+x\beta z$

$x\alpha(y+z)=x\alpha y+x\alpha z$

(3)$(x\alpha y)\beta z=x\alpha(y\beta z)$

many properties of Γ-ring were obtained by many research such as [2]

*Let A,B be subsets of a Γ-ringM and Λ a subset of Γ we denote AΛB the subset of M consisting of all finite sum of the form $\sum a_i\lambda_i b_i$ where $a_i \in$ **A,** $b_i \in$ **B** and $\lambda_i \in \Lambda$.Aright ideal(resp.,left ideal) of a Γ-ring M is an additive subgroup I of M such that IΓM$\subset$I(resp.,MΓI$\subset$I).If I is a right and left ideal inM,then we say that I is an ideal .M is called a 2-torsion free if 2x=0 implies x=0 for all x$\in$M.AΓ-ringM is called prime if aΓMΓb=0 implies a=0 or b=0 and M is called completely prime if a Γb=0 implies a=0 or b=0(a,b $\in$M),Since a Γb Γa Γb $\subset$a ΓM Γb,then every completely prime Γ-ring is prime.AΓ-ring M is called semi-prime if aΓMΓa=0 implies a=0and M is called completely semi-prime if a Γa=0 implies a=0(a$\in$M)*

Let R be a ring,.A left(right) centralizer of R is an additive mapping T:R →R which satisfies T(xy)=T(x)y(T(xy)=xT(y)*) for all x,y$\in$R.A Jordan centralizer be an additive mapping T which satisfies T(x$\circ$y)=T(x) $\circ$y=x$\circ$T(y).*
*A Centralizer of R is an additive which is both left and right centralizer.An easy computation shows that every centralizer is also a Jordan centralizer.Many Papers work about the problem every Jordan centralizer be centralizer such as in[8] .In [6] Rajaa work this problem on some kind of Γ-ring.In this paper we define Jordan *centralizer on Γ-ring with involution* and study this concept on some kind of Γ-ring with involution.*
Now ,we shall give the following definition which are basic in this paper.

***Definition1.2:**- Let M be a Γ-ring with involution* and let T:M → M be an additive map ,T is called*

***Left* centralizer** of M,if for any a,b $\in$M and $\alpha \in \Gamma$, the following condition satisfy T(aαb)=T(a) αb*,*

***Right* centralizer** of M,if for any a,b $\in$M and $\alpha \in \Gamma$, the following condition satisfy*
T(aαb)=aαT(b),*

***Jordan left* centralizer** if for all a$\in$M and $\alpha \in \Gamma$, the following condition satisfy T(aαa)=T(a) αa**

***Jordan Right* centralizer** if for all a$\in$M and $\alpha \in \Gamma$, the following condition satisfy T(aαa)= a* αT(a)*

***Jordan* centralizer** of M,if for any a,b $\in$M and $\alpha \in \Gamma$, the following condition satisfy T(aαb+b αa)=T(a) αb*+b*αT(a)=a* αT(b)+T(b) αa*.*

Now we shall prove the following Lemmas which are necessarily to prove our main result in this paper.

***Lemma 1.3:**-Let M be a 2-torsion freeΓ-ring with involution* and let T:M → M be an additive mapping which satisfies T(aαa)=T(a) αa*,(resp., T(aαa)= a* αT(a)) for all a$\in$M and $\alpha \in \Gamma$,then the following statement holds for all a,b,c$\in$M and α, $\beta \in \Gamma$,*

(i) T(aαb+b αa)=T(a) αb+T(b) αa**
(resp., T(aαb+b αa)=a α T(b)+ b *α T(a))*

(ii) Especially if M is 2-torsion free and aαbβc=aβbαc for all a,b,c$\in$M and α,$\beta \in \Gamma$ then
T(aαbβa)=T(a) αbβa* (resp., T(aαbβa)= a* αb*β T(a))*

(iii) T(aαbβc+cαbβa)=T(a) αbβc*+T(c) αb*βa*.*
*(resp., T(aαbβc+cαbβa)=a *αb*βT(c)+c* αb*βT(a)*

***Proof:**-(i) Since T(aαa)=T(a) αa* for all a$\in$M and $\alpha \in \Gamma$,…….(1)*

Replace a by a+b in (1),we get

$T(a\alpha b+b\alpha a)=T(a)\alpha b^*+T(b)\alpha a^*$……………………………….(2)

(ii) by replacing b by $a\beta b+b\beta a$, $\beta\in\Gamma$

$W=T(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a)$

$=T(a)\alpha(a\beta b+b\beta a)^*+T(a\beta b+b\beta a)\alpha a^*$

$= T(a)\alpha(a\beta b)^*+T(a)\alpha(b\beta a)^*+(T(a)\beta b^*+T(b)\beta a^*)\alpha a^*$

*Since * is involution,then*

$W = T(a)\alpha(b^*\beta a^*)+T(a)\alpha(a^*\beta b^*)+T(a)\beta b^*\alpha a^*+T(b)\beta a^*\alpha a^*$

Since $a\alpha b\beta c=a\beta b\alpha c$*,then*

$W= T(a)\alpha(a^*\beta b^*)+2T(a)\alpha(b^*\beta a^*)+T(b)\beta a^*\alpha a^*$

On the other hand

$W = T(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a)$

$=T(a\alpha(a\beta b)+a\alpha(b\beta a)+(a\beta b)\alpha a+(b\beta a)\alpha a)$

$=T(a\alpha a\beta b+b\beta a\alpha a)+2T(a\alpha b\beta a)$

$=T(a)\alpha a^*\beta b^*+T(b)\beta a^*\alpha a^*+2T(a\alpha b\beta a)$

By comparing these two expression of W, we get

$2T(a\alpha b\beta a)= 2T(a)\alpha b^*\beta a^*$

Since M is 2-torsion free ,then

$T(a\alpha b\beta a)= T(a)\alpha b^*\beta a^*$……………………………….(3)

(iii)In (3) replace a by a+c ,to get

$T(a\alpha b\beta c+c\alpha b\beta a)=T(a)\alpha b^*\beta c^*+T(c)\alpha b^*\beta a^*$………………(4)

***<u>Theorem 1.4:</u>-** Let M be a 2-torsion free completely prime* Γ*-ring which satisfy the condition* $x\alpha y\beta z=x\beta y\alpha z$ *for all* $x,y,z\in M$, $\alpha,\beta\in\Gamma$*, and let* $T:M\rightarrow M$ *be an additive mapping which satisfies* $T(a\alpha a)=T(a)\alpha a^*$ *,for all* $a\in M$ *and* $\alpha\in\Gamma$*,then* $T(a\alpha b)=T(b)\alpha a^*$*, for all* $a,b\in M$ *and* $\alpha\in\Gamma$ *or M is commutative* Γ*-ring.*

***<u>Proof</u>:**-By [Lemma 1.3,(iii)] ,we have*

$T(a\alpha b\beta c+c\alpha b\beta a)=T(a)\alpha b^*\beta c^*+T(c)\alpha b^*\beta a^*$

Replace c by $b\alpha a$

$W=T(a\alpha b\beta(b\alpha a)+(b\alpha a)\alpha b\beta a)$

$=T(a)\alpha b^*\beta a^*\alpha b^*+T(b\alpha a)\alpha b^*\beta a^*$

On the other hand

$W= T((a\alpha b)\beta(b\alpha a)+(b\alpha a)\alpha(b\beta a))$

$= T(a)\alpha b^*\beta b^*\alpha a^*+T(b\alpha a)\beta a^*\alpha b^*$

By comparing these two expression of W, we get

$T(b\alpha a)\beta(a\alpha b-b\alpha a)^* + T(a)\alpha b^*\beta(b\alpha a-a\alpha b)^*=0$

$T(b\alpha a)\beta(a\alpha b-b\alpha a)^* - T(a)\alpha b^*\beta(a\alpha b-b\alpha a)^*=0$

$(T(b\alpha a)-T(a)\alpha b^*)\beta(a\alpha b-b\alpha a)^*=0$………………….(5)

Since M is completely prime Γ*-ring ,then*

either $T(b\alpha a)-T(a)\alpha b^*=0$ *or* $(a\alpha b-b\alpha a)=0$

if $T(b\alpha a)-T(a)\alpha b^*=0$*then* $T(b\alpha a)=T(a)\alpha b^*$ *so T is an anti-left *centralizers.*

and if $a\alpha b-b\alpha a=0$ *for all* $a,b\in M$ *and* $\alpha\in\Gamma$*,then M is commutative* Γ*-ring*◙

***<u>Theorem 1.5:</u>-** Let M be a 2-torsion free completely prime* Γ*-ring which satisfy the condition* $x\alpha y\beta z=x\beta y\alpha z$ *for all* $x,y,z\in M$, $\alpha,\beta\in\Gamma$*, and and let* $T:M\rightarrow M$ *be an additive mapping which satisfies* $T(a\alpha a)=a^*\alpha T(a)$*for all* $a\in M$ *and*

$\alpha \in \Gamma$,then T(aαb)= b α T(a) for all a,b$\in$M and $\alpha \in \Gamma$ or M is commutative Γ-ring.*

***Proof:**- From[Lemma 1.3,(iii)],we have for all a,b,c$\in$M and α, $\beta \in \Gamma$,*

T(a αb βc+c αb βa)=a αb *βT(c)+c* αb* βT(a)……………(6)*

In (6) replace c by a αb, then

W= T(a αb β(a αb) +(a αb) αb βa)

=aαb*βT(a αb)+ b* αa* βb* αT(a)*

on the other hand

W = T(a αb β(aαb) +(a αb β (bα a))

=bαa*βT(aαb)+a*αb*βb*αT(a)*

by comparing these two expression of W ,we get

(aαb-bαa)β(T(aαb)-b*αT(a))=0…………………………….(7)*

since M is completely prime Γ-ring,then

either (T(bαa)-bαT(a))=0 $\Rightarrow$ T(aαb)=b*αT(a) and so T is an anti- right *centralizers or aαb-bαa=0 $\Rightarrow$ aαb=bαa $\Rightarrow$M is commutative Γ-ring*◙

***Corollary 1.6:**- Every Jordan* centralizer of 2-torsion free completely prime Γ-ring M which satisfy the condition xαyβz=xβyαz for all x,y,z$\in$M , α, $\beta \in \Gamma$, is an anti-*centralizer on M or M is commutative.*

2-JORDAN CENTRALIZERS ON SOME GAMMA RING*

***Theorem 2.1:**- Let M be a 2-torsion free Γ-ring which satisfy the condition xαyβz=xβyαz for all x,y,z$\in$M , α, $\beta \in \Gamma$ and has a commutator right non-zero divisor and let T:M→M be an additive mapping which satisfies T(aαa)= T(a) α a* for all a$\in$M and $\alpha \in \Gamma$,then T(aαb)= T(b) α a* for all a,b$\in$M and $\alpha \in \Gamma$.*

***Proof:**- from (5),we have*

(T(bαa)- T(a)αb)β(aαb-b α a)*=0*

if we suppose that

δ(b,a)= T(bαa)- T(a)αb and [a,b]*=(aαb-b α a)**

then δ(b,a) β [a,b]=0 for all a,b$\in$M and α,$\beta \in \Gamma$……………(9)*

Since M has a commutator right non-zero divisor ,then $\exists$x,y$\in$M, $\alpha \in \Gamma$ such that if for every c$\in$M, $\beta \in \Gamma$

cβ[x,y]=0$\Rightarrow$c=0

*since * is involution ,we have δ(y,x) β [x,y]=0 and so δ(x,y)=0………………(10)*

replace a by a+x

δ(b,a+x) β [a+x,b]=0 and so by () and ()*

δ(b,x) β [a,b]+ δ(b,a) β [x,b]*=0……………..(11)*

Now replace b by b+y

δ(b+y,x) β [a,b+y]+ δ(b+y,a) β [x,b+y]*=0*

and so by (10) and (11),we get

δ(b,x) β [a,b]+ δ(y,x) β [a,b]*+δ(b,x) β [a,y]*+δ(y,x) β [a,y]*+*

δ(b,a) β [x,b]+ δ(y,a) β [x,b]*+ δ(b,a) β [x,y]*+ δ(y,a) β [x,y]*=0*

by (11),we get

δ(a,b) β [x,y]- δ(x,y) β [a,y]*=0*

then

$\delta(a,b)\ \beta\ [x,y]^*=0$,and so $\delta(a,b)=0$ for all $a,b\in M$ and $\alpha\in\Gamma$
$T(a\alpha b)=T(b)\alpha a^*\Rightarrow T$ Is anti- left *centralizer of M.

<u>Theorem 2.2:</u>- Let M be a 2-torsion free Γ-ring with involution which satisfy the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$, $\alpha,\ \beta\in\Gamma$ and has a commutator left non-zero divisor and let $T:M\to M$ be an additive mapping which satisfies $T(a\alpha a)=a^*\ \alpha\ T(a)$ for all $a\in M$ and $\alpha\in\Gamma$, then $T(a\alpha b)=a^*\ \alpha\ T(b)$ for all $a,b\in M$ and $\alpha\in\Gamma$.

<u>Proof</u>:- From[Lemma 1.3,(iii)],we have
$T(a\ \alpha b\ \beta c+c\ \alpha b\ \beta a)=a^*\ \alpha b^*\ \beta T(c)+c^*\ \alpha b^*\ \beta T(a)$……………(12)
In (12) replace c by $b\ \alpha a$,then
$W=T(a\ \alpha b\ \beta(b\ \alpha a)+(b\ \alpha a)\ \ \alpha b\ \beta a)$
$=a^*\alpha b^*\beta T(b\ \alpha a)+b^*\alpha a^*\beta b^*\alpha T(a)$
on the other hand
$W=T(a\ \alpha(b\ \beta b)\alpha a+(b\ \alpha a)\ \ \alpha(b\ \beta a))$
$=a^*\alpha b^*\beta b^*\alpha T(a)+b^*\alpha a^*\beta T(b\alpha a)$
by comparing these two expression of W ,we get
$a^*\alpha b^*\beta(T(b\alpha a)-b^*\alpha T(a))-b^*\alpha a^*\beta(T(b\alpha a)-b^*\alpha T(a))=0$
then if we suppose $B(b,a)=(T(b\alpha a)-b^*\alpha T(a))$
$[a,b]^*\beta\ B(b,a)=[a,b]^*\beta B(a,b)=0$ for all $a,b\in M$, α,
$\beta\in\Gamma$…………………….(13)
Since M has a commutator left non-zero divisor then $\exists\, x,y\in M$, $\alpha\in\Gamma$ such that if for every $c\in M$, $\beta\in\Gamma$, $[x,y]\ \beta c=0\Rightarrow c=0$
then by (13),we have
$[x,y]\ \beta\ B(x,y)=0\Rightarrow B(x,y)=0$…………………(14)
in (13) replace a by a+x
$[a+x,b]^*\ \beta\ B(a+x,b)=0$
then by (13)
$[x,y]^*\beta\ B(a,b)+[a,b]^*\beta\ B(x,b)=0$…………………(15)
Now replace b by b+y
$[x,b+y]^*\beta\ B(a,b+y)+[a,b+y]^*\beta\ B(x,b+y)=0$
then by using (14) and (15),we get
$[x,y]^*\beta B(a,b)=0$
and since [x,y] is a commutator left non-zero divisor then
$B(a,b)=0\Rightarrow T(a\alpha b)=a^*\alpha T(b)$ which is mean that T is an anti- right *centralizer

<u>Corrolary2.3</u>:- Let M be a 2-torsion free Γ-ring with involution which satisfy the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$, $\alpha,\ \beta\in\Gamma$, has a commutator non-zero divisor and let $T:M\to M$ be a Jordan *centralizer then T is *centralizer◙

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