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Some Results on Invariant Best Approximations in p-Normed Spaces

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بعض النتائج عن التقارب الأفضل الثابت في فضاءاتp-Normed

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الخلاصة

في هذا البحث سنقدم مبرهنتين في فضاءات p-normed.الأولى حول وجود النقطة الصامدة في التطبيقات متعددة القيم غير التوسعية ومن ثم نستخدمها في برهان نتيجة (Theorem 2.3) والتي هي حول وجود التقارب الأفضل الثابت. كذلك سنحتاج إلى إعطاء النظرية المساعدة (1.14)

Abstract

In this paper we give two theorems in p-normed spaces. The first one is about the existence of fixed point of multivalued nonexpansive mapping and then we use these theorems in the proof of the result(Theorem2.3)about existence of invariant best approximation. Note that, we need to give Lemma (1.14)

1. Introduction and Preliminaries

Fixed point theorems have been used at many places in approximation theory.One of them is while existence of best approximation is proved .Later on, number of results were developed using fixed point theorem to prove the existences of best approximation. Kaneko [4] has proved the coincidence point theorem for complete metric space. Latif and Tweddle[9]obtained the coincide-nece point theorem for Banach spaces. In this paper we extend the work of Lami Dozo[6] and Singh[14] to multivalued nonexpansive mapping in the setting of p-normed space.Here and throughout the paper the symbols $\xrightarrow{w}$ and $\rightarrow$ denote weak and strong convergence, respectively.

To prove our results we need to give the following:

Definition (1.1). Let X be a linear space over the field of real numbers. A p-norm on X is a real valued function $\|.\|_p$ on X with $0 < p \leq 1$, satisfying the following conditions:

(1) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$
(2) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$
(3) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

for all x, y $\in$ X and all scalars α. The pare (X, $\|.\|_p$)is called a p-normed space [5,pp.164].It is a metric space with $d_p(x,y) = \|x - y\|_p$ for all x,y$\in$X[5, pp.165] ,defining a translation invariant metric d_p on X. If p = 1, we obtain a normed linear space. It is well known that the topology of every Hausdorff locally bounded topological linear space given by some p-norms, $0 < p \leq 1$[7].The sp-aces l_p,$0 < p \leq 1$ are p-normed spaces. Note that,a p-normed space is not neces-sarily a locally convex space (such as the spaces l_p, $0 < p \leq 1$)[3].

Definition(1.2). Let X be a p-normed space, a set M in X is said to be convex if $\lambda x + (1 - \lambda)y \in M$, whenever x, y $\in$ M and $0 \leq \lambda \leq 1$[2].

Definition (1.3). Let X be a p-normed space. A set M in X is said to be starshaped, if there exists at least one point q $\in$M such that the line segment joining x to q is contained in M (that is, $\lambda x + (1 - \lambda)q \in M$ and $0 < \lambda < 1$).In this case, q called starcenter of M[2].

A convex set is obviously starshaped with respect to each of it's points, but starshaped need not be convex, (see [2]).

Let X be a p-normed space and M a nonempty subset of X. We denoted by:

2^X the collection of all nonempty subsets of X.

CB(X) the collection of all nonempty closed bounded subsets of X.
K(X) the collection of all nonempty compact subsets of X.

The Hausdorff metric H_p on CB(X) induced by the p-norm of X is defined by:

$$H_p(A, B) = \max\{ \sup_{a \in A} d_p(a, B), \sup_{b \in B} d_p(A, b)\}$$

Where $d_p(x,B) = \inf \{\|x - y\|_p : y \in B\}$ for each $x \in X$ [1].

Definition(1.4). Let X be a p-normed space and M be a nonempty subset of X. Let $x \in X$. An element $y \in M$ is called a best M-approximation to x, if :

$$\|x - y\|_p = d_p(x, M).$$

The set of best M-approximation to x is denoted by $P_M(x)$ and defined as

$P_M(x) = \{z \in M : \|x - z\|_p = d_p(x, M)\}$[5].

The set $P_M(x)$ is always a bounded subset of X and it is closed or convex if M is closed or convex (see[5]).

Definition (1.5). Let X be a p-normed space, M be a nonempty subset of X and f: $M \to X$ be a single-valued mapping.A multivalued mapping T:$M\to$ CB(X) is said to be a multivalued f-contraction mapping if for a fixed constant k, $0 \le k \le 1$ and for all $x, y \in X$,

$$H_p(T(x), T(y)) \le k \| f(x) - f(y) \|_p$$

If f = I (the identity mapping on X), then each multivalued f-contraction mapping is multivalued contraction mapping[1].

Definition (1.6). Let X be a p-normed space, M be a nonempty subset of X and f : $M \to X$ be a single-valued mapping. A multivalued mapping T: $M \to$ CB(X) is said to be a multivalued f-nonexpansive mapping if for all $x, y \in X$,

$$H_p(T(x), T(y)) \le \| f(x) - f(y) \|_p[1].$$

Definition (1.7). Let X be a p-normed space, M be a nonempty subset of X and f : $M \to X$ be a single-valued mapping and T : $M \to$ CB(X),then M is called an invariant set of T if $M \subseteq T(M)$[15,pp.112].

Definition (1.8). Let X be a p-normed space, M be a nonempty subset of X and $T : M \to CB(X)$.An element $x \in X$ is called a fixed point of multivalued mapping T if $x \in T(x)$ [15,pp.184].

Definition (1.9). Let X be a p-normed space, M be a nonempty subset of X and $T : M \to CB(X)$.An element $x \in X$ is called a coincidence point of f and T if $f(x) \in T(x)$ [1].

Note that, we denoted by F(T) the set of fixed points of T and by $C(f \cap T)$ the set of coincidence points of f and T.

Definition(1.10). A complete p-normed space X is said to be satisfy Opial's property if for every sequence$\{x_n\}$in X weakly convergent to $x \in X$, the inequ-ality :

$$\lim_{n\to\infty}\inf \|x_n - x\|_p < \lim_{n\to\infty}\inf \|x_n - y\|_p$$

holds for all $x \neq y$[2].

A complete p-normed space satisfying Opial's property is called Opial's space.It is well known that every Hilbert space satisfies Opial's property [11].The spaces l_p,$(1 < p < \infty)$satisfying Opial's property[5]. However,there are Banach spaces which do not satisfy Opial's property,such as $L_p[0, 2\pi]$,$(p \neq 2)$ [3].

Definition(1.11). Let X be a p-normed space, a multivalued mapping $T: M \to 2^X$ is said to be demiclosed, if for every sequence$\{x_n\} \subset M$ and $y_n \in T(x)$,
$n = 1,2,3,\ldots$such that $x_n \xrightarrow{w} x$ and $y_n \to y$,we have $x \in M$ and $y \in T(x)$ [1].

Definition(1.12). A subset M of a normed space X is called a weakly comp- act if for every sequence $\{x_n\}$ in M contains a subsequence which converge weakly in M[13,pp.168].

In[4],Kaneko has proved the following coincidence point theorem for com- plete metric space.

Theorem(1.13). Let (X, d) be a complete metric space, $f : X \to X$ be a cont- inuous mapping and let $T : X \to CB(X)$ which commutes with f and $T(X) \subseteq f(X)$. Suppose that f is contraction mapping, then there exists an $x_0 \in X$ such that

$$f(x_0) \in T(x_0).$$

In [10], Latif and Tweddle proved the following result which is an extension of Lami Dozo[6].

Lemma(1.14). Let M be a nonempty weakly compact subset of a Banach space X which satisfying Opail's property. Let $f : M \to X$ be a weakly continuous mapping and $T : M \to K(X)$ an f-nonexpansive mapping. Then (f-T) is demiclosed [9].

In [8], Latif and Tweddle obtained the following coincidence point result in the setting of Banach spaces.

Theorem(1.15). Let X be a Banach space and M be a weakly compact sub- set which is starshaped with respect to $q \in M$. Let $f : M \to M$ be a weakly cotinuous mapping such that $f(M) = M$ and $f(q) = q$ and let $T : M \to K(M)$ be a multvalued f-nonexpansive mapping which commutes with f.Then $C(f \cap T) \neq \emptyset$,provided that

(1) $(f \cap T)$ is demiclosed, or

(2) X satisfies Opail's property.

2. Main results

The main objective of this section is to prove Lemma (1.14) ,Theorem (1.15) in the setting of p-normed space, prove best approximation theorem, and a corollary in the setting of a complete p-normed spaces satisfying Opial's pr- operty.

First we prove Lemma (1.14) in the setting of p-normed spaces.

Lemma (2.1). Let M be a weakly compact subset of a complete p-normed space X which satisfying Opail's property. Let $T: M \to K(X)$ be an nonexpans-ive multivalued mapping. Then T is demiclosed mapping.

Proof. Let $\{x_n\} \subset M$ and $y_n \in T(x_n)$ be such that $x_n \xrightarrow{w} x$ and $y_n \to y$.It is obvious that $x \in M$. Since $y_n \in T(x_n)$, we get

$$y_n = x_n - u_n \text{ for some } u_n \in T(x_n) \quad (2.1).$$

Since T(x) is compact set, there is a $v_n \in T(x_n)$ such that

$$\| u_n - v_n \|_p \le H_p(T(x_n),T(x)).$$

So, by using the nonexpansiveness of T, we have

$$H_p(T(x_n),T(x)) \le \| x_n - x \|_p.$$

Thus $$\| u_n - v_n \|_p \le \| x_n - x \|_p \quad (2.2).$$

from (2,1)and(2.2), passing to the limit with respect to n, we obtain

$$\liminf_{n\to\infty} \| x_n - x \|_p \ge \liminf_{n\to\infty} \| u_n - v_n \|_p \ge \liminf_{n\to\infty} \| x_n - y_n - v_n \|_p \quad (2.3)$$

T(x) is compact and $y_n \to y$, so for a convenient subsequence still denoted by $\{v_n\}$,we have $v_n \to v \in T(x)$.So from (2.3) we have

$$\liminf_{n\to\infty} \| x_n - x \|_p \ge \liminf_{n\to\infty} \| x - y - v \|_p. \quad (2.4)$$

Since X satisfies Opial's property and $x_n \xrightarrow{w} x$, then

$$\liminf_{n\to\infty} \| x_n - x \|_p < \liminf_{n\to\infty} \| x - y - v \|_p. \quad (2.5)$$

From (2.4)and(2.5), we obtain $x = y + v$.Thus $y = x - v \in T(x)$ which proves that T is demiclosed. ■

We have the following fixed point result for a complete p-normed space.

Theorem(2.2). Let X be a p-normed space and M be a weakly compact subset which is starshaped with respect to $q \in M$. Let $T : M \to K(M)$ be a multivalued nonexpansive mapping.If T is demiclosed mapping or X is Opail's space, then $F(T) \neq \emptyset$.

Proof. Consider a sequence$\{k_n\}$of real numbers such that $0 < k_n < 1$ and $k_n \to 1$ as $n \to \infty$. Define a mapping T_n by setting

$$T_n(x) = k_n T(x) + (1 - k_n)q \text{ for all } x \in M.$$

Since T maps M into K(M), we observe that for each $n \ge 1$, T_n maps M into K(M).Since M is starshaped with respect to $q \in M$ and $T(M) \subseteq M$. Also $T_n(M) \subseteq M$. Let $x, y \in M$, then

$$H_p(T_n(x), T_n(y)) = k_n H_p(T(x), T(y)),$$

By the nonexpansiveness of T, we have

$$H_p(T_n(x), T_n(y)) \le k_n \| x - y \|_p$$

Showing that T_n is an contraction mapping. Now, let I be an identity mapping on X, then I is continuous and commutes with $T_n \subseteq X = I(X)$.Thus all the conditions of Theorem(1.15)are satisfied and hence there is an $x_n \in M$ such that

$$x_n \in T_n(x_n).$$

So by the definition of $T_n(x_n)$, there is a $u_n \in T_n(x_n)$ such that

$$x_n = k_n u_n + (1 - k_n)q$$

$$x_n = k_n u_n + (1 - k_n)q + k_n x_n - k_n x_n$$

$$k_n x_n - k_n u_n = k_n x_n + (1 - k_n)q - x_n$$

$$x_n - u_n = x_n + (\frac{1}{k_n} - 1)q - \frac{1}{k_n} x_n$$

$$\|x_n - u_n\|_p = (\frac{1}{k_n} - 1)^p \|q - x_n\|_p$$

$$\|x_n - u_n\|_p = (\frac{1}{k_n} - 1)^p \{\|q\|_p - \|x_n\|_p\}.$$

Since $T(M) \subseteq M$ is bounded and $x_n \in T_n(x_n) \subseteq M$,we have $\|x_n\|_p$ is bounded ,so by the fact $k_n \to 1$,we have $\|x_n - u_n\|_p \to 0$. Since M is weakly compact , there is a subsequence$\{x_{n_i}\}$of sequence which converges weakly to $x_0 \in M$. As $x_{n_i} - u_{n_i} \in T(x_{n_i})$, if T is demiclosed, then $0 \in T(x_0)$ and hence $x_0 \in T(x_0)$. If X satisfies Opail's property,then it follows from Lemma (2.14) that T is demiclosed and hence T have a fixed point $x_0 \in M$ as in the previous case. ■

Theorem(2.3). Let X be a complete p-normed space. Let $T : X \to CB(X)$be an nonexpansive mapping such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$. Let M be a nonempty T-invariant subset of X and $x_0 \in F(T)$. Assume that $P_M(x_0)$is nonempty, weakly compact and starshaped with respect to $q \in M$. If T is demiclosed on $P_M(x_0)$, then $P_M(x_0) \cap F(T) \neq \varnothing$.

Proof. Let $D = P_M(x_0)$ and let $u \in D$. Then $u \in M$ and $\|x_0 - u\|_p = d_p(x_0, M)$.Let $v \in T(u) \subset M$. Then we have

$$\|v - x_0\|_p \leq H_p(T(u), T(x_0))$$

so by using the nonexpansiveness of T, we get

$$H_p(T(u), T(x_0)) \leq \|u - x_0\|_p \Rightarrow \|v - x_0\|_p \leq \|u - x_0\|_p$$

which gives that $v \in D$ and thus $T(u) \subset D$. Therefore T carries D into CB(D).Now ,let q be the starshaped of D, then for each $x \in D$ and any $k(0 < k < 1)$,

$$(1-k)q + kx \in D.$$

Take $\{k_n\}$ sequence of real numbers such that $0 < k_n < 1$ and $k_n \rightarrow 1$ as $n \rightarrow \infty$.

Now, for each n define a multivalued mapping T_n by setting

$$T_n(x) = k_n T(x) + (1 - k_n)q, \text{ for all } x \in D$$

Clearly, each T_n maps D into CB(D). Let I be an identity mapping on X, then each T_n commutes with I for each n and $T_n(D) \subseteq D = I(D)$. Let x, $y \in D$. Then by the definition of T_n and the nonexpansiveness of T, we have

$$H_p(T_n(x), T_n(y)) = kn\, H_p(T(x), T(y)) \leq k_n \|x - y\|_p$$

which proves that each T_n is an contraction mapping. Also, Since D is a comp-lete metric space, it follows from Theorem (1.15) that for each $n \geq 1$, there exists $x_n \in D$ such that $x_n \in T_n(x_n)$ such that

$x_n = k_n y_n + (1 - k_n)q$

$x_n = k_n y_n + (1 - k_n)q + k_n x_n - k_n x_n$

$k_n x_n - k_n y_n = (1 - k_n)q - (1 - k_n)\, x_n$

$x_n - y_n = (1 - \frac{1}{k_n})(q - x_n)$

$\|x_n - y_n\|_p = (1 - \frac{1}{k_n})^p \|(q - x_n)\|_p \rightarrow 0$ as $n \rightarrow \infty$.

Since D is weakly compact, there exists a subsequences of$\{x_n\}$, still denoted by $\{x_n\}$, and we have $x_n \xrightarrow{w} z \in D$. Now, as $x_n - y_n \in T(x_n)$and T is demiclos-ed we conclude that $0 \in T(z)$ and hence $z \in T(z)$. ■

Now,by Lemma (2.1),we have the following result on invariant approximation.

Corollary(2.4). Let X be a complete p-normed space and satisfying Opail's property. Let $T : X \rightarrow K(X)$be a nonexpansive mapping such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$. Let M be a nonempty T-invariant subset of X and $x_0 \in F(T)$. Assume that $P_M(x_0)$is nonempty, weakly compact and starshaped with respect to $q \in P_M(x_0)$, then $P_M(x_0) \cap F(T) \neq \emptyset$.

Proof. Let $D = P_M(x_0)$ and let $u \in D$. Then $u \in M$ and

$$\|x_0 - u\|_p = d_p(x_0, M).$$

Let $v \in T(u) \subset M$. Then we have

$$\|v - x_0\|_p \leq H_p(T(u), T(x_0))$$

so by using the nonexpansiveness of T, we get

$$H_p(T(u), T(x_0)) \leq \|u - x_0\|_p$$

Thus

$$\|v - x_0\|_p \leq \|u - x_0\|_p = d_p(x, M)$$

which gives that $v \in D$ and thus $T(u) \subset D$. Therefore T carries D into K(D). Thus by Lemma(2.1), T is demiclosed on D. Now the result follows from Theorem(2.4). ■

Remark(2.5). Theorem(2.4) extends Theorem 3 [8].

References

[1] A.Bano and A.R.Khan. (2003). Radovi Matematicki. 12 : 27-36 .

[2] W. G., Jr., Dotson. (1973).Proc. Amer. Math. Soc. 38(1): 155–156

[3] N.Hussain and B.E.Rhoades.(2006).Hindawi Publishing Corporation, Fixed Point Theory and Applications ID24543:1-9.

[4] H. Kaneko. (1985).Boll.U.M.I. 6: 29-33.

[5] G.KÄothe.(1969).Topological Vector SpaceI.Springer-Verlag.Berlin.

[6] E. Lami Dozo. (1973).Proc. Amer. Math. Soc. 38 : 286-292.

[7] A.Latif and A.Bano. (2002).Tamkang J.Math. 33 (1) : 89-92.

[8] A.Latif. (2001). Arch. Math. (Brno) 37: 71–75.

[9] A.Latif and I. Tweddle. (1999). Bull. Austral. Math. Soc. 59: 111-117.

[10] A.Latif and I. Tweddle. (1999). Demonstratio Math. 32 (3) : 565-574.

[11] L.Narici and E.Bekenstein. (1985). Topological Vector Space. Marcel Dekker and INC. New York.

[12] Z. Opial. (1967). Bull. Amer. Math. Soc. 73 : 531-537.

[13] A. H. Siddiqi.(1986).Functional Analysis With Applications. Vimal Offset Printers. Meerute, Tata Mcgraw-Hill Publishing Company Limited limited. New Delhi.

[14] S.P. Singh. (1979).J. Approx. Theory 25 : 89-90

[15] E. Zidler. (1986).Nonlinear Functional Analysis and Application.(Fixed Point Theorems I). Springer Verlage. New York.

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