Iraqi journal of civil engineering (IJCE)
P-ISSN: 1992-7428 ; E-ISSN: 2706-686X

Vol. 18 , No. 2 (2024) , pp. 52 ~ 72
DOI: 10.37650/ijce.2024.180205

Iraqi Journal of Civil Engineering

Journal homepage: https://ijce.uoanbar.edu.iq/

Solving Clamped Kirchhoff Plate Bending Problems Using Superposition of Sinusoidal and Polynomial Basis Functions in the Ritz Variational Method

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PAPER INFO

Paper history
Received: 10/06/2024
Revised: 23/10/2024
Accepted: 18 /11/2024

Keywords:
Ritz variational method,
Kirchhoff-Love hypothesis,
Kirchhoff plate theory,
clamped plate,
bending moment.

ABSTRACT

Thin plate bending analysis is an important research subject due to the extensive use of plates in the different fields of engineering and the need for accurate solutions. This article uses the Ritz variational method and a superposition of trigonometric and polynomial basis functions to solve the Kirchhoff-Love plate bending problems (KLPBPs). The unknown displacement function in the Ritz variational functional (RVF) to be minimized is sought as linear combinations of basis functions $F_m(x)$ and $G_n(y)$ that are found by superposing sine series and third degree polynomial functions with the polynomial parameters determined such that all boundary conditions of deformation and force are satisfied. The displacement is thus expressed in terms of unknown displacement parameters A_{mn} which are found upon minimization of RVF with respect to A_{mn}. The minimization process gave a matrix stiffness equation in A_{mn} with the stiffness matrix and force matrix found from $F_m(x)$ and $G_n(y)$ and their derivatives. The algebraic equation is solved, and the deflection and bending moments obtained. The problems considered were clamped (CCCC) plates under uniform and hydrostatic distribution of loads and plates with opposite edges clamped, the rest simply supported (CSCS) under uniformly distributed loading. Comparison of the solutions by Generalized Integral transform method, Levy-Nadai series method, and symplectic eigenfunction superposition confirms that the present results are accurate.

1. Introduction

1.1. Preliminaries

Plates are common parts of buildings, aeroplanes, naval vessels, machines, and spacecrafts that have transverse dimensions that are smaller than their in-plane dimensions. They are three-dimensional (3D) in isometric view, and can be subjected to transverse or axially applied forces. They are classified based on their

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material properties as homogeneous or heterogeneous, isotropic or anisotropic, elastic or inelastic. Based on their geometry, plates are classified as: rectangular, circular, elliptical, oval, quadrilateral, skewed, parallelogramic or rhombic. They could be made of different materials or laminated or functionally graded material (FGMs) in which case they are composite plates, laminated plates or functionally graded material (FGM) plates. They could have flat or curved surfaces.

The plate geometry is commonly referenced using the middle plane, which is a plane surface equidistant from the top and bottom surfaces of the plate. The middle surface is thus the xy plane ($z = 0$). The plate behaviour is determined largely by the transverse dimension (depth or thickness) rather than the two in-plane dimensions (length and width). The ratio of least in-plane dimension, a, to the thickness, h, has been used to classify plates as thin when $a/h \geq 20$, moderately thick when $10 < a/h < 20$, thick when $3 < a/h < 10$, and very thick when $a/h < 3$. (Steele and Balch, 2009).

1.2. Literature Review

Early studies of plates were done by Kirchhoff, Germain, Love, Poisson and others who formulated equations that were valid for thin plates (Aginam et al., 2012).

Kirchhoff's plate theory (KPT) is a pioneering study using Kirchhoff-Love hypothesis of normality of cross-sectional lines to the middle surface which impliedly meant the disregard of shear stresses (Ike et al., 2017). The normality hypothesis required that cross-sectional lines originally straight and normal to the middle plane before bending would remain straight and normal to the middle plane after bending determination (Ike et al., 2017; Ike, 2023a, 2023b, 2023c; 2023d; Ike and Oguaghamba, 2023). Impliedly, shear stresses which are responsible for such cross-sectional distortions are neglected, and this restricts the scope of the resulting KPT to thin plates for which such shear stresses do not significantly affect the behaviours in flexure, vibration or buckling (Koc, 2023), Nwoji et al. (2018a, 2018b, 2018c).

Research efforts that are directed at improvement to plate theories yielded shear deformation theories developed by Reissner (1945), Mindlin (1951), higher order shear deformation theories developed by Reddy, (1984), Levinson (1980), Krishna Murty (1987), Ghugal and Gajbhiye (2016), Ghugal and Sayyed (2010), Soltaini et al. (2019), Nareen and Shimpi (2015). Refined shear deformation theories have been developed by Shimpi (2002), Shimpi et al. (2007), Rouzegar and Abdoli-Sharifpoor (2015), Ghugal and Shimpi (2002) among others. Recently, several researchers have worked on formulations for moderately thick and thick plates in bending, vibration and buckling. Such recent research efforts on the shear deformable plates include: Ike (2017a, 2017b, 2018a); Nwoji et al. (2018a, 2018c); Onah et al. (2020).

This work is focused on thin plates.

A review of literature shows that solution methods for thin plate analysis are broadly classified as approximate and analytical methods.

The approximate methods seek to solve the governing partial differential equation (GPDE) in a non-exact manner such that the GPDE is satisfied at certain points on the domain and not at all points. In some approximate methods, all the boundary conditions may not be satisfied by the results.

The approximate solution techniques include:

- Finite Difference Method (FDM)
- Finite Element Method (FEM) (Karttunen et al., 2017)
- Collocation Method (Guo et al., 2019)
- Weighted Residual Method (Ike et al., 2020)
- Ritz Variational Method (Lytvyn et al., 2018; Ike, 2021)
- Galerkin Variational Method (Ike 2017c; Ike, 2015)
- Meshless methods (Du et al., 2022)
- Kantorovich method (Ike, 2023d; Ike and Oguaghamba, 2023)

Analytical methods seek solutions to the GPDE for plates such that all boundary conditions are satisfied at the boundaries, and the GPDE is satisfied at all points on the domain.

Commonly applied analytical methods for rectangular thin plates with two opposite edges on simple supports are series expansion methods (Fogang, 2023), Integral transform methods (Ike, 2022; Ike 2023a; Ike, 2024a; Ike et al., 2021; Mama et al., 2017, 2020), and Symplectic elasticity and eigenfunction superposition methods (Cui, 2007; Ma, 2008; Lim et al., 2007; Zhong and Li 2009; Wang et al., 2016; Su et al., 2023; Zheng et al. 2021).

Evans (1939) presented moments and deflections for CCCC plates under uniformly distributed loads. Taylor and Govidjee (2004) presented accurate solutions of clamped thin plates by closed form solutions of the GPDE. Imrak and Gerdemeli (2007) have satisfactorily solved the isotropic rectangular thin plate with clamped edges. Xu et al. (2020) also solved thin plate bending problems. Rezaiee-Pajand and Karkon (2014) studied the finite element analysis of thin plates in bending using hybrid stress and analytical functions. They developed two different thin plate bending elements, a triangular thin plate bending element with 9 degrees of freedom (DoF) and a quadrilateral thin plate bending element with 12 degrees of freedom (DoF). They developed the thin plate bending elements using hybrid variational principle, and solved illustrative problems to verify the high accuracy of the finite elements.

Rouzegar and Abdoli Sharipfpoor (2015) have developed a finite element formulation based on two variable refined plate bending theory (RPBT) for solving the flexural problems of isotropic and orthotropic plates. Their formulation predicted the vanishing of transverse shear stresses at the plate top and bottom surfaces and was found to apply to both thick plates as well as thin plates.

Al-Ali (2016) used polynomial basis functions to derive accurate polynomial solutions for thin plate bending under different loadings, geometries and boundary conditions. Gao et al. (2019) investigated the bending problems of SCSF and CCCF Kirchhoff plates under hydrostatic loading distribution.

Li et al. (2015) studied the bending and free vibration problems of Kirchhoff plates using a unified analytical method and presented accurate solutions for deflections, bending moments and natural vibration frequencies that agreed with previous solutions. Ibearugbulem et al. (2019) applied Taylor-Maclaurin series basis functions for finding solutions to the rectangular Kirchhoff plate bending problems (KPBPs) with simply supported edges. Aginam et al. (2012) and Oba et al. (2018) used the principle of minimum total potential energy to obtain accurate solutions for deflections in rectangular Kirchhoff plate bending problems (KPBPs).

Recently, symplectic eigenfunction superposition methods have been investigated and used for solving a wide range of plate problems under flexure, vibration and buckling. Li et al. (2015) developed a new symplectic eigenfunction superposition method (SESM) to the static bending and free vibration analysis of KPBPs concerning cases of plates supported at the corner points. The SESM used the Hamiltonian system, and gave accurate solutions. Unlike conventional semi-increase methods, SESM does not need prior determination of the shape functions, which is a major advantage. The SESM is thus extensive in scope of utilization to plate problems. Their work was validated by comparison with previous solutions. They also obtained new solutions to previously unsolved plate problems. Wang et al. (2016) have also used SESM to develop new solutions for Kirchhoff plate buckling problems, but their work was not extended to flexural analysis.

Su et al. (2023) studied SESM for unified solutions to KPBPs with all edges free. Fogang (2023) utilized the Fourier series expansion technique for finding accurate bending solutions to KPBPs subjected to edge bending moments. Ike (2015), Osadebe et al. (2016), Nwoji et al. (2017a), and Ike (2023b) have applied Galerkin's methods to develop accurate solutions for deflections and bending moments in KPBPs for various boundary conditions. Nwoji et al. (2017b), Ike (2017c, 2021, 2023d), Onah et al. (2017) and Ike and Mama (2018) used various strands of the Kantorovich methodology to obtain accurate solutions for the flexural analysis of KPBPs under various boundary conditions. Integral transform methods have been used for KPBPs by Ike (2022), Ike et al. (2021), Mama et al. (2020) and Ike (2024a). Ritz variational method (RVM) has been studied for KPBPs by Ike (2018b) and Nwoji et al. (2018b). Musa et al. (2020) used RVM for the bending solutions of thin plates resting on nonhomogeneous variable foundations for cases of mixed boundary conditions.

Lytryn et al. (2018) developed a computer algorithm for solutions to the fourth order inhomogeneous equation for plate. Xi and Li (2021) used RVM to solve thin plate flexural analysis problems with complex boundary configurations. Zerfu and Ekaputri (2017) utilized the RVM for the approximate deflection solutions of thin quadrilateral plates under applied headings.

This study develops using first principles, systematic, step by step approach, the RVM for solving CCCC and CSCS KPBPs using displacement basis functions that are derived by superposing sinusoidal and polynomial functions. The study extends the work of Ding Zhou (1993) who applied the dimensionless variant of the method to thin plates with three simply supported edges and one clamped edge. This study is a continuation of the previous work of Ike (2024b) which applied the "Ritz variational method for the analysis of thin rectangular plate bending problems with adjacent edges clamped and simply supported using the superposition of trigonometric series and polynomial basis functions". However, unlike the study presented in Ike (2024b), this study applies the superposition of trigonometric and trigonometric basis functions to the bending analysis of fully clamped thin plates under uniformly distributed loads and hydrostatic load distribution. It also studies plates with opposite sides clamped and the remaining sides simply supported for the case of uniformly distributed loading.

2. Ritz Variational Functional (RVF) for the Flexural Analysis of Thin Plates

2.1. Thin Plate Flexural Problem Investigated

The thin plate flexural problems investigated are shown in Figures 1 and 2.

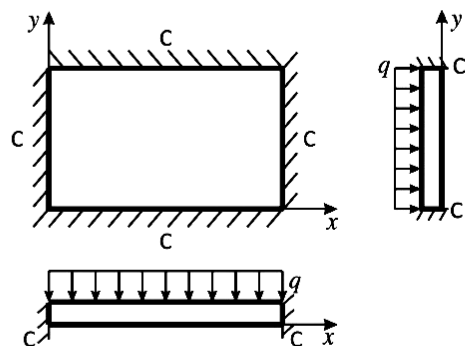

Fig. 1 Clamped Kirchhoff plate subjected to uniformly distributed load of intensity, *q*

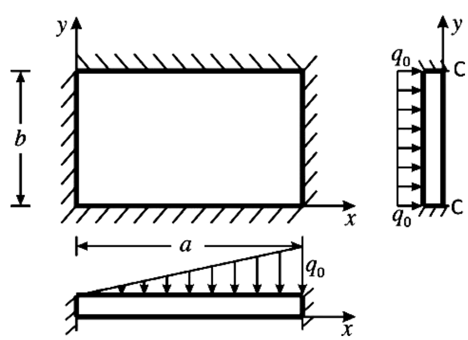

Fig. 2 Clamped Kirchhoff plate under hydrostatically distributed loading $q(x, y) = \frac{q_0 x}{a}$

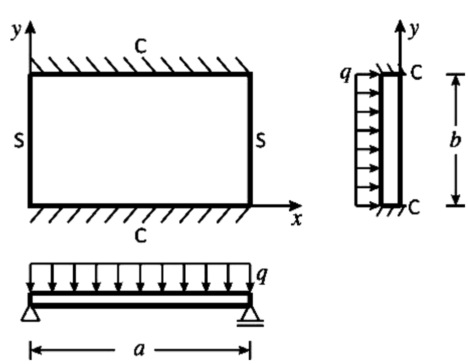

Fig. 3 CSCS Kirchhoff plate subjected to uniformly distributed load *q*

Kirchhoff Plate Theory (KPT)

KPT is a classical thin plate theory based on the Kirchhoff-Love's hypothesis which are assumptions similar to the Bernoulli-Navier's hypothesis used for the classical theory of thin beams. It is a small-deflection theory.

Fundamental Assumptions

The fundamental assumptions of KPT for homogeneous, isotropic, linear elastic plates are (Gujar and Ladhare, 2015)

(i) Straight lines that are initially normal to the middle surface of the plate remain straight and normal to the deformed middle surface of the plate and unstretched.
(ii) The transverse displacement is very small compared to the plate thickness. The slope of the deflected middle surface is small; hence the square of the slope is negligible compared to unity.
(iii) The normal stresses and in-plane shear stress are zero at the middle surface for small deflection cases (where the maximum deflection wmax is much smaller than the thickness, h).
(iv) The transverse normal stress σzz is very small relative to other stress components and can be neglected without significant errors $(\sigma_{zz} \approx 0)$.
(v) The middle surface is unstrained after deformation and it is a neutral plane.

These assumptions reduce the 3D plate problem to a two-dimensional (2D) approximate theory. The non-vanishing stresses are reduced to σxx, σyy and τxy, and other stresses vanish.

Disadvantages of KPT

KPT is governed by a fourth order PDE associated with two edge conditions rather than three edge conditions required in a 3D problem. This raises the Poisson-Kirchhoff boundary conditions paradox.

Despite the disadvantages, KPT is still commonly used for thin plate bending analysis because of the simplicity of the governing equations. The results have been found to be accurate for thin plates KPT is the theoretical plate model for this work.

2.2. Deriving Ritz Variational Functional and Total Potential Energy Functional *Π*

Displacement Field Components

The displacement field components are:

$$u = -z\frac{\partial w}{\partial x}, \quad v = -z\frac{\partial w}{\partial y}, \quad w = w(x, y) \tag{1}$$

where *u*, *v*, and *w* are the displacement field components in the *x*, *y* and *z* Cartesian coordinate directions respectively.

Strains

The normal strains ε_{xx}, ε_{yy}, and shear strains γ_{xy} are found from the displacement field components using the strain-displacement relations of the linear elasticity theory.

Thus,

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{2}$$

Stress-Strain Relations

For linear elastic homogeneous, isotropic materials, and plane stress the material constitutive relations are:

$$\sigma_{xx}=\frac{E}{1-\mu^2}(\varepsilon_{xx}+\mu\varepsilon_{yy})$$
$$\sigma_{yy}=\frac{E}{1-\mu^2}(\varepsilon_{yy}+\mu\varepsilon_{xx}) \quad (3)$$
$$\tau_{xy}=G\gamma_{xy}=\frac{E}{2(1+\mu)}\gamma_{xy}$$

wherein σ_{xx} and σ_{yy} are the normal stresses, τ_{xy} is the shear stress.
E is the Young's modulus of the plate material.
μ is the Poisson's ratio, *G* is the shear modulus.

Stress-Displacement Equations

Substituting the strain-displacement equations in the stress-strain relations give the stress-displacement equations as follows:

$$\sigma_{xx}=\frac{-Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)$$
$$\sigma_{yy}=\frac{-Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right) \quad (4)$$
$$\tau_{xy}=-2Gz\frac{\partial^2 w}{\partial x\partial y}=\frac{-Ez}{1+\mu}\frac{\partial^2 w}{\partial x\partial y}$$

Internal Force Resultants

The bending moments M_{xx}, M_{yy} and twisting moment M_{xy} are found as integration problems across the plate thickness.
Thus,

$$M_{xx}=\int_{-h/2}^{h/2}\sigma_{xx}z\,dz=-D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)$$
$$M_{yy}=\int_{-h/2}^{h/2}\sigma_{yy}z\,dz=-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)$$
$$M_{xy}=\int_{-h/2}^{h/2}\tau_{xy}z\,dz=-\frac{Gh^3}{6}\frac{\partial^2 w}{\partial x\partial y} \quad (5)$$

where $D=\frac{Eh^3}{12(1-\mu^2)}$ (6)

D is the modulus of flexural rigidity of the plate.

Strain Energy of Thin Plate

The strain energy U_b of thin plate under plane stress bending deformation is the volume integral over the plate domain given by:

$$U_b=\frac{1}{2}\int_{-h/2}^{h/2}\int_0^a\int_0^b(\sigma_{xx}\varepsilon_{xx}+\sigma_{yy}\varepsilon_{yy}+\tau_{xy}\gamma_{xy})dxdydz \quad (7)$$

since $\varepsilon_{zz}\approx 0,\ \gamma_{xz}=\gamma_{yz}=0$ (8)

Hence,

$$U_b = \frac{1}{2}\int_{-h/2}^{h/2}\int_0^a\int_0^b \left\{ \frac{-Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\left(-z\frac{\partial^2 w}{\partial x^2}\right) - \frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)\left(-z\frac{\partial^2 w}{\partial y^2}\right) \right.$$

$$\left. \left(-\frac{Ez}{1+\mu}\frac{\partial^2 w}{\partial x \partial y}\right)\left(-2z\frac{\partial^2 w}{\partial x\partial y}\right)\right\} dxdydz \quad (9)$$

Simplifying the integrand gives:

$$U_b = \frac{1}{2}\int_{-h/2}^{h/2}\int_0^a\int_0^b \left\{ \frac{-Ez^2}{1-\mu^2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \mu\left(\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \mu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)\right] + \frac{2Ez^2}{1+\mu}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\} dxdydz \quad (10)$$

Simplifying

$$U_b = \frac{1}{2}\int_{-h/2}^{h/2}\frac{Ez^2}{1-\mu^2}dz\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\mu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)\right\}dxdy + \frac{1}{2}\int_{-h/2}^{h/2}\frac{2Ez^2}{1+\mu}dz\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x\partial y}\right)dxdy \quad (11)$$

It is observed that

$$\int_{-h/2}^{h/2}\frac{Ez^2}{1-\mu^2}dz = \frac{Eh^3}{12(1-\mu^2)} = D$$

$$\int_{-h/2}^{h/2}\frac{Ez^2}{1+\mu}dz = \frac{Eh^3}{12(1+\mu)} \quad (12)$$

Hence,

$$U_b = \frac{D}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right)\right\}dxdy$$

(13)

$$U_b = \frac{D}{2}\int_0^a\int_0^b\left\{((\nabla^2 w)^2 - 2(1-\mu)(w_{xx}w_{yy} - (w_{xy})^2)\right\}dxdy \quad (14)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (15)

$$w_{xx} = \frac{\partial^2 w}{\partial x^2},\quad w_{yy} = \frac{\partial^2 w}{\partial y^2},\quad w_{xy} = \frac{\partial^2 w}{\partial x\partial y} \quad (16)$$

The work done (W_a) by the applied load $q(x,y)$ is the double integral.

$$W_a = \int_0^a\int_0^b q(x,y)w(x,y)dxdy \quad (17)$$

Total Potential Energy Functional $\prod$

The total potential energy functional $\prod$ is expressed in terms of U_b and W_a as:

$$\prod = U_b - W_a$$

$$\prod = \frac{D}{2}\int_0^a\int_0^b\left\{(\nabla^2 w)^2 - 2(1-\mu)(w_{xx}w_{yy} - w_{xy}^2)\right\}dxdy - \int_0^a\int_0^b q(x,y)w(x,y)dxdy \quad (18)$$

$$\prod = \frac{D}{2}\int_0^a\int_0^b\left\{(\nabla^2 w)^2 - 2(1-\mu)(w_{xx}w_{yy} - w_{xy}^2) - \frac{2q(x,y)w(x,y)}{D}\right\}dxdy \quad (19)$$

The method adopted in the work seeks to minimize the functional $\prod$ with respect to $w(x,y)$.

3. Method of Superposing Sinusoidal and Polynomial Basis Functions

3.1. General Form of the Basis Functions

The general forms of the basis functions $F_m(x)$ and $G_n(y)$ are (Ike, 2024b):

$$F_m(x) = \sin\left(\frac{m\pi x}{a}\right) + P_m(x) \quad (20a)$$

$$F_m(x) = \sin\left(\frac{m\pi x}{a}\right) + A_{m_0} + A_{m_1}\left(\frac{x}{a}\right) + A_{m_2}\left(\frac{x}{a}\right)^2 + A_{m_3}\left(\frac{x}{a}\right)^3 \quad (20b)$$

$$P_m(x) = A_{m_0} + A_{m_1}\left(\frac{x}{a}\right) + A_{m_2}\left(\frac{x}{a}\right)^2 + A_{m_3}\left(\frac{x}{a}\right)^3 \quad (20c)$$

where $m = 1, 2, 3, 4, 5, \ldots$

$$G_n(y) = \sin\left(\frac{n\pi y}{b}\right) + P_n(y) \quad (21a)$$

$$G_n(y) = \sin\left(\frac{n\pi y}{b}\right) + B_{n_0} + B_{n_1}\left(\frac{y}{b}\right) + B_{n_2}\left(\frac{y}{b}\right)^2 + B_{n_3}\left(\frac{y}{b}\right)^3 \quad (21b)$$

where $n = 1, 2, 3, 4, 5, \ldots$

$$P_n(y) = B_{n_0} + B_{n_1}\left(\frac{y}{b}\right) + B_{n_2}\left(\frac{y}{b}\right)^2 + B_{n_3}\left(\frac{y}{b}\right)^3 \quad (21c)$$

A_{m_0}, A_{m_1}, A_{m_2}, A_{m_3} are the four coefficients of the polynomial coordinate function $P_m(x)$ which are found to render $P_m(x)$ a suitable shape function that satisfies the deformation and force boundary conditions at the plate edges $x = 0$, and $x = a$.

Similarly, B_{n_0}, B_{n_1}, B_{n_2}, and B_{n_3} are the four coefficients of the polynomial function $P_n(y)$. The polynomial constants are found such that $G_n(y)$ satisfies the deformation and force boundary conditions at the plate edges $y = 0$ and $y = b$.

3.2. Basis Function Coefficients for Clamped (CCCC) Plates and CSCS Plates

The edge conditions investigated are:

(i) Plate is clamped at the four edges *x=0*, *x=a*, *y=0*, and *y=b*.

(ii) Plate is clamped at two opposite edges *y=0*, and *y=b;* simply supported at the other opposite edges *x=0* and *x=a*.

(iii) Plate is clamped at edges *x=0* and *x=a*.

Figure 4 presents the cross-sectional view of thin plate clamped at *x=0* and *x=a*.

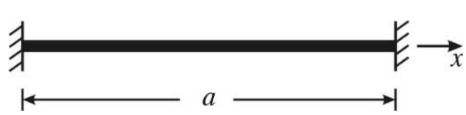

Fig. 4 Cross sectional view of thin plate clamped at *x=0* and *x=a*

The boundary conditions are

$$w(0, y) = \sum_m \sum_n A_{mn} F_m(0) G_n(y) = 0 \quad (22)$$

$$w(0, y) = \sum_m \sum_n A_{mn} F_m(a) G_n(y) = 0 \quad (23)$$

$$w'(0, y) = \sum_m \sum_n A_{mn} F'_m(0) G_n(y) = 0 \quad (24)$$

$$w'(a,y)=\sum_m\sum_n A_{mn}F'_m(a)G_n(y)=0 \quad (25)$$

Hence,

$$F_m(0)=F_m(a)=0$$
$$F'_m(0)=F'_m(a)=0 \quad (26)$$

$$F_m(0)=\sin 0+A_{m_0}+A_{m_1}0=0 \quad (27)$$

$$A_{m_0}=0 \quad (28)$$

$$F'_m(x)=\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)+A_{m_1}\frac{1}{a}+2A_{m_2}\frac{x}{a^2}+3A_{m_3}\frac{x^2}{a^3} \quad (29)$$

$$F'_m(0)=\frac{m\pi}{a}\cos 0+\frac{A_{m_1}}{a}+2A_{m_2}0+3A_{m_3}0=0 \quad (30)$$

$$\frac{m\pi}{a}+\frac{A_{m_1}}{a}=0 \quad (31)$$

$$A_{m_1}=-m\pi \quad (32)$$

$$F_m(a)=\sin(m\pi)+(-m\pi)+A_{m_2}+A_{m_3}=0 \quad (33)$$

$$A_{m_2}+A_{m_3}=m\pi-\sin(m\pi)=m\pi \quad (34)$$

$$F'_m(a)=\frac{m\pi}{a}\cos(m\pi)+(-m\pi)\frac{1}{a}+2A_{m_2}\left(\frac{1}{a}\right)+3A_{m_3}\left(\frac{1}{a}\right)=0 \quad (35)$$

Multiplying by *a* gives:

$$m\pi\cos(m\pi)-m\pi+2A_{m_2}+3A_{m_3}=0 \quad (36)$$

$$2A_{m_2}+3A_{m_2}=m\pi(1-\cos(m\pi)) \quad (37)$$

From Equation (34)

$$2A_{m_2}+2A_{m_3}=2m\pi \quad (38)$$

Hence,

$$A_{m_3}=m\pi-m\pi\cos(m\pi)-2m\pi \quad (39)$$

$$A_{m_3}=-m\pi\cos(m\pi)-m\pi \quad (40)$$

$$A_{m_3}=-m\pi(1+\cos(m\pi)) \quad (41)$$

$$A_{m_3}=-m\pi(1+(-1)^m) \quad (42)$$

Then,

$$A_{m_2}=m\pi-A_{m_3}=m\pi-(-m\pi(1+(-1)^m) \quad (43)$$

$$A_{m_2}=m\pi(2+(-1)^m) \quad (44)$$

Hence,

$$F_m(x)=\sin\left(\frac{m\pi x}{a}\right)+A_{m_1}\left(\frac{x}{a}\right)+A_{m_2}\left(\frac{x}{a}\right)^2+A_{m_3}\left(\frac{x}{a}\right)^3 \quad (45)$$

where $A_{m_1}=-m\pi$

$$A_{m_2}=m\pi(2+(-1)^m)$$
$$A_{m_3}=-m\pi(1+(-1)^m) \quad (46)$$

Using similar procedure for thin plate with clamped edges at $y=0$ and $y=b$,

$$G_n(y)=\sin\left(\frac{n\pi y}{b}\right)+B_{n_1}\left(\frac{y}{b}\right)+B_{n_2}\left(\frac{y}{b}\right)^2+B_{n_3}\left(\frac{y}{b}\right)^3 \quad (47)$$

where

$$B_{n_1}=-n\pi \quad (48)$$

$$B_{n_2} = n\pi(2+(-1)^n)$$

$$B_{n_3} = -n\pi(1+(-1)^n)$$

(ii)Plate is clamped at $y=0,\ y=b$ and simply supported at $x=0$ and $x=a$

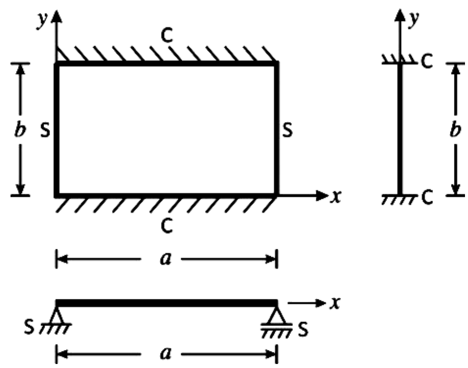

Fig. 5 CSCS plate

For simply supported edges $x=0$ and $x=a$ the boundary conditions are

$$F_m(0) = F_m(a) = 0$$
$$F_m''(0) = F_m''(a) = 0 \tag{49}$$

$$F_m''(x) = -\left(\frac{m\pi}{a}\right)^2 \sin\left(\frac{m\pi x}{a}\right) + 2A_{n_2}\left(\frac{1}{a^2}\right) + 6A_{m_3}\left(\frac{x}{a^3}\right) \tag{50}$$

$$F_m''(0) = -\left(\frac{m\pi}{a}\right)^2 \sin 0 + 2A_{n_2}\left(\frac{1}{a^2}\right) = 0 \tag{51}$$

$$A_{m_2} = 0 \tag{52}$$

$$F_m(0) = \sin 0 + A_{m_0} = 0 \tag{53}$$

$$A_{m_0} = 0 \tag{54}$$

$$F_m''(a) = \sin m\pi + A_{m_1} + A_{m_3} = 0 \tag{55}$$

$$A_{m_1} = -A_{m_3} \tag{56}$$

$$F_m''(a) = -\left(\frac{m\pi}{a}\right)^2 \sin m\pi + 6A_{m_3}\left(\frac{a}{a^3}\right) = 0 \tag{57}$$

$$-\left(\frac{m\pi}{a}\right)^2 \sin m\pi + \frac{6A_{m_3}}{a^2} = 0 \tag{58}$$

$$A_{m_3} = 0 \tag{59}$$

$$A_{m_1} = 0 \tag{60}$$

Then, $F_m(x) = \sin\left(\frac{m\pi x}{a}\right)$ (61)

For clamped edge $y=0,\ y=b,$

$$G_n(y) = \sin\left(\frac{n\pi y}{b}\right) + B_{n_1}\left(\frac{y}{b}\right) + B_{n_2}\left(\frac{y}{b}\right)^2 + B_{n_3}\left(\frac{y}{b}\right)^3 \tag{62}$$

$$B_{n_1} = -n\pi$$

$$B_{n_2} = n\pi(2+(-1)^n)$$

$$B_{n_3} = -n\pi(1+(-1)^n) \tag{63}$$

3.3. Ritz Variational Functional (RVF)

RVF is expressed using the basis functions as:

$$\Pi = \frac{D}{2}\int_0^a\int_0^b \left\{ \left(\nabla^2 \left(\sum_m \sum_n A_{mn} F_m(x) G_n(y) \right) \right)^2 - 2(1-\mu)\left(\frac{\partial^2}{\partial x^2} \sum_m \sum_n A_{mn} F_m(x) G_n(y) \right) * \right.$$
$$\left. \frac{\partial^2}{\partial y^2}\left(\sum_m \sum_n A_{mn} F_m(x) G_n(y) \right) - \left(\frac{\partial^2}{\partial x \partial y}\left(\sum_m \sum_n A_{mn} F_m(x) G_n(y) \right) \right)^2 \right\} dxdy -$$
$$\int_0^a\int_0^b q(x,y) \sum_m \sum_n A_{mn} F_m(x) G_n(y) dxdy \tag{64}$$

Simplifying,

$$\Pi = \sum_m \sum_n A_{mn} \left\{ \frac{D}{2}\int_0^a\int_0^b \left\{ (\nabla^2 F_m(x) G_n(y))^2 - 2(1-\mu)\left[(F_m''(x) G_n(y) F_m(x) G_n''(y)) - \right.\right.\right.$$
$$\left.\left. (F_m'(x) G_n'(y))^2 \right] \right\} dxdy - \sum_m \sum_n A_{mn} \int_0^a\int_0^b q(x,y) F_m(x) G_n(y) dxdy \tag{65}$$

3.4. Minimization of the Ritz Variational Functional

The principle of minimization of the total potential energy functional is equivalent to equilibrium. For minimization of RVF, the first variation of $\prod$ would vanish.

$$\partial \prod = 0 \tag{66}$$

The following system of algebraic equations result:

$$\sum_m \sum_n K_{mn}^{ij} A_{mn} = F_{ij} \tag{67}$$

Here,

K_{mn}^{ij} is the stiffness matrix

F_{ij} is the force matrix

$$K_{mn}^{ij} = D\int_0^a\int_0^b \left\{ \left(\nabla^2 (F_m(x) G_n(y) \right)\left(\nabla^2 F_i(x) G_j(y) \right) - (1-\mu)\left[(F_m''(x) F_i(x) G_n(y) G_j''(y)) + \right.\right.$$
$$\left.\left. (F_i''(x) F_m(x) G_j(y) G_n''(y) - 2F_i'(x) F_m'(x) G_j'(y) G_n'(y) \right] \right\} dxdy \tag{68}$$

$$F_{ij} = \int_0^a\int_0^b q(x,y) F_i(x) G_j(y) dxdy \tag{69}$$

$$K_{mn}^{ij} = D\int_0^a\int_0^b \left\{ \left[F_m''(x) F_i(x) G_n(y) G_j''(y) + F_m''(x) G_n(y) F_i(x) G_j''(y) + F_m(x) G_n''(y) F_i''(x) G_j(y) + F_m''(x) G_n(y) F_i(x) G_j''(y) \right] - \right.$$
$$\left. (1-\mu)(F_n''(x) F_i(x) G_j''(y) G_n(y) + F_i''(x) F_m(x) G_j(y) G_n''(y) - 2F_i'(x) F_m'(x) G_j'(y) G_n'(y)) \right\} dxdy \tag{70}$$

For uniformly distributed load with intensity $q(x,y) = q_0$,

$$F_{ij} = \int_0^a\int_0^b q_0 F_i(x) G_j(y) dxdy = F_{ij} = q_0 \int_0^a\int_0^b F_i(x) G_j(y) dxdy \tag{71}$$

For hydrostatic load distribution, $q(x,y) = q_0 x/a$,

$$F_{ij} = \int_0^a \int_0^b \frac{q_0 x}{a} F_i(x) G_j(y) dx dy = \frac{q_0}{a} \int_0^a x F_i(x) dx \int_0^b G_j(y) dy \quad (72)$$

4. Results

4.1. Results for CCCC Clamped KPBP

KPBP with CCCC boundaries as shown in Figure 6, is considered.

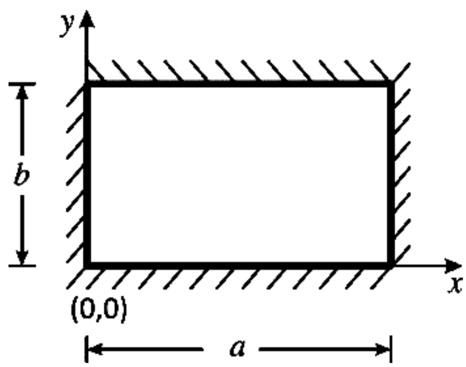

Fig. 6 CCCC Kirchhoff plate

The functions $F_m(x)$ and $G_n(y)$ are given by Equations (45) and (47).

The centre deflections and bending moments are presented in Tables 1 and 2 for uniformly distributed and hydrostatic load distributions respectively.

Table 1 – Deflections and Bending Moments at the Centre of CCCC KPBP under Uniformly Distributed Load.

a/*b*	Method/Reference	$\frac{Dw(0.5a,0.5b)}{qb^4}$	$\frac{M_{xx}(0.5a,0.5b)}{qb^2}$	$\frac{M_{yy}(0.5a,0.5b)}{qb^2}$
1	Present	0.00126541	0.0229074	0.0229062
	Batista (2011)	0.00126532	0.0229051	0.0229051
	Cui (2007)	0.00126532	0.0229054	0.0229052
	Ike (2022)	0.00126725	0.0231	0.0231
	Timoshenko & Woinowsky-Krieger (1959) Evans (1939)	0.00126	0.0231	0.0231
1.2	Present	0.00172495	0.0228416	0.0299727
	Batista (2011)	–	–	–
	Cui (2007)	0.00172487	0.0228406	0.0299717
	Ike (2022)	0.0017283	0.0288	0.0299
	Timoshenko & Woinowsky-Krieger (1959) Evans (1939)	0.00172	0.0288	0.0299
	Imrak and Gerdemeli (2007)	0.00172833		
1.5	Present	0.00219658	0.0202682	0.0367722
	Batista (2011)	0.00219652	0.0202680	0.0367714
	Cui (2007)	0.00219653	0.0202681	0.0367715
	Ike (2022)	0.002203	0.0203	0.0368
	Timoshenko & Woinowsky-Krieger	0.00220	0.0203	0.0368

	(1959) Evans (1939)			
	Imrak and Gerdemeli (2007)	–	–	–
1.7	Present	0.00238207	0.0182691	0.0392702
	Batista (2011)	`–	–	–
	Cui (2007)	0.00238203	0.0182692	0.0392697
	Ike (2022)	0.002382	0.0182	0.0392
	Timoshenko & Woinowsky-Krieger (1959) Evans (1939)	0.00238	0.0182	0.0392
	Imrak and Gerdemeli (2007)	–	–	–
2	Present	0.00253298	0.0158078	0.0411553
	Batista (2011)	–	–	–
	Cui (2007)	0.00253296	0.0158080	0.0411550
	Ike (2022)	0.002536	0.0158	0.0412
	Timoshenko & Woinowsky-Krieger (1959) Evans (1939)	0.00254	–	–
	Imrak and Gerdemeli (2007)	0.00253297	0.0158	0.0412
3	Present	0.002615	0.012690	0.04190
	Batista (2011)	0.00261723	0.0126928	0.0419013
	Cui (2007)	–	–	–
	Ike (2022)	–	–	–
	Timoshenko & Woinowsky-Krieger (1959) Evans (1939)	–	–	–
	Imrak and Gerdemeli	–	–	–
4	Present	0.002607	0.01247	0.0417
	Batista (2011)	0.00260659	0.0124713	0.0416988
	Cui (2007)	–	–	–
	Ike (2022)	–	–	–
	Timoshenko & Woinowsky-Krieger (1959) Evans (1939)	–	–	–
	Imrak and Gerdemeli (2007)	–	–	–
5	Present	0.002604	0.0125	0.0417
	Batista (2011)	0.00260423	0.0124941	0.0416666
	Cui (2007)	–	–	–
	Ike (2022)	–	–	–
	Timoshenko & Woinowsky-Krieger (1959) Evans (1939)	–	–	–
	Imrak and Gerdemeli (2007)	–	–	–
10	Present	0.002605	0.0125	0.0417
	Batista (2011)	0.00260417	0.0125000	0.0416667
	Cui (2007)	–	–	–
	Ike (2022)	–	–	–
	Timoshenko & Woinowsky-Krieger (1959) Evans (1939)	–	–	–
	Imrak and Gerdemeli (2007)	–	–	–

Table 2 – Deflections at the Centre of CCCC KPBP under Hydrostatic Load Distributed Over the Entire Domain.

Aspect ratio *b/a*	Centre deflection $\left(w_c D/qa^4\right)$		
	Present study	Ike (2022)	Timoshenko and Woinowsky-Krieger (1959)
0.5	0.000080	0.000080	0.000080
2/3	0.000217	0.000217	0.000217
1.0	0.00063	0.00063	0.00063
1.5	0.0110	0.0110	0.0110
∞	0.0130	0.0130	0.0130

Table 3 – Bending Moments at the Centre of CCCC KPBP under Hydrostatic Load Distribution Over the Entire Plate (for μ=0.30).

Aspect ratio *b/a*	Present study		Ike (2022)	Timoshenko and Woinowsky-Krieger (1959)
	M_{xx}/qa^2	M_{yy}/qa^2	M_{xx}/qa^2	M_{yy}/qa^2
0.5	1.98×10^{-3}	5.15×10^{-3}	1.98×10^{-3}	5.15×10^{-3}
2/3	4.51×10^{-3}	8.17×10^{-3}	4.51×10^{-3}	8.17×10^{-3}
1.0	11.5×10^{-3}	11.5×10^{-3}	11.5×10^{-3}	11.5×10^{-3}
1.5	18.4×10^{-3}	10.2×10^{-3}	18.4×10^{-3}	10.2×10^{-3}
∞	20.8×10^{-3}	6.3×10^{-3}	20.8×10^{-3}	6.3×10^{-3}

Table 4 – Deflection and Bending Moment Parameters for CSCS KPBP under Uniformly Distributed Loading (for μ=0.30).

b/a	Method/Reference	$\frac{Dw(0.5a, 0.5b)}{qa^4}$	$\frac{M_{xx}(0.5a, 0.5b)}{qa^2}$	$\frac{M_{yy}(0.5a, 0.5b)}{qa^2}$
1	Present	0.00192	0.0244	0.0332
	Timoshenko & Woinowsky-Krieger (1959)	0.00192	0.0244	0.0332
	Batista (2011)	0.00191714	0.0243874	0.0332449
1.5	Present	0.00532	0.05849	0.0460
	Timoshenko & Woinowsky-Krieger (1959)	0.00531	0.0585	0.0460
	Batista (2011)	0.00532645	0.0584804	0.0459444
2	Present	0.08445	0.08687	0.04736
	Timoshenko & Woinowsky-Krieger (1959)	0.00844	0.0869	0.0474
	Batista (2011)	0.00844500	0.088681	0.0473622
3	Present	0.01168	0.011436	0.0421
	Timoshenko & Woinowsky-Krieger (1959)	0.01168	0.01144	0.0419
	Batista (2011)	0.01168129	0.01143571	0.0421263
4	Present	0.01267	0.12225	0.039
	Timoshenko & Woinowsky-Krieger (1959)	–	–	–

	Batista (2011)	0.01266531	0.01222547	0.0389927
5	Present	0.01293	0.1243	0.038
	Timoshenko & Woinowsky-Krieger (1959)	–	–	–
	Batista (2011)	0.01293098	0.1243191	0.0379205
a*/*b	**Method/Reference**	$\frac{Dw(0.5a,0.5b)}{qa^4}$	$\frac{M_{xx}(0.5a,0.5b)}{qa^2}$	$\frac{M_{yy}(0.5a,0.5b)}{qa^2}$
1.5	Present	0.02476	0.0178	0.04063
	Timoshenko & Woinowsky-Krieger (1959)	0.00247	0.0179	0.0406
	Batista (2011)	0.00247571	0.0178003	0.0406276
2	Present	0.00261	0.01417	0.0421
	Timoshenko & Woinowsky-Krieger (1959)	0.00260	0.0142	0.0420
	Batista (2011)	0.01261080	0.0141717	0.0420629
3	Present	0.00215	0.0125	0.04183
	Timoshenko & Woinowsky-Krieger (1959)	–	–	–
	Batista (2011)	0.00261488	0.0124986	0.0418311
4	Present	0.00261	0.0125	0.04168
	Timoshenko & Woinowsky-Krieger (1959)	–	–	–
	Batista (2011)	0.00260519	0.0124751	0.0416780
5	Present	0.0026	0.0125	0.04167
	Timoshenko & Woinowsky-Krieger (1959)	–	–	–
	Batista (2011)	0.00260412	0.0124974	0.0416654

5. Discussion

This paper has developed the solutions for CCCC and CSCS Kirchhoff plate bending problems using the superposition of sinusoidal and third order polynomial basis functions in the Ritz variational method. The problems studied were:

(i) CCCC thin plate under uniformly distributed load.
(ii) CCCC thin plate under hydrostatic load distribution.
(iii) CSCS thin plate under uniformly distributed load.

The results for deflections and bending moments at the plate centre for CCCC thin plate under uniformly distributed load for *a*/*b* ranging from 1.0 to 10, and for $\mu = 0.30$ are presented, in Table 1, along with previous results by Batista (2011), Cui (2007), Ike (2022) and Timoshenko and Woinowsky-Krieger (1959). Table 1 illustrates that the present results are identical with results by Timoshenko and Woinowsky-Krieger (1959), Batista (2011), Cui (2007) and Ike (2022).

Table 2 shows the present results for centre deflections of CCCC KPBP under hydrostatic load distribution for aspect ratios (*b*/*a*) ranging from $b/a = 0.5,\ 2/3$, 1.0, 1.5 and ∞. Table 2 also compares the results with results from Timoshenko and Woinowsky-Krieger (1959) and Ike (2022). Table 2 illustrates that the present results are identical with previous results by Timoshenko and Woinowsky-Krieger (1959) and Ike (2022).

Table 3 presents the bending moment coefficient values for M_{xx}, M_{yy} at the plate centre for hydrostatically loaded CCCC KPBP for $\mu = 0.30$ and aspect ratios $b/a = 0.5,\ 2/3$, 1.0, 1.5, ∞. Table 3 confirms that the present results for bending moments at the plate centre for hydrostatically loaded CCCC KPBPs are almost identical with results by Timoshenko and Woinowsky-Krieger (1959) and Ike (2022), with differences that are less than 0.1%.

Table 4 presents the centre deflections and bending moments M_{xx}, M_{yy} coefficient values for CSCS KPBPs subjected to uniformly distributed load over the entire domain; for values of $b/a = 1.5$, 1.5, 2, 3, 4, 5 and for $a/b = 1.5, 2$, 3, 4, 5. Table 4 illustrates that the present solutions are almost identical with previous solutions by Timoshenko and Woinowsky-Krieger (1959) and Batista (2011), with differences that are less than 0.1%. The differences are attributed to approximations and round off errors during calculations

6. Conclusion

This article has derived the solutions of CCCC and CSCS KPBPS using the superposition of sine and third-degree polynomial basis functions in the Ritz variational method (RVM). The derived basis functions used in the RVM satisfied all boundary conditions and the resulting Ritz variational functional (RVF) became expressed in terms of the displacement parameters A_{mn}. RVF was minimized with respect to A_{mn} to obtain the equilibrium equations in stiffness form.
In conclusion,

(i) The present solutions for CCCC plates were identical with previous solutions obtained using Levy-Nadai series method, Generalized Integral Transform Method and Symplectic Eigenfunction Superposition Method.

(ii) The present solutions for CSCS plates under uniform load are identical with previous solutions that used the Levy-Nadai series method.

(iii) The present results for deflections at the center of the fully clamped KPBP under uniformly distributed load are close to the previous symplectic elasticity results by Cui (2007), Levy-Nadai trigonometric series expansion results by Timoshenko and Woinowsky-Krieger (1959) and Generalized Integral transformation method results by Ike (2022) for aspect ratios, of 1, 1.2, 1.5, 1.7, 2, 3, 4, 5, 10.

(iv) For fully clamped KPBP, the present results for center bending moments Mxx and Myy (for all aspect ratios considered) are also almost identical to the previous results obtained by symplectic elasticity methods, by Cui (2007), Levy-Nadai methods by Timoshenko and Woinowsky-Krieger (1959), and Evans (1939), and GITM by Ike (2022).

(v) For fully clamped KPBP under hydrostatic load, the present results for center deflections and bending moments Mxx, Myy (for b/a = 0.5, 2/3, 1, 1.5, and ∞) are identical with previous results obtained by Ike (2022) and Timoshenko and Woinowsky-Krieger (1959).

(vi) For uniformly loaded KPBP clamped along y = 0, y =b and simply supported along x = 0, x = a, the present results for center deflections and center bending moments Mxx, Myy (for b/a = 1, 1.5, 2, 3, 4, 5, and, a/b = 1.5, 2, 3, 4, 5) are almost identical with previous results by Timoshenko and Woinowsky-Krieger (1959) and Batista (2011).

Notations

xy	in-plane coordinates
z	transverse coordinate
q	intensity of uniformly distributed load
a, b	in-plane dimensions of plate
a	in-plane dimension in the x direction
b	in-plane dimension in the y direction
u	displacement component in x direction
v	displacement component in y direction
w	displacement component in z direction
Π	total potential energy functional
ε_{xx}	normal strain in x direction
ε_{yy}	normal strain in y direction
ε_{zz}	normal strain in z direction
γ_{xy}	shear strain
γ_{xz}, γ_{yz}	transverse shear strain

σ_{xx}	normal stress in x direction
σ_{yy}	normal stress in y direction
σ_{zz}	normal stress in z direction
τ_{xy}	shear stress
τ_{xz}, τ_{yz}	transverse shear stress
μ	Poisson's ratio
E	Young's modulus of elasticity
G	shear modulus
M_{xx}	bending moment
M_{yy}	bending moment
M_{xy}	twisting moment
h	plate thickness
D	flexural rigidity of plate
U_b	strain energy of thin plate under bending deformation
∇^2	Laplacian operator
w_{xx}	second partial derivative of $w(x,y)$ with respect to x
w_{yy}	second partial derivative of $w(x,y)$ with respect to y
w_{xy}	partial mixed derivative of $w(x,y)$ with respect to x and y
W_a	work done by applied load
sin	sine function
$P_m(x)$	third-order polynomial in terms of x
$P_n(x)$	third-order polynomial in terms of y
$F_m(x)$	basis function in the x direction
$G_n(y)$	basis function in the y direction
A_{m_0}, A_{m_1}, A_{m_2}, A_{m_3}	unknown coefficients (parameters) of $P_m(x)$
B_{n_0}, B_{n_1}, B_{n_2}, B_{n_3}	unknown coefficients (parameters) of $P_n(y)$
m, n	integers
A_{mn}	generalized displacement parameters of $w(x,y)$
$\sum_m \sum_n$	double summation
δ	first variation of
K_{mn}^{ij}	stiffness matrix
F_{ij}	force matrix
$\int_0^a \int_0^b (\ )dxdy$	double integration over the plate domain with respect to x and y
C	clamped support
S	simple support
CCCC	thin plate with all four edges clamped
CSCS	thin plate with two opposite edges clamped and the other two opposite edges simply supported
KPT	Kirchhoff Plate Theory
2D	two-dimensional
3D	three-dimensional
GPDE	Governing Partial Differential Equation
PDE	Partial Differential Equation
RVF	Ritz Variational Functional
RVM	Ritz Variational Method
SESM	Symplectic Eigenfunction Superposition Method
KPBP(s)	Kirchhoff Plate Bending Problem(s)

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