

The Numerical Solution for Quadratic Optimal Control Problems by Using Chebyshev and Legendre Polynomials

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Abstract

The purpose of this paper is to solve quadratic optimal control problems (QOCP) numerically with the assist of once Chebyshev and Legendre polynomials as basic functions to find the solution for optimal control (QOC) approximately. We will explain the algorithms of solution by examples and use the Mathcad's Program to reach the exact result.

Introduction

The optimal control problem is to find a control $u^*(t)$ which minimizes a given performance index while satisfying the system state equations and constraints. [1]

We use the approximation methods to solve the optimal control problem depending on the Chebyshev polynomials in the first time and Legendre polynomials, after that we will approximate these solutions of continuous time linear. To reach the approximate solutions we use the linear multi- term differential equations of $u(t)$ and $x(t)$ for both Chebyshev and Legendre polynomials and make the terms of these equations as square matrix to find these values by matrices system. When we use these polynomials in approximate solutions, the results were evaluated by using index with $n = 5$.

We will explain these algorithms by taking some examples for the quadratic control problems.

The linear quadratic problem is stated as follows;

Minimize the quadratic continuous time

$$\text{Cost function } J = \int_{t_0}^{t_f} (x^T Q x + u^T R u) dt \quad \dots(1)$$

Subject to the linear system state equations;

$$\dot{x}(t) = D x(t) + E u(t), \quad \dots(2)$$

where the initial condition $x(0) = x_0$ and the matrices $(D, E, Q$ and $R)$ are constants. [3]

Chebyshev Polynomials of the First Kind of Degree n [2]

The Chebyshev polynomials $T_n(x)$ can be obtained by means of Rodrigue's formula

$$T_n(x) = \frac{(-2)^n n!}{(2n)!} \sqrt{1-x^2} \frac{d^n}{dx^n} (1-x^2)^{n-1/2} \quad n = 0, 1, 2, \dots$$

We find that $T_0(x) = 1$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$\vdots$

In this paper we use recurrence formula for $T_n(x)$. When the first two Chebyshev polynomials $T_0(x)$, $T_1(x)$ are known, all other polynomials $T_n(x)$, $n \geq 2$ can be obtained by means of the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \dots(3)$$

Therefore, the polynomials will be on the forms:

$$T_2(x) = 2xT_1(x) - T_0(x)$$

$$T_3(x) = 2xT_2(x) - T_1(x)$$

$$T_4(x) = 2xT_3(x) - T_2(x)$$

$\vdots$

Algorithm of Solution

To begin in solution we have to approximate both states variables $x(t)$ and control variables $u(t)$ by using Chebyshev polynomials as follows:

$$x(t) \approx \sum_{i=0}^n a_i T_i(t) \quad \dots(4)$$

$$u(t) \approx \sum_{i=0}^n b_i T_i(t) \quad \dots(5)$$

Where $0 \leq t \leq 1$, and a_i, b_i are unknown parameters.

Then by expanding $x(t)$ and $u(t)$ into five order ($n = 5$) in eqs. (4) and (5), we get:

$$x(t) = a_0 T_0(t) + a_1 T_1(t) + a_2 T_2(t) + a_3 T_3(t) + a_4 T_4(t) + a_5 T_5(t) \quad \dots(6)$$

$$u(t) = b_0 T_0(t) + b_1 T_1(t) + b_2 T_2(t) + b_3 T_3(t) + b_4 T_4(t) + b_5 T_5(t) \quad \dots(7)$$

Where $T_i(t)$ are Chebyshev polynomials, which can be found by (3).

We can evaluate $a_i, b_i (i = 0, \dots, 5)$ as follows:

We claim that $t = 0, t = 0.5$ and $t = 1$ in eqs. (6) and (7) to find four equations $x(0), x(0.5), x(1), u(0), u(0.5)$ and $u(1)$. Differentiating (6) with respect to t , and put $t = 0$ and substituting the result in (2), seven equations have been obtained with twelve variables.

We need also to find variables, $\dot{x}(0), \ddot{x}(0), x^{(3)}(0), x^{(4)}(0), x^{(5)}(0)$.

After that, we use Gauss elimination procedure for solving the above system to find $a_i, b_i (i = 0, \dots, 5)$.

Finally, substitute a_i, b_i in (6) and (7) and put them in (1) to find the approximating solution.

Example1

$$\text{Minimize } J = \int_0^1 (x^2 + u^2) dt \quad \dots(8)$$

$$\text{Subject to } \dot{x} = u, \quad x(0) = 1 \quad \dots(9)$$

$$\text{The exact value for } x \text{ and } u \text{ are } x(t) = \frac{\cosh(1-t)}{\cosh 1}, \quad u(t) = \frac{-\sinh(1-t)}{\cosh 1}$$

The optimal value of the performance in this problem is $J = 0.761594156$. [4]

Solution

Now we use the previous algorithm to solve this function approximately with Chebyshev polynomials into ($n = 5$)

$$x(t) \approx \sum_{i=0}^5 a_i T_i(t) \quad \dots(10)$$

$$u(t) \approx \sum_{i=0}^5 b_i T_i(t) \quad \dots(11)$$

Where $0 \leq t \leq 1$ and $T_i(t)$ are Chebyshev polynomials

Then (10) and (11) will be

$$\begin{aligned} x(t) &= a_0 T_0(t) + a_1 T_1(t) + a_2 T_2(t) + a_3 T_3(t) + a_4 T_4(t) + a_5 T_5(t) \\ &= (a_0 - a_2 + a_4) + (a_1 - 3a_3 + 5a_5)t + (2a_2 - 8a_4)t^2 + \\ &\quad (4a_3 - 20a_5)t^3 + 8a_4 t^4 + 16a_5 t^5 \quad \dots(12) \end{aligned}$$

$$\begin{aligned} u(t) &= b_0 T_0(t) + b_1 T_1(t) + b_2 T_2(t) + b_3 T_3(t) + b_4 T_4(t) + b_5 T_5(t) \\ &= (b_0 - b_2 + b_4) + (b_1 - 3b_3 + 5b_5)t + (2b_2 - 8b_4)t^2 + \\ &\quad (4b_3 - 20b_5)t^3 + 8b_4 t^4 + 16b_5 t^5 \quad \dots(13) \end{aligned}$$

Now, to evaluate the control points $a_i, b_i (i = 0, \dots, 5)$, we have to find the following values from (12) and (13):

$$\left. \begin{aligned} x(0) &= a_0 - a_2 + a_4 \\ x(0.5) &= a_0 + 0.5a_1 - 0.5a_2 - a_3 - 0.5a_4 + 0.5a_5 \\ x(1) &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 \\ u(0) &= b_0 - b_2 + b_4 \\ \dots(14) \\ u(0.5) &= b_0 + 0.5b_1 - 0.5b_2 - b_3 - 0.5b_4 + 0.5b_5 \\ u(1) &= b_0 + b_1 + b_2 + b_3 + b_4 + b_5 \end{aligned} \right\} \quad \dots(15)$$

Then we find $\dot{x}(0)$:

$$\left. \begin{aligned} \dot{x}(0) &= a_1 - 3a_3 + a_5 \\ \text{After that, we substitute eqs. (15) and (13) into (9):} \\ \dot{x}(0) = u(0) &= -a_1 + 3a_3 - 5a_5 + b_0 - b_2 + b_4 = 0 \\ \ddot{x}(0) = \ddot{u}(0) &= -4a_2 + 16a_4 + b_1 - 3b_3 + 5b_5 = 0 \\ x^{(3)}(0) = \ddot{u}(0) &= -24a_3 + 120a_5 + 4b_2 - 16b_4 = 0 \\ x^{(4)}(0) = u^{(3)}(0) &= -192a_4 + 24b_3 - 120b_5 = 0 \\ x^{(5)}(0) = u^{(4)}(0) &= -1920a_5 + 192b_4 = 0 \end{aligned} \right\} \quad \dots(16)$$

$$x^{(6)}(0) = u^{(5)}(0) = -1920b_5 = 0$$

Transform eqs. (14-16) to the matrix form as follows:

$$EG = H \quad \dots(17)$$

Where,

$$E =$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & -0.5 & -1 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.5 & -0.5 & -1 & -0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 3 & 0 & -5 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 16 & 0 & 0 & 1 & 0 & -3 & 0 & 5 \\ 0 & 0 & 0 & -24 & 0 & 120 & 0 & 0 & 4 & 0 & -16 & 0 \\ 0 & 0 & 0 & 0 & -192 & 0 & 0 & 0 & 0 & 24 & 0 & -120 \\ 0 & 0 & 0 & 0 & 0 & -1920 & 0 & 0 & 0 & 0 & 192 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1920 \end{bmatrix}$$

$$, G = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 0.7307628258 \\ 0.64805427 \\ -0.76159415 \\ -0.3376980397 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we use the Mathcad's Program to find values of G.

Now we will list the values of $a_i, b_i (i = 0, \dots, 5)$ over and above J^*

$$\begin{aligned} a_0 &= 1.264 & b_0 &= -0.955 \\ a_1 &= -0.857 & b_1 &= 1.114 \\ a_2 &= 0.269 & b_2 &= -0.195 \\ a_3 &= -0.032 & b_3 &= 0.038 \\ a_4 &= 4.76 \times 10^{-3} & b_4 &= -1.78 \times 10^{-3} \\ a_5 &= -1.78 \times 10^{-4} & b_5 &= 0 \end{aligned}$$

$$\text{And } J^* = 0.761408638457$$

Example2

$$\text{Minimize } J = \frac{1}{2} \int_0^1 (2x^2 + u^2) dt \quad \dots(18)$$

$$\text{Subject to } \dot{x} = \frac{x}{2} + u, \quad x(0) = 1 \quad \dots(19)$$

$$\text{The exact value for } x \text{ and } u \text{ are } x(t) = \frac{2e^{3t} + e^3}{e^{3t/2}(2 + e^3)}, \quad u(t) = \frac{2(e^{3t} - e^3)}{e^{3t/2}(2 + e^3)} \dots(20)$$

The optimal value of the performance in this problem is $J = 0.864164497$. [4]

Solution

We will solve this example in the same method of the previous example that depends on Chebyshev polynomials with ($n = 5$).

The change will be in eqs. (16) because we will substitute eqs. (15) and (13) into (19):

$$\left. \begin{aligned} \dot{x}(0) &= \frac{x(0)}{2} + u(0) = a_0 - 2a_1 - a_2 + 6a_3 + a_4 - 10a_5 + 2b_0 - 2b_2 + 2b_4 = 0 \\ \dot{\dot{x}}(0) &= \frac{\dot{x}(0)}{2} + \dot{u}(0) = a_1 - 8a_2 - 3a_3 + 32a_4 + 5a_5 + 2b_1 - 6b_3 + 10b_5 = 0 \\ x^{(3)}(0) &= \frac{\dot{\dot{x}}(0)}{2} + \dot{\dot{u}}(0) = 4a_2 - 48a_3 - 16a_4 + 240a_5 + 8b_2 - 32b_4 = 0 \\ x^{(4)}(0) &= \frac{x^{(3)}(0)}{2} + u^{(3)}(0) = 24a_3 - 384a_4 - 120a_5 + 48b_3 - 240b_5 = 0 \\ \dots(21) \\ x^{(5)}(0) &= \frac{x^{(4)}(0)}{2} + u^{(4)}(0) = 192a_4 - 3840a_5 + 384b_4 = 0 \\ x^{(6)}(0) &= \frac{x^{(5)}(0)}{2} + u^{(5)}(0) = 1920a_5 + 3840b_5 = 0 \end{aligned} \right\}$$

Therefore the terms of the matrices E, H will change according to eqs. (21) and eq. (20) respectively.

Then we use also the Mathcad's Program to find values of G.

Now we will list the values of $a_i, b_i (i = 0, \dots, 5)$ over and above J^*

$$\begin{aligned} a_0 &= 1.619 & b_0 &= -2.705 \\ a_1 &= -1.557 & b_1 &= 3.5 \\ a_2 &= 0.639 & b_2 &= -0.996 \\ a_3 &= -0.111 & b_3 &= 0.22 \\ a_4 &= 0.021 & b_4 &= -0.019 \\ a_5 &= -8.75 \times 10^{-4} & b_5 &= 4.375 \times 10^{-4} \end{aligned}$$

And $J^* = 0.86485361679$

Legendre Polynomials of the Degree n

The Legendre polynomials are the everywhere regular solutions of Legendre's equation, [5]

$$(1 - x^2)\ddot{u} - 2x\dot{u} + mu = [(1 - x^2)\dot{u}]' + mu = 0 \quad \dots(22)$$

which are possible only if

$$m = n(n + 1), \quad n = 0, 1, 2, \dots$$

The solution for a particular value of n is $P_n(x)$. It is a polynomial of degree n . If n is even/ odd then the polynomial is even/ odd. They are normalized such that $P_n(x) = 1$.

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2 - 1)/2 \\ P_3(x) &= (5x^3 - 3x)/2 \\ P_4(x) &= (35x^4 - 30x^2 + 3)/8 \\ &\vdots \end{aligned}$$

In this paper, we use the recurrence formula

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x) \quad \dots(23)$$

Therefore, the polynomials will be on the form

$$\begin{aligned} P_2(x) &= \frac{3}{2} x P_1(x) - \frac{1}{2} P_0(x) \\ P_3(x) &= \frac{5}{3} x P_2(x) - \frac{2}{3} P_1(x) \\ P_4(x) &= \frac{7}{4} x P_3(x) - \frac{3}{4} P_2(x) \\ &\vdots \quad [6] \end{aligned}$$

Algorithm of Solution

This algorithm is similar to the previous algorithm that depends on Chebyshev polynomials except that we put $P_n(x)$ instead of $T_n(x)$ as follows:

$$x(t) \approx \sum_{i=0}^n a_i P_i(t) \quad \dots(24)$$

$$u(t) \approx \sum_{i=0}^n b_i P_i(t) \quad \dots(25)$$

Where $0 \leq t \leq 1$, and a_i, b_i are unknown parameters and $P_n(x)$ are Legendre polynomials.

Then by expanding $x(t)$ and $u(t)$ into five order ($n = 5$) in eqs. (24) and (25), we get:

$$x(t) = a_0 P_0(t) + a_1 P_1(t) + a_2 P_2(t) + a_3 P_3(t) + a_4 P_4(t) + a_5 P_5(t) \quad \dots(26)$$

$$u(t) = b_0 P_0(t) + b_1 P_1(t) + b_2 P_2(t) + b_3 P_3(t) + b_4 P_4(t) + b_5 P_5(t) \quad \dots(27)$$

Where $P_i(t)$ are Legendre polynomials, which can be found by (23).

We can evaluate $a_i, b_i (i = 0, \dots, 5)$ as follows:

We also claim that $t = 0, t = 0.5$ and $t = 1$ in eqs. (26) and (27) to find four equations $x(0), x(0.5), x(1), u(0), u(0.5)$ and $u(1)$. Differentiating (26) with respect to t , and put $t = 0$ and substituting the result in (2), seven equations have been obtained with twelve variables.

We need also to find the variables, $\dot{x}(0), \ddot{x}(0), x^{(3)}(0), x^{(4)}(0), x^{(5)}(0)$.

After that, we use Gauss elimination procedure for solving the above system to find $a_i, b_i (i = 0, \dots, 5)$.

Finally, substitute a_i, b_i in (26) and (27) and put them in (1) to find the approximating solution.

Now, we will repeat the solution of the previous examples by using the Legendre polynomials.

Solution Of Example 1

We use the previous algorithm to solve this function approximately with Legendre polynomials into ($n = 5$)

$$x(t) \approx \sum_{i=0}^5 a_i P_i(t) \quad \dots(28)$$

$$u(t) \approx \sum_{i=0}^5 b_i P_i(t) \quad \dots(29)$$

Where $0 \leq t \leq 1$ and $P_i(t)$ are Legendre polynomials

Then (28) and (29) will be

$$\begin{aligned} x(t) &= a_0 P_0(t) + a_1 P_1(t) + a_2 P_2(t) + a_3 P_3(t) + a_4 P_4(t) + a_5 P_5(t) \\ &= \left(a_0 - \frac{1}{2}a_2 + \frac{3}{8}a_4\right) + \left(a_1 - \frac{3}{2}a_3 + \frac{15}{8}a_5\right)t + \left(\frac{3}{2}a_2 - \frac{15}{4}a_4\right)t^2 + \\ &\quad \left(\frac{5}{2}a_3 - \frac{35}{4}a_5\right)t^3 + \frac{35}{8}a_4 t^4 + \frac{63}{8}a_5 t^5 \quad \dots(30) \end{aligned}$$

$$\begin{aligned} u(t) &= b_0 P_0(t) + b_1 P_1(t) + b_2 P_2(t) + b_3 P_3(t) + b_4 P_4(t) + b_5 P_5(t) \\ &= \left(b_0 - \frac{1}{2}b_2 + \frac{3}{8}b_4\right) + \left(b_1 - \frac{3}{2}b_3 + \frac{15}{8}b_5\right)t + \left(\frac{3}{2}b_2 - \frac{15}{4}b_4\right)t^2 + \\ &\quad \left(\frac{5}{2}b_3 - \frac{35}{4}b_5\right)t^3 + \frac{35}{8}b_4 t^4 + \frac{63}{8}b_5 t^5 \quad \dots(31) \end{aligned}$$

Now, to evaluate the control points $a_i, b_i (i = 0, \dots, 5)$, we have to find the following values from (30) and (31):

$$\left. \begin{aligned} x(0) &= a_0 - \frac{1}{2}a_2 + \frac{3}{8}a_4 \\ x(0.5) &= a_0 + \frac{1}{2}a_1 - \frac{1}{8}a_2 - \frac{7}{16}a_3 - \frac{37}{128}a_4 + \frac{23}{256}a_5 \\ x(1) &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 \\ u(0) &= b_0 - \frac{1}{2}b_2 + \frac{3}{8}b_4 \\ \dots(32) \\ u(0.5) &= b_0 + \frac{1}{2}b_1 - \frac{1}{8}b_2 - \frac{7}{16}b_3 - \frac{37}{128}b_4 + \frac{23}{256}b_5 \\ u(1) &= b_0 + b_1 + b_2 + b_3 + b_4 + b_5 \end{aligned} \right\} \quad \dots(33)$$

Then we find $\dot{x}(0)$:

$$\dot{x}(0) = a_1 - \frac{3}{2}a_3 + \frac{15}{8}a_5$$

After that, we substitute eqs. (15) and (13) into (9):

$$\left. \begin{aligned} \dot{x}(0) = u(0) &= -a_1 + \frac{3}{2}a_3 - \frac{15}{8}a_5 + b_0 - \frac{1}{2}b_2 + \frac{3}{8}b_4 = 0 \\ \ddot{x}(0) = \dot{u}(0) &= -3a_2 + \frac{15}{2}a_4 + b_1 - \frac{3}{2}b_3 + \frac{15}{8}b_5 = 0 \\ x^{(3)}(0) = \dot{u}(0) &= -15a_3 + \frac{105}{2}a_5 + 3b_2 - \frac{15}{2}b_4 = 0 \\ x^{(4)}(0) = u^{(3)}(0) &= -105a_4 + 15b_3 - \frac{105}{2}b_5 = 0 \\ x^{(5)}(0) = u^{(4)}(0) &= -945a_5 + 105b_4 = 0 \\ x^{(6)}(0) = u^{(5)}(0) &= 945b_5 = 0 \end{aligned} \right\} \dots(34)$$

Transform eqs. (32-34) to the matrix on the form of eq. (17)

Where,

E =

$$E = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{8} & -\frac{7}{16} & -\frac{37}{128} & \frac{23}{256} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{8} & -\frac{7}{16} & -\frac{37}{128} & \frac{23}{256} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & \frac{3}{2} & 0 & -\frac{15}{8} & 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 0 & -3 & 0 & \frac{15}{2} & 0 & 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{15}{8} \\ 0 & 0 & 0 & -15 & 0 & \frac{105}{2} & 0 & 0 & 3 & 0 & -\frac{15}{2} & 0 \\ 0 & 0 & 0 & 0 & -105 & 0 & 0 & 0 & 0 & 15 & 0 & -\frac{105}{2} \\ 0 & 0 & 0 & 0 & 0 & -945 & 0 & 0 & 0 & 0 & 105 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 945 \end{bmatrix}$$

$$, G = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 0.7307628258 \\ 0.64805427 \\ -0.76159415 \\ -0.3376980397 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we use the Mathcad's Program to find values of G.

Now we will list the values of $a_i, b_i (i = 0, \dots, 5)$ over and above J^*

$$\begin{array}{ll} a_0 = 1.174 & b_0 = -0.89 \\ a_1 = -0.838 & b_1 = 1.091 \\ a_2 = 0.355 & b_2 = -0.259 \\ a_3 = -0.051 & b_3 = 0.061 \\ a_4 = 8.704 \times 10^{-3} & b_4 = -3.255 \times 10^{-3} \\ a_5 = -3.616 \times 10^{-4} & b_5 = 0 \end{array}$$

And $J^* = 0.761317923467$

Solution of Example2

We use the same method of the previous algorithm to solve example (2) approximately with Legendre polynomials into ($n = 5$).

The change will be in eqs. (34) because we will substitute eqs. (33) and (31) into (19):

$$\left. \begin{array}{l} \dot{x}(0) = \frac{x(0)}{2} + u(0) = a_0 - 2a_1 - \frac{1}{2}a_2 + 3a_3 + \frac{3}{8}a_4 - \frac{15}{4}a_5 + 2b_0 - b_2 + \frac{3}{4}b_4 = 0 \\ \dot{x}(0) = \frac{\dot{x}(0)}{2} + \dot{u}(0) = a_1 - 6a_2 - \frac{3}{2}a_3 + 15a_4 + \frac{15}{8}a_5 + 2b_1 - 3b_3 + \frac{15}{4}b_5 = 0 \\ x^{(3)}(0) = \frac{\dot{x}(0)}{2} + \dot{u}(0) = 3a_2 - 30a_3 - \frac{15}{2}a_4 + 105a_5 + 6b_2 - 15b_4 = 0 \\ x^{(4)}(0) = \frac{x^{(3)}(0)}{2} + u^{(3)}(0) = 15a_3 - 210a_4 - \frac{105}{2}a_5 + 30b_3 - 105b_5 = 0 \\ \dots(35) \\ x^{(5)}(0) = \frac{x^{(4)}(0)}{2} + u^{(4)}(0) = 105a_4 - 1890a_5 + 210b_4 = 0 \\ x^{(6)}(0) = \frac{x^{(5)}(0)}{2} + u^{(5)}(0) = 945a_5 + 1890b_5 = 0 \end{array} \right\}$$

Therefore the terms of the matrices E, H will change according to eqs. (21) and eq. (20) respectively.

Then we use also the Mathcad's Program to find values of G.

Now we will list the values of $a_i, b_i (i = 0, \dots, 5)$ over and above J^*

$$\begin{array}{ll} a_0 = 1.404 & b_0 = -2.372 \\ a_1 = -1.49 & b_1 = 3.368 \\ a_2 = 0.837 & b_2 = -1.313 \\ a_3 = -0.177 & b_3 = 0.351 \\ a_4 = 0.037 & b_4 = -0.035 \\ a_5 = -1.778 \times 10^{-3} & b_5 = 8.889 \times 10^{-4} \end{array}$$

And $J^* = 0.864283480578$

Conclusion

The Chebyshev and Legendre polynomials are considered in this paper, in order to use them to approximate the variables by using these polynomials.

we use the Chebyshev polynomials in the first solution and after that we use Legendre polynomials, to reach to the approximate solution J^* .

If we compare the results with respect to the exact solution that given, we can use these polynomials to solve quadratic optimal control problem numerically, with accurate results.

References

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المستخلص

الغرض الاساسي من هذا البحث هو حل مسائل السيطرة المثلى التربيعية عدديا باستخدام متعددة الحدود جيبيشيف وليجيندر واستخدامها كدوال اساسية لايجاد حلول تقريبية لمثل هذه المسائل، مع بيان خوارزميات هذه الحلول على بعض الامثلة وبأستخدام برنامج الماث كاد للوصول الى نتائج مضبوطة.