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Buckling Analysis of Euler-Bernoulli Beams Resting on Two-Parameter Elastic Foundations: Closed Form Solutions

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ABSTRACT

The buckling analysis of Euler-Bernoulli beam resting on two-parameter elastic foundation (EBBo2PEF) has important applications in the analysis and design of foundation structures, buried gas pipeline systems and other soil-structure interaction systems under compressive loads. This study investigates the buckling analysis of EBBo2PEFs. The governing differential equation of elastic stability (GDiES) is derived in this work using first principles equilibrium method. In general, the GDiES is an inhomogeneous equation with variable parameters for non-prismatic beams under distributed transverse loadings. However, when transverse loads are absent and the beam is prismatic the GDiES becomes a fourth order ordinary differential constant parameter homogeneous equation. General solution to GDiES is obtained in this work using the classical trial exponential function method of solving equations. Two cases of end supports were considered: simply supported ends and clamped ends. Boundary conditions (BCs) were used to obtain the characteristic buckling equations whose eigenvalues were used to determine the critical buckling loads for two cases of BCs considered. It was found that the method gave exact solutions for each of the BCs. The critical elastic buckling load coefficients for dimensionless beam-foundation parameter $\widehat{K}_1$ and $\widehat{K}_2$ ranging from $\widehat{K}_1 = 0, 1, 100$; $\widehat{K}_2 = 0, 0.5, 1, 2.5$ for simply supported EBBo2PEFs were identical with previous results that used Stodola-Vianello iteration methods and finite element method. Similarly, the critical buckling load coefficients for $\widehat{K}_1 = 0, 1, 50, 100$ and $\widehat{K}_2 = 0, 0.5, 1, 2.5$ are identical with previous results that used Ritz variational method.

1. Introduction

The theme of beam on elastic foundation is used for the analysis of foundation structures, railroads, airport, runways, dam embankments and buried gas pipeline systems. Buckling is one of the most disadvantageous types of instability for beams under compressive forces, and must be studied and considered for safety in their design (Timoshenko & Gere,1985).

Beam on elastic foundation theories (BoEFTs) have been derived using variational methods or equilibrium methods. Irrespective of the method used, the equations are seen to incorporate the elastic foundation's reaction

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into the beam model equation used to describe the beam. The effect of the elastic foundation thus modifies the governing equation of the beam under buckling. The simplest beam model is the thin beam model that is valid for thickness to span ratios that are less than 0.05. Such a model was derived by Euler and Bernoulli for bending and is called Eulier-Bernoulli beam theory (EBBT). EBBT assumes that the beam cross-sections are orthogonal to the middle plane before, and after deformation, and the middle plane is free of axial deformation. This renders EBBT unable to consider transverse shear strains which are responsible for causing distortions of the cross-sections. This limits the application of EBBT to thin or slender beams for which transverse shear deformations are negligible. EBBT has been extended to buckling and vibration of thin beams.

Research efforts to extend beam theory to moderately thick beams and consider transverse shear deformation led to the developments of Timoshenko beam theory (TBT), and shear deformation beam theories by Dahake and Ghugal (2013), Levinson (1981), and Sayyad and Ghugal (2011) for beams with thickness to span ratios greater than 0.05. This work explores thin beams and hence used EBBT.

Elastic foundations have been modeled using two fundamental approaches; discrete or lumped parametrization and continuously distributed parametrization. The mathematical theory of elasticity is used in continuously distributed parameter models to formulate the soil reactions. They are consequently difficult to formulate, and are rarely used due to extreme rigours of mathematics involved.

Discrete parameter models are commonly used and have been much researched on because of their simple mathematical forms.

The lumped parameter models that are common include: (i) Winkler model, a one parameter model, (ii) Pasternak, Hetenyi, Vlasov, Filonenko-Borodich, which are two parameter models, and (iii) Kerr (1985) three-parameter model.

The Winkler-Zimmerman model shown in Figure 1 assumes the soil is a system of vertical, closely spaced, independent, linear elastic springs that has a stiffness which is proportional to the deflection at any point on the beam.

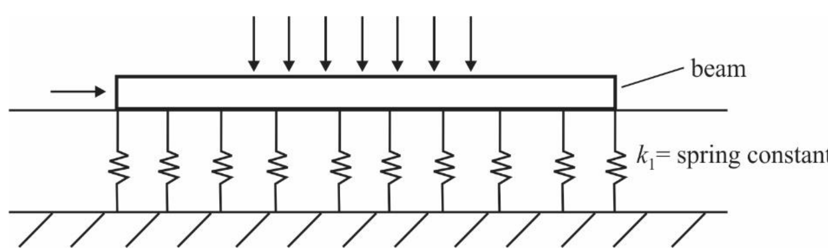

Fig. 1 Winkler-Zimmerman foundation model

Winkler-Zimmerman elastic foundation model has been criticized because the deformation of the foundation occurs only in the vicinity of the applied load, resulting in issues of discontinuity in the deflection (Akhazhanov et al., 2023). The model thus disregards the deformation of areas that are in the vicinity of the applied point loads. Another reason for criticizing the Winkler method is the lack of certainty in the method for finding the Winkler parameter.

The Winkler-Zimmerman foundation model is the simple equation:

$$r_f(x) = k_1 w(x) \tag{1}$$

where $r_f(x)$ is the foundation reaction, k_1 is the Winkler constant, $w(x)$ is the beam transverse deflection.

Two-parameter foundation model is adopted in this study and shown in Figure 2.

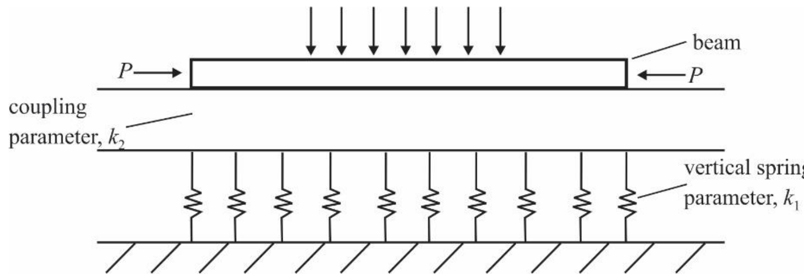

Fig. 2 Thin beam resting on two-parameter foundation model

The two-parameter foundation model was formulated to account for the discontinuity defects of the one-parameter model. The second parameter k_2 is defined as a parameter coupling the set of vertical springs as is represented in the Figure 2, and thus representing the coupling of the vertical springs to ensure continuity of the deflections.

The two-parameter elastic foundation model equation is given by the simple equation:

$$r_f(x) = k_1 w(x) - k_2 \frac{d^2 w(x)}{dx^2} \tag{2}$$

Wstawska, Magnucki and Kedzia (2022) presented analytical and numerical studies of thin axially compressed beam resting on a variable Winkler foundation. Wstawska, Magnucki and Kedzia (2019) used approximate methods of solving boundary value problems for the stability analysis of homogeneous beam on elastic foundation.

Palacio-Betancur and Aristizabal-Ochoa (2019) studied the static flexure, buckling and vibration analysis of non-prismatic beam-columns with semi-rigid connections rested on elastic foundation. Atay and Coskun (2009) utilized the variational iteration method (VIM) to develop closed form solutions for the elastic stability of uniform homogeneous beams resting on Winkler foundation for various cases of boundary conditions.

Dutta et al. (2021) derived a first order three noded slender beam element using fifth order displacement functions for the solutions of slender beam on Pasternak foundation. They used a MATLAB computer program and obtained accurate results, when compared with the literature.

Al-Ejbari , Faris, and Al-Jumaily (2007) presented a theoretical analysis for approximately solving the in-plane large displacement elastic buckling problems of non-prismatic structures on Winkler foundations. The work modeled the structures as beam-columns resting on Winkler foundations. The axial forces are considered in the Euler approach to the flexural formulation of the beam-column. Finite difference and finite element methods were used in their work as the numerical tools for solving the geometrically nonlinear buckling problem.

Baraldi (2019) presented a simple numerical model for sandwich composites resting on an elastic layer. The thin beam hypothesis and an approximate expression of the Green's function of an elastic layer on a rigid base was adopted in the work. A simple finite element boundary integral equation method was used for the static and buckling analysis of the thin beam in frictionless contact with the elastic layer. Their work used a mixed variational formulation that assumed that both beam displacements and contact reactions between the beam and the layer are independent fields. Numerical analysis reveals that a beam on thick layer unbounded to the rigid base has similar results as a beam on a half-plane.

Kuliński and Przybylski (2015) studied buckling load analysis of stepped beams resting on elastic foundations. Mohammed, Hareb and Eqal (2021) studied the stability problem of functionally graded material (FGM) thin beams resting on Winkler-Pasternak elastic foundations. Loya et al. (2023) investigated the buckling loads behaviour of cracked Euler-Bernoulli columns embedded in Winkler medium. Nguyen et al. (2023) used the Legendre-Ritz method to find solutions for the buckling of porous beams on elastic foundation. Mellal et al. (2023) studied the buckling of porous functionally graded (FG) beams supported on variable elastic foundation by using higher order shear deformation beam theory to model the porous FG beam.

Abramian et al. (2021) studied the buckling loads for a vibrating clamped thin beam resting on non-homogeneous elastic foundations for various types of non-homogeneities and for various types of damping models. They used Rayleigh variational method, but did not however derive exact buckling solutions for axially loaded beams on homogeneous elastic foundations.

Hamed et al. (2020) investigated stability analysis of sandwich beams on elastic foundation under varying inplane loads. Patel (2019) presented dynamic stability of axially loaded beams on elastic foundations problem.

Taha (2014) presented recursive differentiation method (RDM) for solving boundary value problems (BVPs) and showed its application to the analysis of beam-columns resting on elastic foundation. RDM relies on Taylor series expansion methods and was found useful as a numerical solutions method for BVPs. Later, Hassan and Hadima (2015), used the RDM for the analysis of non-uniform beams rested on elastic foundations. Other applications of RDM for the analysis of vibration and buckling problems of beam on elastic foundation were done by Hassan and Doha (2015).

Hetenyi (1946), Timoshenko and Gere (1985) and Wang et al. (2005) have presented exact solutions for buckling problems by seeking mathematically rigorous solutions to the governing equations. In their studies, they obtained analytical expressions that satisfied the governing equations of elastic stability at all points on the domain and simultaneously satisfied the displacement and force boundary conditions at the supported ends. Hassan (2008) also presented exact buckling solutions to thin beam on elastic foundations of the Winkler type but did not consider two parameter foundations.

Akgöz et al. (2016) used discrete singular convolution method for the static flexural analysis of beam on elastic foundation, but did not consider buckling analysis. Anghel and Mares (2019) presented an integral formulation for the dynamic and buckling analysis of beams on elastic foundation and used collocation methods to obtain accurate vibration and buckling solutions. Their solutions needed a large number of collocation points to achieve accuracy, at the expense of tedious algebraic work.

Aristizabal-Ochoa (2013) studied the EBBoEF problem for generalized boundary conditions as the EBBoEF is analogous to the slender column on an elastic foundation problem. Naidu and Rao (1995) presented buckling solutions for uniform beams on two parameter foundations using finite element methodology (FEM). FEM was also used for Timoshenko beams in elastic foundation by Soltani (2020).

Rao and Raju (2002) derived analytical solutions for dynamic and buckling analysis of thin beam on Pasternak foundations. Hariz et al. (2022) studied the buckling analysis of Timoshenko beams on two-parameter elastic foundation.

Ike (2018a) studied the finite sine transformation method (FSTM) for the natural transverse vibration analysis of EBBoWF with Dirichlet boundary conditions. The sinusoidal kernel funcrtion in the integral transformation satisfies the Dirichlet boundary conditions, rendering the FSTM suitable for the problem. However, the study did not consider buckling problems. Ike (2018b) used point collocation method (PCM) for the approximate solutions of static flexural analysis of thin beam on Winkler foundation. PCMs aim to solve the BVP at discrete, known collocation points and not over the entire domain rendering the method approximate. The study did not consider buckling analysis. Ike (2022) used Reissner functional minimization method for bending analysis of EBBoWF but did not consider buckling and two-parameter foundations.

Ike (2023a) used generalized integral transformation method (GITM) for free transverse vibration solutions of EBBoWF for different boundary conditions. The GITM used eigenfunctions of vibrating beams with equivalent end supports to drive exact analytical solutions to the vibration problem. However, buckling analysis was not considered in the study. Ike (2023b) used Stodola-Vianello iteration method (SVIM) to determine exact buckling load solutions of simply supported Euler-Bernoulli beams on Pasternak foundation (EBBoPF). The exact buckling shape functions for the Dirichlet boundary conditions were used to find the exact buckling loads for any mode of buckling. Exact critical buckling loads were found at the first mode.

Ike (2023c) used Stodola-Vianello iteration method and algebraic polynomial shape functions that satisfied simple end supports to derive approximate but accurate buckling load solutions for simply supported Euler-Bernoulli beams on two-parameter elastic foundations (EBBo2PEFs). Ike (2023d) explored SVIM for the determination of critical buckling load solutions for simply supported EBBoWFs. The work used fourth order polynomial shape functions that were constructed to satisfy the simply supported BCs leading to a one-parameter displacement shape function. Accurate buckling solutions were obtained by assuming convergence after the first iteration. Ike (2023e) used exact sinusoidal shape functions in the SVIM to define the exact buckling loads for any mode for a simply supported EBBoWF. Ike (2024a) used Fourier series method to derive exact buckling solutions to the BVP of simply supported EBBo2PEFs. The Fourier series method was found particularly ideal for the study because the infinite Fourier sine series adopted satisfied all the deformation and force boundary conditions and was also made to satisfy the domain equations. Ike (2024b) used Ritz variational method to derive the accurate solutions to the buckling of EBBo2PEFs for different boundary conditions, including simple end supports, clamped-clamped ends and clamped-free ends. The RVM adopted is based on the principle of minimizing the total potential energy functional for the buckling EBBo2PEF. Buckling shape functions that satisfied the boundary conditions for simply supported, clamped ends and clamped-free ends were constructed using polynomial functions and used in the total potential energy functional to express Π in terms of the generalized displacement parameters of the buckling displacement function. Subsequently, Π was minimized to obtain an eigenvalue problem which was solved for nontrivial solution conditions to obtain the buckling loads from the roots of the characteristic buckling equation. Ike (2024c) developed critical buckling load solutions of Euler-Bernoulli beams on two-parameter elastic foundations using Galerkin variational method (GVM) and polynomial basis functions. Polynomial basis functions were derived in the work for pinned-pinned supported, clamped-clamped ends, clamped-simply supported ends and used in the GVM to express the governing PDE as an algebraic equation in terms of the displacement parameters. Reasonably accurate buckling load solutions were derived for the boundary conditions considered.

Ike et al. (2018) explored Picard's successive iteration method for the elastic solutions of Euler columns with pinned ends but did not consider elastic foundation interaction effects. Ike et al. (2023a) used SVIM for the accurate critical buckling load analysis of EBBo2PEF with clamped-clamped ends. The SVIM was used to express the governing equations in iterative form as a system of algebraic equations. One-parameter polynomial shape function was constructed form fifth degree polynomial to satisfy the clamped-clamped boundary conditions and used in the SVIM to derive the buckling load for EBBo2PEF using the convergence criteria of the iteration. In their work, one parameter polynomial buckling shape function was found using fifth degree polynomial to satisfy clamped-clamped boundary conditions and utilized in the SVIM to derive the buckling load for EBB0WF using the convergence criteria of the iteration. Ike et al. (2023b) used SVIM for the accurate critical buckling load analysis of EBBoWF with clamped-clamped ends. Ikwueze et al. (2018) explored least squares weighted residual method (LSWRM) for the elastic buckling analysis of Euler column with one end clamped and the other end pinned, but failed to consider elastic foundation interaction effects. Mama et al. (2020) used fifth degree polynomial shape functions in the finite element method to find approximate solutions to the elastic buckling of EBBoWF. Ofondu et al. (2018) have explored the SVIM for the critical buckling load analysis of Euler columns with one end fixed and the other end pinned; but did not consider elastic foundation interaction effects.

In this study the buckling analysis of Euler-Bernoulli beams resting on two-parameter elastic foundations is studied from fundamental principles. The governing differential equations of elastic stability (GDiES) for isotropic, homogenous, prismatic, linearly elastic beams resting on two-parameter elastic foundations is derived for the general case where there are transverse loads and where there is no transverse load. The GDiES is then solved in closed form such that boundary conditions and domain equations are satisfied. The classical method of trial functions is used to develop buckling solutions for TBo2PF with simply supported ends and clamped-clamped ends.

2. Framework of theory and formulation of governing domain equation of stability (GDES)

2.1. Fundamental assumptions

The assumptions are:

(i) the beam material is linearly elastic, homogeneous and isotropic.
(ii) transverse displacements are infinitesimally small in comparison with the beam depth.
(iii) axial (longitudinal) strains are negligible.
(iv) transverse normal strains and shear strains are insignificant, and neglected.
(v) the planes of the cross-section remain plane and orthogonal to the longitudinal middle plane of the beam before and after deformation. This orthogonality hypothesis is called the Euler-Bernoulli hypothesis.

2.2. Formulation of governing differential equation of stability (GDiES)

Figure 3 shows a typical beam on elastic foundation. Figure 4 depicts the free body diagram of an element of a beam resting on an elastic foundation. Figure 4 shows all the forces – internal forces, reaction forces and applied distributed transverse forces on the BoEF.

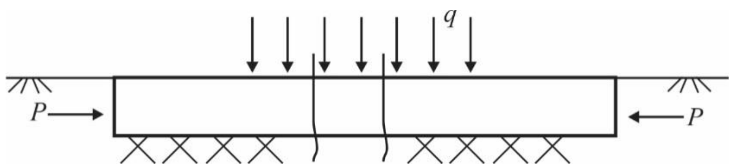

Fig. 3 Buckling of beam on elastic foundation BoEF

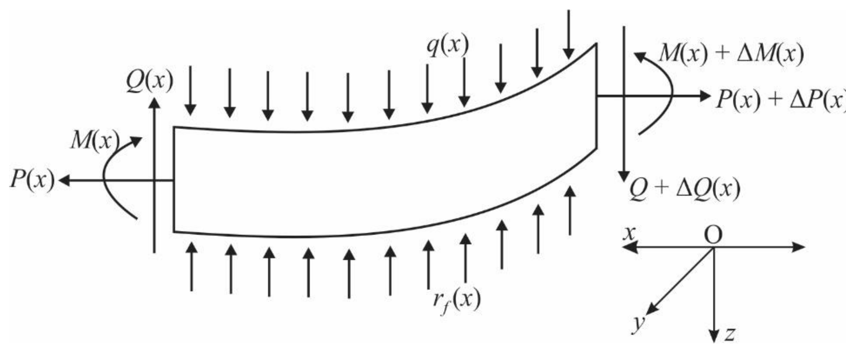

Fig. 4 Free body diagram of an element of a beam on Winkler foundation

For equilibrium in the horizontal direction,

$$\sum F_{ix} = P(x) + \Delta P(x) - P(x) = 0 \quad (3)$$

where $P(x) + \Delta P(x)$ is the axial force on the right hand side of the section, and $P(x)$ is the axial force on the left hand side of the section.

Hence, $\Delta P(x) = 0$ **(4)**

For equilibrium in the vertical direction,

$$\sum F_{iv} = Q(x) + \Delta Q(x) - Q(x) + q(x)\Delta x - r_f(x)\Delta x = 0 \quad (5)$$

$Q(x)$ is the shear force on the left hand side, $Q(x) + \Delta Q(x)$ is the shear force on the right hand side, $r_f(x)$ is the foundation reaction, $q(x)$ is the transverse load intensity.
Simplifying,

$$\Delta Q(x) = \left(r_f(x) - q(x)\right)\Delta x \quad (6)$$

Dividing by Δx, and considering the limit as Δx becomes infinitesimally small, Equation (6) becomes:

$$\lim_{\Delta x \to 0} \frac{\Delta Q(x)}{\Delta x} = \frac{dQ(x)}{dx} = r_f(x) - q(x) \quad (7)$$

For rotational equilibrium, taking moments about the left hand side of the section gives:

$$\sum M_i = M(x) + \Delta M(x) - M(x) - \left(Q(x) + \Delta Q(x)\right)\Delta x - q(x)\frac{(\Delta x)^2}{2} + r_f(x)\frac{(\Delta x)^2}{2} - \left(P(x) + \Delta P(x)\right)\Delta w(x) = 0 \quad (8)$$

Simplifying Equation (8) and dividing by Δx, gives:

$$\frac{\Delta M(x)}{\Delta x} - P(x)\frac{\Delta w(x)}{\Delta x} - Q(x) - \Delta Q(x) - q(x)\frac{\Delta x}{2} + r_f(x)\frac{\Delta x}{2} - \Delta P(x)\frac{\Delta w(x)}{\Delta x} = 0 \quad (9)$$

Further simplification of Equation (9) by considering that $\Delta P(x) = 0$ from Equation (4) gives upon considering the limit as $\Delta x \to 0$,

$$\lim_{\Delta x \to 0}\left(\frac{\Delta M(x)}{\Delta x} - P(x)\frac{\Delta w(x)}{\Delta x} - Q(x) - \Delta Q(x) - q(x)\frac{\Delta x}{2} + r_f(x)\frac{\Delta x}{2}\right) = \frac{dM(x)}{dx} - P(x)\frac{dw(x)}{dx} - Q(x) = 0 \quad \ldots(10)$$

Combining the three equations of equilibrium gives:

$$\frac{dQ(x)}{dx} = \frac{d}{dx}\left(\frac{dM(x)}{dx} - P(x)\frac{dw(x)}{dx}\right) = r_f(x) - q(x) \quad (11)$$

Hence, the GDiES becomes:

$$\frac{d}{dx}\left(\frac{dM(x)}{dx} - P(x)\frac{dw(x)}{dx}\right) = r_f(x) - q(x) \quad (12)$$

The GDiES can be expressed in terms of $w(x)$ using the Euler-Bernoulli theory EBT for thin beams in flexure. From EBT, $M(x)$ is related to $w(x)$ as:

$$M(x) = -EI\frac{d^2 w(x)}{dx^2} \quad (13)$$

where E is the Young's modulus, I is the moment of inertia.
Then, the GDiES becomes:

$$\frac{d}{dx}\left(\frac{d}{dx}\left(-EI\frac{d^2 w(x)}{dx^2}\right) - P\frac{d^2 w(x)}{dx^2}\right) = r_f(x) - q(x) \quad (14)$$

Simplifying,

$$\frac{d^2}{dx^2}\left(-EI\frac{d^2 w(x)}{dx^2}\right) - \frac{d}{dx}\left(P\frac{d^2 w(x)}{dx^2}\right) = r_f(x) - q(x) \quad (15)$$

Multiplying by (–1) gives:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w(x)}{dx^2}\right)+\frac{d}{dx}\left(P\frac{d^2w(x)}{dx^2}\right)=q(x)-r_f(x) \tag{16}$$

When the beam is homogeneous, and prismatic and the axial load $P(x)$ is not a function of x, then the GDiES simplifies to the fourth order differential equation:

$$EI\frac{d^4w(x)}{dx^4}+P\frac{d^2w(x)}{dx^2}=q(x)-r_f(x) \tag{17}$$

For two-parameter foundations, Equation (2) is used in Equation (17). The GDiES for thin beam on two-parameter foundation is:

$$EI\frac{d^4w}{dx^4}+P\frac{d^2w}{dx^2}+k_1w(x)-k_2\frac{d^2w}{dx^2}=q(x) \tag{18}$$

When there is no transverse force, $q(x) = 0$, and GDiES becomes homogeneous as follows:

$$EI\frac{d^4w}{dx^4}+(P-k_2)\frac{d^2w}{dx^2}+k_1w(x)=0 \tag{19}$$

3. Closed form solution method

The paper seeks an exact solution to the problem that satisfies the GDiES at all points on the solution domain, and on the boundaries using the classical method of trial functions. The GDiES is divided by EI to have:

$$\frac{d^4w}{dx^4}+\left(\frac{P-k_2}{EI}\right)\frac{d^2w}{dx^2}+\frac{k_1}{EI}w(x)=0 \tag{20}$$

Let, $\alpha=\frac{P}{EI}$, $\beta_1=\frac{k_1}{EI}$, $\beta_2=\frac{k_2}{EI}$ **(21)**

$$w^{iv}(x)+(\alpha-\beta_2)w''(x)+\beta_1w(x)=0 \tag{22}$$

where $w''(x)=\frac{d^2w(x)}{dx^2}$; $w^{iv}(x)=\frac{d^4w(x)}{dx^4}$ **(23)**

By the method of trial functions, the solution for $w(x)$ is sought in the form of an exponential function of the form:

$$w(x)=He^{sx} \tag{24}$$

where s is a parameter to be determined and H is the amplitude of $w(x)$, where H does not depend upon x.

If $w(x)$ is the trial solution, the GDiES becomes:

$$s^4 He^{sx} + (\alpha - \beta_2) s^2 He^{sx} + \beta_1 He^{sx} = 0 \quad (25)$$

Simplifying,

$$\left(s^4 + (\alpha - \beta_2) s^2 + \beta_1\right) He^{sx} = 0 \quad (26)$$

For meaningful solutions, $He^{sx} = w(x) \neq 0$
The auxiliary equation for nontrivial solutions become:

$$s^4 + (\alpha - \beta_2) s^2 + \beta_1 = 0 \quad (27)$$

This is a quadratic equation in s^2. The roots are:

$$s^2 = \frac{-(\alpha - \beta_2) \pm \sqrt{(\alpha - \beta_2)^2 - 4\beta_1}}{2} \quad (28a)$$

Or,

$$s^2 = \frac{-(\alpha - \beta_2)}{2} \pm \frac{\sqrt{(\alpha - \beta_2)^2 - 4\beta_1}}{2} \quad (28b)$$

$$s^2 = -\left(\frac{\alpha - \beta_2}{2} \mp \sqrt{\left(\frac{\alpha - \beta_2}{2}\right)^2 - \beta_1}\right) \quad (28c)$$

$$s^2 = -A^2$$
$$s^2 = -B^2 \quad (29)$$

$$A^2 = \frac{\alpha - \beta_2}{2} - \sqrt{\left(\frac{\alpha - \beta_2}{2}\right)^2 - \beta_1} \quad (30a)$$

$$B^2 = \frac{\alpha - \beta_2}{2} + \sqrt{\left(\frac{\alpha - \beta_2}{2}\right)^2 - \beta_1} \quad (30b)$$

Then the four roots are:

$$s_{1,2} = \pm Ai$$
$$i = \sqrt{-1} \quad (31)$$
$$s_{3,4} = \pm Bi$$

The basis of linearly independent solution becomes: $e^{Aix}, e^{-Aix}, e^{Bix}, e^{-Bix}$

Hence, the general solution is:

$$w(x) = \bar{c}_1 e^{Aix} + \bar{c}_2 e^{-Aix} + \bar{c}_3 e^{Bix} + \bar{c}_4 e^{-Bix} \quad (32)$$

Using Euler's formula,

$$w(x) = \bar{c}_1(\cos Ax + i\sin Ax) + \bar{c}_2(\cos Ax - i\sin Ax) + \bar{c}_3(\cos Bx + i\sin Bx) + \bar{c}_4(\cos Bx - i\sin Bx) \quad \ldots(33)$$

Further simplification gives:

$$w(x) = c_1 \cos Ax + c_2 \sin Ax + c_3 \cos Bx + c_4 \sin Bx \quad (34)$$

where $c_1 = \bar{c}_1 + \bar{c}_2,\ c_2 = i(\bar{c}_1 - \bar{c}_2),\ c_3 = \bar{c}_3 + \bar{c}_4,\ c_4 = i(\bar{c}_3 - \bar{c}_4)$ (35)

The constants c_1, c_2, c_3 and c_4 are dependent on the end support conditions of the problem.

4. Results and discussion

4.1. Case 1: Thin beam on 2 parameter foundation (TBo2PF) with simply supported ends

The boundary conditions (BCs) are

$$w(0) = 0,\ w(l) = 0,\ w''(0) = 0, w''(l) = 0 \quad (36)$$

Applying the BCs gives:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -A^2 & 0 & -B^2 & 0 \\ \cos Al & \sin Al & \cos Bl & \sin Bl \\ -A^2\cos Al & -A^2\sin Al & -B^2\cos Bl & -B^2\sin Bl \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (37)$$

Solving, this reduces to:

$$\begin{pmatrix} \sin Al & \sin Bl \\ -A^2\sin Al & -B^2\sin Bl \end{pmatrix} \begin{pmatrix} c_2 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (38)$$

For nontrivial solutions, the characteristic equation is:

$$\begin{vmatrix} \sin Al & \sin Bl \\ -A^2\sin Al & -B^2\sin Bl \end{vmatrix} = 0 \quad (39)$$

Expanding the determinant,

$$(A^2 - B^2)\sin Al \sin Bl = 0 \quad (40)$$

Solving, $\sin Al = 0$ (41)

$$\boldsymbol{Al} = \sin^{-1} 0 = n\pi \quad (42)$$

$$\boldsymbol{A} = n\pi / l \quad (43)$$

Or, $\sin B\, l = 0$

$Bl = n\pi$

$B = \frac{n\pi}{l}$

So, $$A^2 = \left(\frac{n\pi}{l}\right)^2 = \left(\frac{\alpha - \beta_2}{2}\right) - \sqrt{\left(\frac{\alpha - \beta_2}{2}\right)^2 - \beta_1} \quad (44)$$

$$\left(\frac{n\pi}{l}\right)^2 - \left(\frac{\alpha - \beta_2}{2}\right) = -\sqrt{\left(\frac{\alpha - \beta_2}{2}\right)^2 - \beta_1} \quad (45)$$

Squaring,

$$\left(\left(\frac{n\pi}{l}\right)^2 - \left(\frac{\alpha - \beta_2}{2}\right)\right)^2 = \left(-\sqrt{\left(\frac{\alpha - \beta_2}{2}\right)^2 - \beta_1}\right)^2 \quad (46)$$

$$\alpha = \frac{P}{EI} = \beta_2 + \frac{\beta_1 l^2}{(n\pi)^2} + \left(\frac{n\pi}{l}\right)^2 \quad (47)$$

$$P = EI\left(\beta_2 + \frac{\beta_1 l^2}{(n\pi)^2} + \left(\frac{n\pi}{l}\right)^2\right) = \frac{EI}{l^2}\left(\beta_2 l^2 + \frac{\beta_1 l^4}{(n\pi)^2} + (n\pi)^2\right) \quad (48)$$

$$P_n = \frac{EI}{l^2} K(\beta_1 l^4, \beta_2 l^2, n) \quad (49)$$

$$K(\beta_1 l^4, \beta_2 l^2, n) = \left(\frac{\beta_2 l^2}{(n\pi^2)} + \frac{\beta_1 l^4}{(n\pi)^4} + 1\right)(n\pi)^2 \quad (50)$$

Identical results for P_n is obtained using the result for B.

Equation (50) is used to calculate $K(\beta_1 l^4, \beta_2 l^2, n = 1)$ for previous work done by Soltani and Asgarin (2019). Table 1 illustrates the close agreement of present solution and the previous Soltani and Asgarin's results, for $\beta_2 l^2 = 0, 2, 4, 6, 8$ and $\beta_1 l^4 = 0, 20, 40, 60, 80$.

Table 2 presents $K(\beta_1 l^4, \beta_2 l^2, n = 1)$ for $\beta_1 l^4 = \widehat{K}_1 = 0,\ 1,\ 50,\ 100,\ 1000,\ 10{,}000$ and $\frac{\beta_2 l^2}{\pi^2} = \widehat{K}_2 = 0,\ 0.5,\ 1,\ 2.5$ for the present study and for previous studies by Taha (2014), Anghel and Mares (2019), Naidu and Rao (1995), and Ike (2023a, 2023c, 2024a, 2024b). Table 2 illustrates that present results are identical with previous results by Ike (2023a, 2024a, 2024b) and closely agree with results by Ike (2023c), Taha (2014), Naidu and Rao (1995), and Anghel and Mares (2019).

Table 1 – Critical buckling load coefficients of simply supported thin beam on two-parameter elastic foundation of Pasternak type

Pasternak foundation parameter $\frac{k_2 l^2}{EI} = \beta_2 l^2$	Winkler foundation parameter $\frac{k_1 l^4}{EI} = \beta_1 l^4$									
	Present study	Soltani & Asgarin (2019)	Present study	Soltani & Asgarin (2019)	Present study	Soltani & Asgarin (2019)	Present study	Soltani & Asgarin (2019)	Present study	Soltani & Asgarin (2019)
	0	0	20	20	40	40	60	60	80	80
0	9.8696	9.8694	11.8960	11.8965	13.9225	13.9236	15.9489	15.9507	17.9753	17.9778
2	11.8696	11.8694	13.8960	13.8965	15.9225	15.9236	17.9489	17.9507	19.9753	19.9778
4	13.8696	13.8694	15.8960	15.8965	17.9225	17.9236	19.9489	19.9507	21.9753	21.9778
6	15.8696	15.8694	17.8960	17.8965	19.9225	19.9236	21.9489	21.9507	23.9753	23.9778
8	17.8696	17.8694	19.8960	19.8965	21.9225	21.9236	23.9489	23.9507	25.9753	25.9778

Table 2 – Critical buckling load coefficients $K(\beta_1 l^4, \beta_2 l^2, n = 1)$ for TBo2PF with simply supported ends

$$K = \left(\frac{\beta_2 l^2}{\pi^2} + \frac{\beta_1 l^4}{\pi^4} + 1\right)\pi^2$$

$\hat{K}_1$	$\hat{K}_2 = 0$					
	Taha (2014)	Anghel and Mares (2019)	Ike (2023a, 2024a, 2024b)	Ike (2023c)	Naidu and Rao (1995)	Present study
0	9.8690	9.8678	9.8696	9.8696	9.8696	9.8696
1			9.9709	9.9709	9.9709	9.9709
50			14.9357	14.9357		14.9357
100	20.0015	20.0000	20.0017	20.0017	20.002	20.0017
1000			111.1908	111.1908		111.1908
10,000			1,023.0814	1,023.0814		1,023.0814
$\hat{K}_1$	$K2$=0.5					
	Taha (2014)	Anghel and Mares (2019)	Ike (2023a, 2024a, 2024b)	Ike (2023c)	Naidu and Rao (1995)	Present study
0					14.804	14.8034
1					14.907	
50						
100					24.937	
1000						
10,000						

$\hat{K}_1$	$K2$=1.0					
	Taha (2014)	**Anghel and Mares (2019)**	**Ike (2023a, 2024a, 2024b)**	**Ike (2023c)**	**Naidu and Rao (1995)**	**Present study**
0	19.7385	19.7376	19.7387	19.7520	19.739	19.7387
1					19.841	
50						
100	29.8706	29.8695	29.8713	29.8827	29.871	29.8713
1000						
10,000						
$\hat{K}_1$	$\hat{K}_2 = 2.50$					
	Taha (2014)	**Anghel and Mares (2019)**	**Ike (2023a, 2024a, 2024b)**	**Ike (2023c)**	**Naidu and Rao (1995)**	**Present study**
0	34.5438	34.5415	34.5436	34.5564	34.544	34.5436
1					34.645	
50						
100	44.6759	44.6732	44.6757	44.6871	44.676	44.6757
1000						
10,000						

4.2. Case 2: Thin beam on 2 parameter foundation (TBo2PF) with clamped-clamped ends

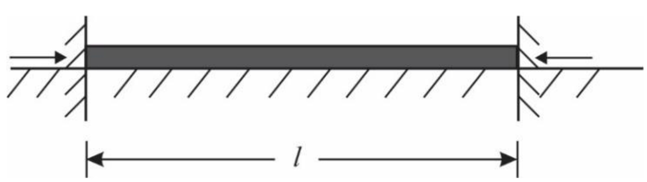

Fig. 5 Buckling of thin beam on 2 parameter foundation with clamped-clamped ends

The BCs of clamped-clamped EBBo2PEF shown in Figure 5 are:

$$\begin{aligned} w(0) &= 0 \\ w'(0) &= 0 \\ w(l) &= 0 \\ w'(l) &= 0 \end{aligned} \tag{51}$$

Differentiating $w(x)$ in Equation (34) with respect to x, gives:

$$w'(x) = -c_1 A \sin Ax + c_2 A \cos Ax - c_3 B \sin Bx + c_4 B \cos Bx \tag{52}$$

The boundary conditions give the homogeneous equations:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & A & 0 & B \\ \cos Al & \sin Al & \cos Bl & \sin Bl \\ -A\sin Al & A\cos Al & -B\sin Bl & B\cos Bl \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (53)$$

Simplifying using, $c_1 = -c_3,\ c_2 = -\dfrac{c_4 B}{A}$ **(54)**

$$\begin{pmatrix} (\cos Bl - \cos Al) & \left(\sin Bl - \dfrac{B}{A}\sin Al\right) \\ (A\sin Al - B\sin Bl) & B(\cos Bl - \cos Al) \end{pmatrix}\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (55)$$

For nontrivial solutions, the characteristic buckling equation is:

$$\begin{vmatrix} (\cos Bl - \cos Al) & \left(\dfrac{A\sin Bl - B\sin Bl}{A}\right) \\ (A\sin Al - B\sin Bl) & B(\cos Bl - \cos Al) \end{vmatrix} = 0 \quad (56)$$

Expanding the determinant gives:

$$B(\cos Bl - \cos Al)^2 - \left(\frac{A\sin Bl - B\sin Bl}{A}\right)(A\sin Al - B\sin Bl) = 0 \quad (57)$$

Further expansion, simplification and use of trigonometric identities yield the characteristic buckling equation as:

$$2AB(\cos Al\cos Bl - 1) + (A^2 + B^2)\sin Al\sin Bl = 0 \quad (58)$$

The eigenvalues are used to calculate critical buckling load coefficients for $\hat{K}_1 = \frac{k_1 l^4}{EI} = 0,\ 1,\ 50,\ 100$ and $\hat{K}_2 = \frac{k_2 l^2}{\pi^2 EI} = 0,\ 0.50,\ 1.0$ and 2.50; and presented in Table 3.
Table 3 also depicts critical buckling load solutions by Rao and Raju (2002), Naidu and Rao (1995), and Ike (2024). Table 3 confirms that the present exact results are identical with previous results that used RVM, and FEM.

Table 3 – Buckling load parameter (coefficients) of TBo2PF with clamped-clamped ends

$\hat{K}_1 = \dfrac{k_1 l^4}{EI}$	$\hat{K}_2 = \dfrac{k_2 l^2}{\pi^2 EI} = 0$			
	Rao and Raju (2002)	**Naidu and Rao (1995)**	**Ike (2024b)**	**Present study**
0	39.478	39.479	39.4784176	39.4784176
1	39.554	39.555	39.55440849	39.55440849
50			43.27796199	43.27796199
100	47.077	47.077	47.07750638	47.07750638

$\hat{K}_1 = \frac{k_1 l^4}{EI}$	$\hat{K}_2 = 0.50$			
	Rao and Raju (2002)	**Naidu and Rao (1995)**	**Ike (2024b)**	**Present study**
0	44.413	44.414	44.4132198	44.4132198
1	44.489	44.490	44.48921069	44.48921069
50			48.21276419	48.21276419
100	52.012	51.942	52.01230858	52.01230858
$\hat{K}_1 = \frac{k_1 l^4}{EI}$	$\hat{K}_2 = 1.0$			
	Rao and Raju (2002)	**Naidu and Rao (1995)**	**Ike (2024b)**	**Present study**
0	49.348	49.349	49.348022	49.348022
1	44.424	44.425	44.424012	44.424012
50			53.14756639	53.14756639
100	56.9471	56.877	56.94711078	56.94711078
$\hat{K}_1 = \frac{k_1 l^4}{EI}$	$\hat{K}_2 = 2.50$			
	Rao and Raju (2002)	**Naidu and Rao (1995)**	**Ike (2024b)**	**Present study**
0	64.152	64.153	64.15242861	64.15242861
1	64.228	64.229	64.22841949	64.22841949
50			67.95197299	67.95197299
100	71.751	71.681	71.75151738	71.75151738

5. Conclusion

This study has investigated the buckling analysis of EBBo2PEF under inplane compressive loads. The differential equation of elastic stability was derived for the problem using the equilibrium method. The GDiES was found to be in general a nonhomogeneous equation with variable parameters when the beam is non-prismatic and transverse distributed loads are present.

For case of prismatic beam and absence of distributed transverse loading, the GDiES becomes a fourth order homogeneous equation in terms of the buckling deflection $w(x)$. The work considered two cases of boundary conditions (BCs), namely: (i) EBBo2PEF with simple supports at $x = 0$, and $x = l$; (ii) EBBo2PEF with clamped supports at $x = 0$ and $x = l$

In conclusion,

(i) the critical buckling load coefficients for $\hat{K}_1 = 0, 1, 50, 100, 1000, 10000$ and $\hat{K}_2 = 0, 0.5, 1, 2.5$ for EBBo2PEF with simply supported ends are found to be identical with previous results for the same foundation beam parameters by Ike (2023a, 2024a, 2024b) and Naidu and Rao (1995), and agrees with solution by Anghel and Mares (2019).

(ii) the expression for buckling load coefficient was found to depend upon the buckling mode number, and the two parameters of the elastic foundation.

(iii) the least buckling load corresponds to the first buckling mode, n = 1, and is the critical buckling load.

(iv) for EBBo2PEFs with clamped-clamped ends, the critical buckling load coefficients for $\hat{K}_1 = 0, 1, 100$; $\hat{K}_2 = 0, 0.5, 1, 2.5$ are identical with previous solutions by Ike (2024b) and agree closely with past solutions by Rao and Raju (2002), and Naidu and Rao (1995).

Notations / Symbol / Nomenclature

x	longitudinal (axial) coordinate
y	coordinate for the direction of width
z	transverse coordinate
P	axial load
$Q(x)$	shear force distribution
$M(x)$	bending moment distribution
r_f	reaction of foundation on the beam
Δ	change in
$\sum$	sum of, summation
$q(x)$	applied transverse load distribution intensity
lim, Lt	limit of
$\rightarrow$	tends to
E	Young's modulus
I	moment of inertia
$u(x)$	displacement in x direction
$w(x)$	transverse displacement in the z direction
$v(x)$	displacement in y direction
k_1, k_2	foundation parameters of the two-parameter elastic foundation
α	parameter defined in terms of P and EI
β_1	parameter defined in terms k_1 and EI
β_2	parameter defined in terms k_2 and EI
$\frac{d}{dx}$	first ordinary differential coefficient with respect to x
$\frac{d^n}{dx^n}$	nth ordinary differential coefficient with respect to x
H	amplitude of trial function
s	unknown parameter of the exponential trial function
A, and B	eigenvalues defined in terms of α, β_1 and β_2
i	imaginary number / complex number
$\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$	constants of integration
c_1, c_2, c_3, c_4	modified constants of integration
l	span of beam, length of beam
n	buckling mode number
$K(\beta_1 l^4, \beta_2 l^2, n)$	buckling load coefficient (parameter)
$\hat{K}_1$	modified parameter defined in terms of k_1, l and EI
$\hat{K}_2$	modified parameter defined in terms of k_2, l and EI
BCs	boundary condition(s)
GDES	governing domain equation of stability
GDiES	governing differential equation of stability
EBT	Euler-Bernoulli theory
FG	functionally graded
FGM	functionally graded material
RDM	recursive differentiation method
FEM	finite element method
FSTM	finite sine transformation method
PCM	point collocation method
SVIM	Stodola-Vianello iteration method
GITM	generalized integral transformation method
EBBoWF	Euler-Bernoulli beam on Winkler foundation
EBBoPF	Euler-Bernoulli beam on Pasternak foundation
EBBo2PEFs	Euler-Bernoulli beam on two parameter elastic foundations

PSIM Picard's successive iteration methodology

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