STUDY OF A NONLINEAR MODEL FOR TWO PREY AND TWO PREDATORS SPECIES

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Abstract

In this work we propose and analyze a model, where prey species is supposed to live in two distinct habitats with group defense. One of the predators tends to switch between the habitats. The boundedness of the positive solutions and local stability for all possible equilibrium points of the model are studied for the case, where the switching indicator is $n = 1$, furthermore, a suitable Lyapunov function has been defined to construct a basin of attraction for the interior equilibrium point. Numerical simulations are used to support the analytical results, such that some examples of locally stable and unstable equilibrium points furthermore stable limit cycle will be given.

Key words: Prey, Predator, Switching, Group defense.

1. INTRODUCTION:

Predators tend to feed themselves in a habitat for some time and then migrate to another habitat. This phenomenon of migrating from one habitat (A) to another habitat (B), is called the phenomenon of switching. This phenomenon has many causes, including the small number of prey in the habitat (A), or the small size of prey and the inability to defend itself in the new habitat (B). Predator prefers to catch a prey in a habitat in which there is a large number of prey. There are many examples that predators prefer the habitat, where a large number of prey species live all the time ([3], [6]). Many mathematical models have been studied in the prey-predator field, involving one or two predators with two prey species. ([4], [7], [5]).

In [2], Bhattacharyya and Mukhopadhyay proposed and studied two models of prey–predator, only one of the two models involve prey group defense. The prey species are supposed to live in two distinct habitats with group defense and the predator species tends to switch between the habitats.

In this paper, a model of two prey and two predators will be proposed and studied. The model involves prey group defense and the two prey live in two distinct habitats, and only one of two predators tends to switch between the habitats. The models are studied for n = 1 of the switching index.

2. THE MATHEMATICAL MODEL:

According to the system bellow [2]:

$$\begin{cases} \dot{x}_1 = x_1\left[g_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1 x_2^n y_1}{x_1^n+x_2^n}\right], \\ \dot{x}_2 = x_2\left[g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\alpha_1 x_1^n y_1}{x_1^n+x_2^n}\right], \\ \dot{y}_1 = y_1\left[-\mu_1+\frac{\delta_1 x_1 x_2^n}{x_1^n+x_2^n}+\frac{\delta_2 x_1^n x_2}{x_1^n+x_2^n}\right], \end{cases} \qquad (2.1)$$

We have proposed a new system as follows:

$$\begin{cases} \dot{x}_1 = x_1\left[g_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1 x_2^n y_1}{x_1^n+x_2^n} -\beta y_2\right], \\ \dot{x}_2 = x_2\left[g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\alpha_1 x_1^n y_1}{x_1^n+x_2^n}\right], \\ \dot{y}_1 = y_1\left[-\mu_1+\frac{\delta_1 x_1 x_2^n}{x_1^n+x_2^n}+\frac{\delta_2 x_1^n x_2}{x_1^n+x_2^n}\right], \\ \dot{y}_2 = y_2[-\mu_2+\gamma x_1], \end{cases} \qquad (2.2)$$

Where all parameters are positive, $x_i, i = 1,2,$ $(y_i, i = 1,2,)$ denote prey density in two habitats, (predator density). The prey population is assumed to grow logistically with a specific growth rate $g_i, i = 1,2$ and environmental carrying capacity $k_i, i = 1,2$. α_1 and α_2 represent the predation rate towards the prey x_1 and x_2 respectively by y_1, β represent the predation rate towards the prey x_1 by y_2, δ_1, δ_2 and γ are the corresponding conversion rates. The predation functions $\alpha_1 x_2^n y_1 (x_1^n + x_2^n)^{-1}$ and $\alpha_1 x_1^n y_1 (x_1^n + x_2^n)^{-1}$ model the switching behavior of the predator y_1 in the realm of prey group defense. And the namely that, there will be less predation by y_1 in the habitat has larger prey density. $\mu_i, i = 1,2,$ the per capita death rate of predator $y_i, i = 1,2$.

For $n = 1$ the system (2.2) becomes:

$$\begin{cases} \dot{x}_1 = x_1\left[g_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1 x_2 y_1}{x_1+x_2}-\beta y_2\right],\\ \dot{x}_2 = x_2\left[g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\alpha_2 x_1 y_1}{x_1+x_2}\right],\\ \dot{y}_1 = y_1\left[-\mu_1+\frac{\delta x_1 x_2}{x_1+x_2}\right],\\ \dot{y}_2 = y_2[-\mu_2+\gamma x_1], \end{cases} \qquad (2.3)$$

where

$\delta = \delta_1 + \delta_2,$

3. THE BOUNDEDNESS OF POSITIVE SOLUTIONS

Set $D \coloneqq \{(x_1, x_2, y_1, y_2) \in \mathcal{R}^4, x_i \in (0, k_i), y_i > 0, i = 1,2\}$. (3.1)

Lemma1: If $\delta \le \alpha_1 + \alpha_2, \gamma \le \beta$, then all the trajectories of the system (2.3) start in D are bounded.

Proof: Consider the function, defined on D as follow:

$u(t) = x_1(t) + x_2(t) + y_1(t) + y_2(t),$

then from (2.2), we have:

$$\dot{u} = x_1 g_1\left(1-\frac{x_1}{k_1}\right) + x_2 g_2\left(1-\frac{x_2}{k_2}\right) - (y_1\mu_1 + y_2\mu_2) + (\gamma-\beta)x_1 y_2 + \frac{(\delta-\alpha_1-\alpha_2)x_1 x_2 y_1}{x_1+x_2}.$$

Let ρ be a positive constant, such that $\rho \le max\{\mu_1, \mu_2\}$, we have:

$$\dot{u} + \rho u = x_1 g_1\left(1-\frac{x_1}{k_1}+\frac{\rho}{g_1}\right) + x_2 g_2\left(1-\frac{x_2}{k_2}+\frac{\rho}{g_2}\right) + (y_1(\rho-\mu_1) + y_2(\rho-\mu_2)) + \frac{x_1 x_2}{x_1+x_2}[(\delta-\alpha_1-\alpha_2)y_1 + (\gamma-\beta)y_2].$$

It is clear that

$$\dot{u} + \rho u \le \frac{x_1}{k_1}(k_1 g_1 - g_1 x_1 + k_1\rho) + \frac{x_2}{k_2}(k_2 g_2 - g_2 x_2 + k_2\rho)$$
$$< \frac{x_1}{k_1}(k_1 g_1 + k_1\rho) + \frac{x_2}{k_2}(k_2 g_2 + k_2\rho) < k_1(g_1+\rho) + k_2(g_2+\rho).$$

Then, $0 \le u(t) \le \frac{\alpha}{\rho} + u(0)e^{-\rho t}$, and for t→∞, $0 \le u(t) \le \frac{\alpha}{\rho}$, where $\alpha = k_1(g_1+\rho) + k_2(g_2+\rho)$.

Hence, we obtain that all the positive solutions of the system (2.3) with initial conditions $(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2) \in D$, and satisfy $\delta \le \alpha_1 + \alpha_2, \gamma \le \beta$ are bounded. So that the proof is complete.

4. STABILITY OF EQUILIBRIA:

The system (2.3) has at least four non negative equilibrium points. In this section we shall study the local stability for these points of our system.

1. The equilibrium point $E_1 = (0,0,0,0)$ of the systems (2.3) is always exists. The following Lemm2 illustrates that E_1 is not stable.

Lemma2: If $x_1(0) > 0$ or $x_2(0) > 0$, then no trajectory of the system (2.3) converge to (0,0,0,0).

Proof: Suppose that $x_1(0) > 0$ and $(x_1, x_2, y_1, y_2) \to (0,0,0,0)$, as $t \to \infty$, then $\frac{d}{dt}(\ln x_1) \to g_1$. It is clear that $\frac{d}{dt}(\ln x_1) \geq \frac{g_1}{2}$. If $(x_1, x_2, y_1, y_2) \to (0,0,0,0)$ as $t \to \infty$, then there exist $t_0 > 0$, such that $x_1(t_0) > 0$, $x_1(t) \geq x_1(t_0) exp\left(\frac{g_1(t-t_0)}{2}\right) \to \infty$, as $t \to \infty$. So $x_1 \to \infty$. Similarly if $x_1(0) > 0, x_1 \to \infty$. Which means there is no trajectory of the system (2.3) converges to(0,0,0,0).

Hence, the equilibrium point (0,0,0,0) is unstable. So that the proof is complete.

2. The equilibrium points. $E_2 = (k_1, 0,0,0)$ and $E_3 = (0, k_2, 0,0)$ of the system (2.3) are always exists. The set of eigenvalues of the characteristic equations of the variation matrix of the system (2.3) at the equilibrium points E_2 and E_3 are:

$S(E_2) = \{-g_1, g_2, -\mu_1, \gamma k_2 - \mu_2\}$ and $S(E_3) = \{g_1, -g_2, -\mu_1, -\mu_2\}$ respectively. Note that $S(E_i), i = 2,3,$ has at least one positive eigenvalue and at least two negative eigenvalues, which mean that the two equilibrium points are saddle points (unstable).

3. The equilibrium point $E_4 = (k_1, k_2, 0,0)$ always exists. The set of eigenvalues of the characteristic equations of the variation matrix of the system (2.3) at the equilibrium points E_4 is:

$$S(E_4) = \left\{-g_1, -g_2, \frac{\delta k_1 k_2}{k_1 + k_2} - \mu_1, \gamma k_1 - \mu_2\right\}.$$

Hence, the equilibrium $(k_1, k_2, 0,0)$ will be locally asymptotically stable if and only if it satisfies the following conditions:

$$\begin{cases} k_1 k_2 \delta < (k_1 + k_2)\mu_1 \\ k_1 \gamma < \mu_2 \end{cases} \qquad (4.1)$$

4. The existence of the point of equilibrium $E_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$, where:

$$\begin{cases} \check{x}_1 = \frac{\mu_2}{\gamma}, \check{x}_2 = k_2, \\ \check{y}_2 = \frac{g_1}{\beta}\left(1 - \frac{x_1}{k_1}\right), \end{cases} \qquad (4.2)$$

depends on the parameters of the system (2.3).

The characteristic equation of the Jacobean matrix of (2.3) near the point $E_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$ is:

$\lambda^4 + \check{E}_1 \lambda^3 + \check{E}_2 \lambda^2 + \check{E}_3 \lambda + \check{E}_4 = 0$, where:

$\check{E}_1 = \frac{g_1 \check{x}_1}{k_1} + g_2 + \mu_1 - \delta D,$

$\check{E}_2 = (\mu_1 - \delta D)\left(\frac{g_1\check{x}_1}{k_1} + g_2\right) + \beta\check{x}_1\gamma\check{y}_2 + \frac{g_1 g_2\check{x}_1}{k_1}$

$\check{E}_3 = (\mu_1 - \delta D)\frac{g_1 g_2\check{x}_1}{k_1} + \beta\check{x}_1\gamma\check{y}_2(g_2 + \mu_1 - \delta D),$

$\check{E}_4 = g_2\beta\check{x}_1\gamma\check{y}_2(\mu_1 - \delta D).$

and $D = k_2\mu_2(\mu_2 + \gamma k_2)^{-1}$.

If $E_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$, exists, then necessary and sufficient conditions for $E_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$, to be locally asymptotically stable according to the criteria of Routh-Hurwitz [1], of the forth system (2.3) are:

$$\begin{cases} \check{E}_i > 0,\ i = 1,2,3,4 \\ \check{E}_1\check{E}_2 - \check{E}_3 > 0, \\ \check{E}_3(\check{E}_1\check{E}_2 - \check{E}_3) - \check{E}_4\check{E}_1^2 > 0. \end{cases} \quad (4.3)$$

It is clear that, if $\mu_1 < \delta D$, then $\check{E}_4 < 0$, and hens $\mu_1 < \delta D$ is a necessary condition for $E_5 = (\check{x}_1, \check{x}_2, 0, \check{y}_2)$ to be locally asymptotically stable.

5. The existence of the point of equilibrium $E_6 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$, depends on the parameters of the system (2.3), where:

$$\begin{cases} \tilde{x}_1 = \frac{\mu_1(1+\tilde{x})}{\delta}, \qquad \tilde{x}_2 = \frac{\tilde{x}_1}{\tilde{x}} \\ \tilde{y}_1 = \frac{g_1}{\alpha_1}\left(1 - \frac{\tilde{x}_1}{k_1}\right)(1+\tilde{x}) = \frac{g_2}{\alpha_2}\left(1 - \frac{\tilde{x}_2}{k_2}\right)\frac{(1+\tilde{x})}{\tilde{x}} \end{cases} \quad (4.4a)$$

where $\tilde{x}$ is the real positive roots of the equation:

$$\pi_1\tilde{x}^3 + \pi_2\tilde{x}^2 + \pi_3\tilde{x} + \pi_4 = 0, \quad (4.4b)$$

such that:

$\pi_1 = g_1 k_2\alpha_2\mu_1, \qquad \pi_2 = g_1 k_2\alpha_2(\mu_1 - k_1\delta),$

$\pi_3 = g_2 k_1\alpha_1(k_2\delta - \mu_1), \qquad \pi_4 = -g_2\alpha_1 k_1\mu_1.$

The characteristic equation of the Jacobian matrix of (2.3) near the point $E_6 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$ is:

$\lambda^4 + \tilde{E}_1\lambda^3 + \tilde{E}_2\lambda^2 + \tilde{E}_3\lambda + \tilde{E}_4 = 0,$

such that:

$$\tilde{E}_1 = \sum_{i=1}^{2}\left(\mu_i + \frac{g_i\tilde{x}_i}{k_i} - \frac{\tilde{x}_1\tilde{x}_2\tilde{y}_1\alpha_i}{(\tilde{x}_1 + \tilde{x}_2)^2} - \frac{\gamma\tilde{x}_1 + \delta A\tilde{x}_1\tilde{x}_2}{2}\right)$$

$$\tilde{E}_2 = (\mu_2 - \gamma\bar{x}_1)(\tilde{E}_1 - \mu_2 + \gamma\bar{x}_1) + \frac{\tilde{E}_3}{(\mu_2 - \gamma\tilde{x}_1)} - \frac{\tilde{E}_4}{(\mu_2 - \gamma\tilde{x}_1)^2}$$

$$\tilde{E}_3=(\mu_2-\gamma\tilde{x}_1)\sum_{i=1}^{2}\left[\frac{\tilde{x}_1\tilde{x}_2\tilde{y}_1\alpha_{i-(-1)^i}}{(\tilde{x}_1+\tilde{x}_2)^2}\left(\frac{\delta\tilde{x}_i^2}{(\tilde{x}_1+\tilde{x}_2)}-\frac{g_i\tilde{x}_i}{k_i}\right)+\frac{g_1g_2\tilde{x}_1\tilde{x}_2}{2k_1k_2}+(\delta A\tilde{x}_1\tilde{x}_2-\mu_1)\left(\frac{\tilde{x}_1\tilde{x}_2\tilde{y}_1\alpha_i}{(\tilde{x}_1+\tilde{x}_2)^2}-\frac{g_i\tilde{x}_i}{k_i}\right)\right]+\frac{\tilde{E}_4}{(\mu_2-\gamma\tilde{x}_1)},$$

$$\tilde{E}_4=\frac{(\mu_2-\gamma\tilde{x}_1)\tilde{x}_1\tilde{x}_2\tilde{y}_1}{(\tilde{x}_1+\tilde{x}_2)^2}\sum_{i=1}^{2}\left[\frac{g_i\tilde{x}_i\alpha_{i-(-1)^i}}{k_i}\left(\delta A\tilde{x}_i^2+\delta A\tilde{x}_1\tilde{x}_2-\mu_1\right)-\frac{g_1g_2(\delta A\tilde{x}_1\tilde{x}_2-\mu_1)}{2A^2k_1k_2\tilde{y}_1}-\frac{\alpha_1\alpha_2\delta\tilde{x}_1\tilde{x}_2\tilde{y}_1}{2(\tilde{x}_1+\tilde{x}_2)}\right],$$

and $A=(\tilde{x}_1+\tilde{x}_2)^{-1}$.

If $E_6=(\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,0)$ exists then the necessary and sufficient conditions for $E_6=(\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,0)$, to be locally asymptotically stable according to the criteria of Routh-Hurwitz [1], of the forth system (2.3) are:

$$\begin{cases}\tilde{E}_i>0,\ i=1,2,3,4\\ \tilde{E}_1\tilde{E}_2-\tilde{E}_3>0,\\ \tilde{E}_3\left(\tilde{E}_1\tilde{E}_2-\tilde{E}_3\right)-\tilde{E}_4\tilde{E}_1^2>0.\end{cases}\qquad(4.5)$$

6. The interior equilibrium point

The existence of the point of equilibrium $E_7=(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$, depends on the parameters of the system (2.3). where:

$$\begin{cases}\bar{x}_1=\dfrac{\mu_2}{\gamma}, & \bar{y}_1=\dfrac{g_2(k_2-\bar{x}_2)(\bar{x}_1+\bar{x}_2)}{\alpha_2k_2\bar{x}_1}\\ \bar{x}_2=\dfrac{\mu_1\bar{x}_1}{\delta\bar{x}_1-\mu_1}, & \bar{y}_2=\dfrac{g_1}{\beta k_1}(k_1-\bar{x}_1)-\dfrac{\alpha_2\bar{x}_2\bar{y}_1}{\beta(\bar{x}_1+\bar{x}_2)}.\end{cases}\qquad(4.6)$$

The characteristic equations of the Jacobian matrix of (2.3) near the point $E_7=(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$ is:

$$\lambda^4+\bar{E}_1\lambda^3+\bar{E}_2\lambda^2+\bar{E}_3\lambda+\bar{E}_4=0,$$

$$\bar{E}_1=\sum_{i=1}^{2}\left[\frac{g_i\bar{x}_i}{k_i}-\frac{\bar{x}_1\bar{x}_2\bar{y}_1\alpha_i}{(\bar{x}_1+\bar{x}_2)^2}+\frac{(\mu_1-A\delta\bar{x}_1\bar{x}_2)}{2}\right],$$

$$\bar{E}_2=\frac{g_1g_2\bar{x}_1\bar{x}_2}{k_1k_2}+\beta\gamma\bar{x}_1\bar{y}_2+\sum_{i=1}^{2}\left[\frac{g_i\bar{x}_i}{k_i}\left(\frac{(A\mu_1-\delta\bar{x}_1\bar{x}_2)}{\bar{x}_1+\bar{x}_2}\right)+\frac{\bar{x}_1\bar{x}_2\bar{y}_1\alpha_{i-(-1)^i}}{(\bar{x}_1+\bar{x}_2)^2}\left(\frac{\delta\bar{x}_i^2+\delta\bar{x}_1\bar{x}_2-A\mu_1}{(\bar{x}_1+\bar{x}_2)}-\frac{g_i\bar{x}_i}{k_i}\right)\right],$$

$$\bar{E}_3=\beta\gamma\bar{x}_1\bar{y}_2\left(\frac{g_2\bar{x}_2}{k_2}-\frac{\bar{x}_1\bar{x}_2\bar{y}_1\alpha_2}{(\bar{x}_1+\bar{x}_2)^2}-\frac{\delta\bar{x}_1\bar{x}_2-A\mu_1}{\bar{x}_1+\bar{x}_2}\right)-\frac{g_1g_2\bar{x}_1\bar{x}_2}{k_1k_2}\left(\frac{\delta\bar{x}_1\bar{x}_2-A\mu_1}{\bar{x}_1+\bar{x}_2}\right)$$

$$+\sum_{i=1}^{2}\left[\frac{\bar{x}_1\bar{x}_2\bar{y}_1 g_i\bar{x}_i\alpha_{i-(-1)^i}}{(\bar{x}_1+\bar{x}_2)^3k_i}(\delta\bar{x}_i^2+\delta\bar{x}_1\bar{x}_2-A\mu_1)\right],$$

$$\bar{E}_4=\beta\gamma\bar{x}_1\bar{y}_2\left[\frac{\delta\alpha_2\bar{x}_1^3\bar{x}_2\bar{y}_1}{(\bar{x}_1+\bar{x}_2)^3}+\left(\frac{\delta\bar{x}_1\bar{x}_2}{(\bar{x}_1+\bar{x}_2)}-\mu_1\right)\left(\frac{\alpha_2\bar{x}_1\bar{x}_2\bar{y}_1}{(\bar{x}_1+\bar{x}_2)^2}-\frac{g_2\bar{x}_2}{k_2}\right)\right]$$

and $A=(\bar{x}_1+\bar{x}_2)^{-1}$.

If $E_7=(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$ exists then the necessary and sufficient conditions for $E_7=(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$, to be locally asymptotically stable the criteria of Routh-Hurwitz [1] of the forth system (2.3) are:

$$\begin{cases}\bar{E}_i>0,\ i=1,2,3,4\\ \bar{E}_1\bar{E}_2-\bar{E}_3>0, \\ \bar{E}_3(\bar{E}_1\bar{E}_2-\bar{E}_3)-\bar{E}_3\bar{E}_1^2>0.\end{cases}\qquad(4.7)$$

Now we will define a Lyapunov function appropriately to determine the basin attractions for $E_7=(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$.

It is easy to see that:

i) $$\min_{(x_1,x_2)\in S}\left(\frac{\delta x_1x_2}{x_1+x_2}-\mu_1\right)=\left(\frac{\delta\,\bar{x}_1\bar{x}_2}{\bar{x}_1+\bar{x}_2}-\mu_1\right).\qquad(4.8)$$

ii) $$\min_{(x_1,x_2)\in S}(\gamma x_1-\mu_1)=(\gamma\,\bar{x}_1-\mu_1),\qquad(4.9)$$

where, $S=\{(x_1,x_2):\bar{x}_i\le x_i, i=1,2\}$

Theorem: Assume that the equilibrium point E_7 of (2.3) is locally asymptotically stable and $\bar{x}_i\ge k_i$, then the basin of attraction of E_7 is determined by the set:

$$B=\{(x_1,x_2,y_1,y_2):x_i\ge\bar{x}_i,y_i\le\bar{y}_2,i=1,2\}.$$

Proof: The function

$$V=\sum_{i=1}^{2}\left(x_i-\bar{x}_i-\bar{x}_i\ln\frac{x_i}{\bar{x}_i}+y_i-\bar{y}_i-\bar{y}_i\ln\frac{y_i}{\bar{y}_i}\right)$$

is positive definite .

$$\dot{V}=\sum_{i=1}^{2}\left(\dot{x}_i\left(1-\frac{k_1}{x_i}\right)+\dot{y}_i\left(1-\frac{y_i}{\bar{y}_i}\right)\right)$$

$$=(x_1-\bar{x}_1)G_1+(x_1-\bar{x}_1)G_2+(y_1-\bar{y}_1)G_3(x_1)+(y_1-\bar{y}_1)G_4(x_1,x_2).$$

Such that:

$$G_1=\left[g_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1x_2y_1}{x_1+x_2}-\beta y_2\right],$$

$$G_2=\left[g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\alpha_2x_1y_1}{x_1+x_2}\right],$$

$G_3(x_1,x_2) = \left[-\mu_1 + \frac{\delta x_1 x_2}{x_1+x_2}\right]$ and $G_4(x_1,x_2) = [-\mu_2 + \gamma x_1]$.
It is clear that $G_i < 0, (x_i - \bar{x}_i) > 0, i = 1,2$, so that
$(x_1 - \bar{x}_1)G_1 + (x_1 - \bar{x}_1)G_2 < 0$.
From (4.8) and (4.9) we have:
$0 = G_3(\bar{x}_1,\bar{x}_2) < G_3(x_1,x_2), \quad \forall x_1 > \bar{x}_1, x_2 > \bar{x}_2,$
$0 = G_4(\bar{x}_1,\bar{x}_2) < G_4(x_1,x_2), \quad \forall x_1 > \bar{x}_1, x_2 > \bar{x}_2,$
Since
$(y_1 - \bar{y}_1) < 0, (y_2 - \bar{y}_2) < 0, \forall, y_1 > \bar{y}_1, y_2 > \bar{y}_2,$
We have:

$$(y_1 - \bar{y}_1)G_3(x_1) + (y_1 - \bar{y}_1)G_4(x_1,x_2) < 0, \forall, y_1 > \bar{y}_1, y_2 > \bar{y}_2.$$

So that
$\dot{V} < 0, \forall (x_1,x_2,y_1,y_2) \in B \setminus \{(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)\}$,
and $\dot{V} = 0, \forall (x_1,x_2,y_1,y_2)\ at \{(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)\}$.
So that any trajectory with initial condition $(x_{01},x_{02},y_{01},y_{02}) \in B$ converge asymptotically to $(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$, which means that B is a basin of attraction of $E_7(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$. Thus the proof is complete.

5. NUMERICAL SIMULATIONS:

For the set of parameter values given in following Table:

Table (1)

k_1	k_2	g_1	g_2	μ_1	μ_2	δ	γ	α_1	α_2	β
2.8	2.4	1.5	2	1	0.25	0.6	0.1	0.2	0.5	0.1

the system (2.3) , has E_1, E_2, E_3, E_4 and E_5 as non negative equilibrium points, such that E_1, E_2, E_3, and E_4 are unstable equilibrium points, while $E_5 = (2.6, 2.4, 0, 1.6071)$ is locally asymptotically stable, as shown in Figure (1). However for $\mu_1 = 0.2$, with the rest parameter values given in Table (1), the system (2.3) has E_1, E_2, E_3, E_4, $E_5 = (2.6, 2.4, 0, 1.6071)$, $E_6 = (2.6167, 0.3820, 3.8543, 0)$ and $E_7 = (2.5, 0.3846, 3.8757, 0.5736)$ as non negative equilibrium points, such that E_1, E_2, E_3, E_4, E_5 and E_6 are unstable equilibrium points, while E_7 is locally asymptotically stable, as shown in Figure (2) and Figure (3). We used the point $(2.2, 2.1, 0.8, 1.8)$ as an initial point in a neighborhood of E_5 for the cases, $\mu_1 = 1$ and $\mu_1 = 0.2$. In the first case, the trajectory converges to E_5, see Figure (1) and, in the second case, the trajectory diverges from E_5, and converges to E_7, see Figure (2).

For the parameters given in:

Table (2)

k_1	k_2	g_1	g_2	μ_1	μ_2	δ	γ	α_1	α_2	β
2	2.4	1.5	2	0.8	0.25	0.6	0.1	0.2	0.5	0.1

the system (2.3) , has only four non negative equilibrium points, E_1, E_2, E_3, and E_4 such that E_1, E_2, and E_3 are unstable equilibrium points, while $E_4 = (2, 2.4, 0, 0)$ is locally asymptotically stable, as shown in Figure (4).

Finely for the parameters given in:

Table (3)

k_1	k_2	g_1	g_2	μ_1	μ_2	δ	γ	α_1	α_2	β
2.4	2.4	1.5	1.5	1	0.25	2	0.1	0.5	0.5	0.1

the system (2.3) , has four unstable non negative equilibrium points, E_1, E_2, E_3, E_4 and has three equilibrium points of the type E_6. The equation $(4.4b)$ has three real positive roots, $\tilde{x} = 1$, with the equilibrium point $E_{61} = (1,1,3.5,0)$, which is unstable, $\tilde{x} = 2.3798$ with the equilibrium point $E_{62} = (1.6899,0.7101,3,0)$ and $\tilde{x} = 0.4202$ with the equilibrium point $E_{63} = (0.7101,1.6899,3,0)$.

The system (2.3) has two stable limit cycles, one of them around the equilibrium point E_{62} and the other around E_{63}. Figure (5a-c) for $E_{62} = (1.6899,0.7101,3,0)$ with initial point (1.0999,0.999,3.51,0.01), the trajectory converges to a stable limit cycle, but with initial point $(1.689,0.71,3,0)$, which it close to E_{62} the trajectory diverges from E_{62} and converges to a the stable limit cycle that mentioned above, see Figure (6a-c). That mean there is a stable limit cycle around E_{62}. Similarly we can show that, there is stable limit cycle around E_{63}.

Conclusion

In this paper, a new model for two prey and two predators species has been proposed. The prey species is supposed to live in two distinct habitats and have the ability of prey group defense. One of the two predators tends to exchange the habitats.

We have found that, all the trajectories of the positive solutions of the system (2.3) are bounded under the condition (3.1). The system (2.3) has at least four nonegetive equilibrium points, The equilibrium points $E_1 = (0,0,0,0)$, $E_2 = (\mathrm{k}_1, 0,0,0)$, $\mathrm{E}_3 = (0, \mathrm{k}_2, 0,0)$, and $E_4 = (k_1, k_2, 0,0)$ always exist. E_4 is locally asymptotically stable if it satisfies the conditions (4.1), while E_1, E_2 and E_3 are not stables.

If any one of the rest equilibrium points exists, then it is locally asymptotically stabile under the conditions showed in section (4). Furthermore, a basin of attraction for $\mathrm{E}_7 = (\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2, \bar{\mathrm{y}}_1, \bar{\mathrm{y}}_2)$ has been constructed using a Lyapunov function.

Some examples of locally stable, unstable equilibrium points and stable limit cycle have been given using numerical simulation. Our model with $n > 1$ will be analyzed in a future work.

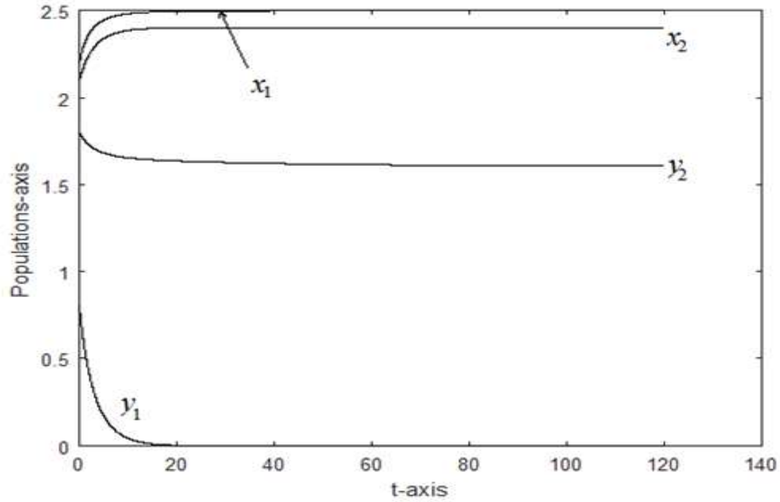

Figure (1) With the parameters given in Table (1), the system (2.3) has $E_5 = (2.6, 2.4, 0, 1.6071)$ as an equilibrium point which is locally asymptotically stable.

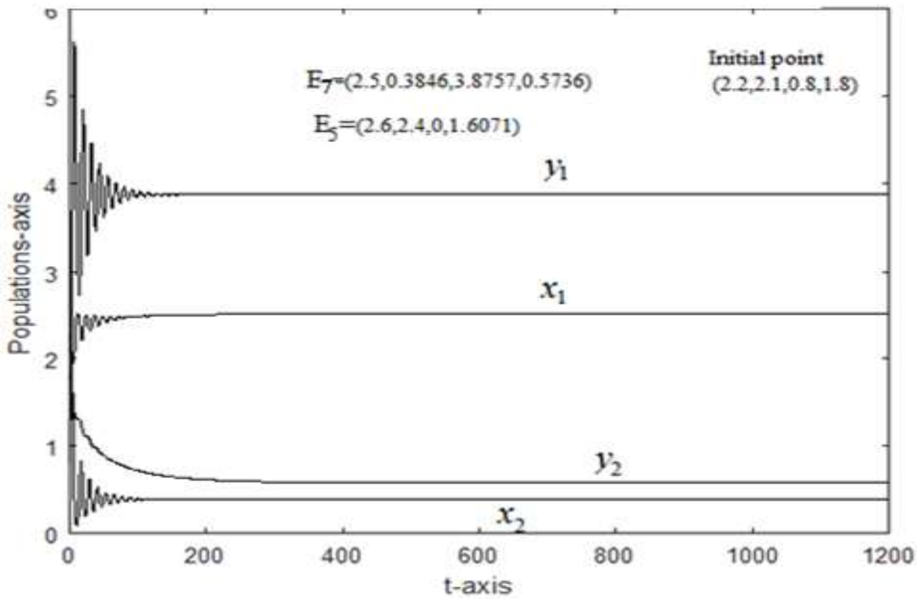

Figure (2) For $\mu_1 = 0.2$, with the rest of the parameters given in Table (1), the system (2.3) has $E_7 = (2.5, 0.3846, 3.8757, 0.5736)$ as an equilibrium point Which is locally asymptotically stable. Here the trajectory starts in the same initial point of Figure (1). The trajectory diverges from $E_5 = (2.6, 2.4, 0, 1.6071)$.

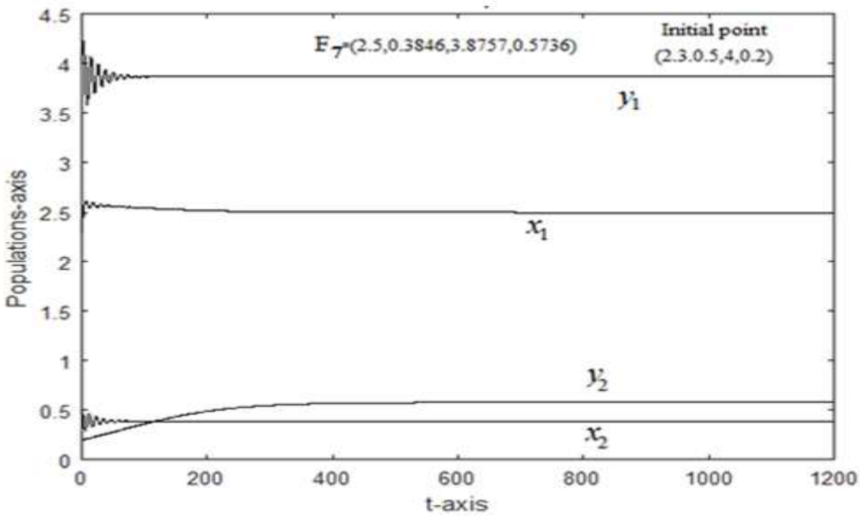

Figure (3) For $\mu_1 = 0.2$, with the rest of the parameters given in Table (1), the system (2.3) has $E_7 = (2.5, 0.3846, 3.8757, 0.5736)$ as an equilibrium point Which is locally asymptotically stable.

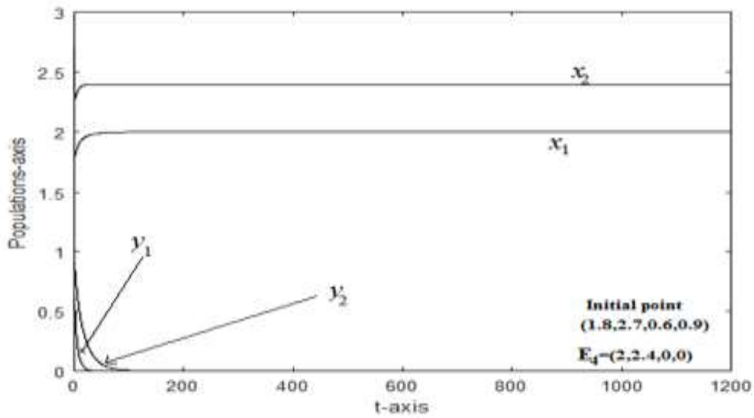

Figure (4) With the parameters given in Table (2), the system (2.3) has $E_4 = (2, 2.4, 0, 0)$ as an equilibrium point which is locally asymptotically stable.

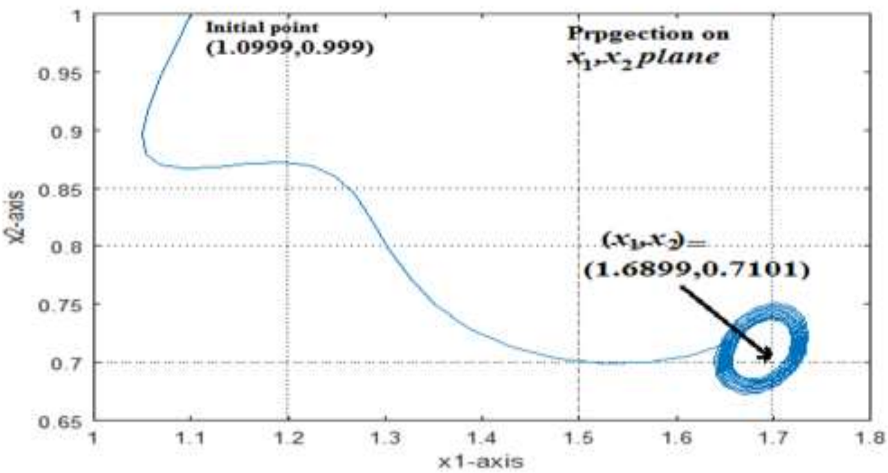

Figure (5a) With initial point (1,0999,0.999,3.51,0.01), the trajectory converges to a stable limit cycle around $E_{62} = (1.6899, 0.7101, 3, 0)$, where E_{62} is as an equilibrium point of the system (2.3) with the parameters given in Table (3). Projection on $x_1 x_2 plane$.

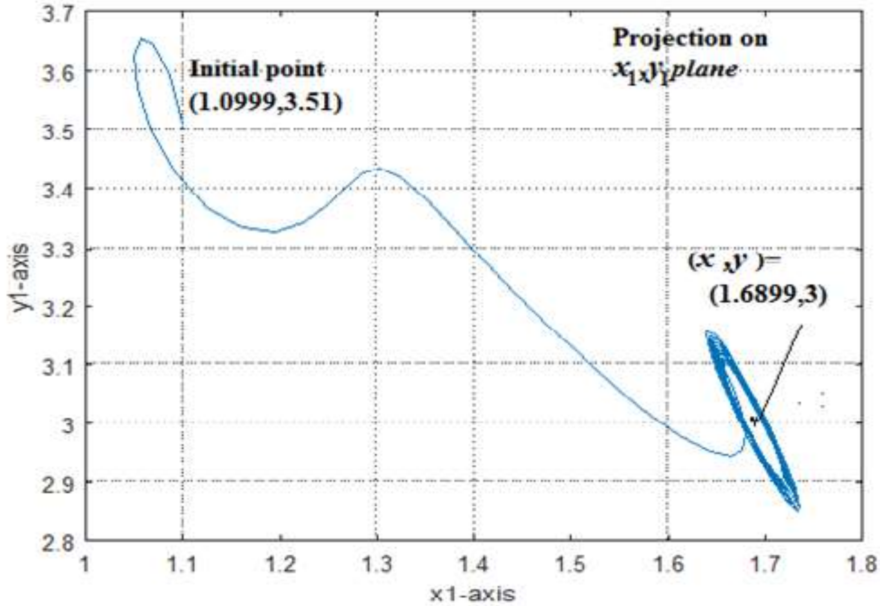

Figure (5b) With initial point (1.0999,0.999,3.51,0.01), the trajectory converges to a stable limit cycle around $E_{62} = (1.6899, 0.7101, 3, 0)$, where E_{62} is as an equilibrium point of the system (2.3) with the parameters given in Table (3). Projection on x_1y_1plane.

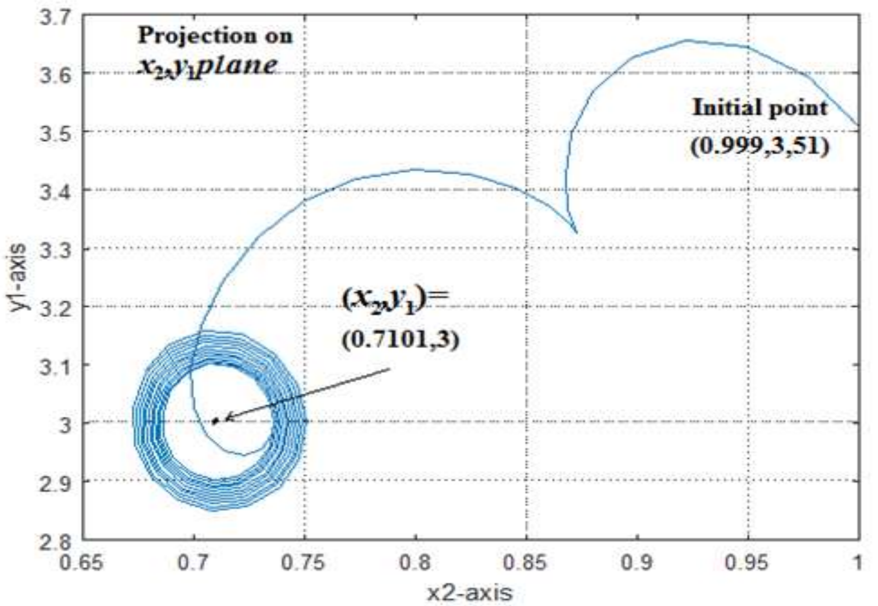

Figure (5c) With initial point (1.0999,0.999,3.51,0.01), the trajectory converges to a stable limit cycle around $E_{62} = (1.6899, 0.7101, 3, 0)$, where E_{62} is as an equilibrium point of the system (2.3) with the parameters given in Table (3). Projection on x_2y_1plane.

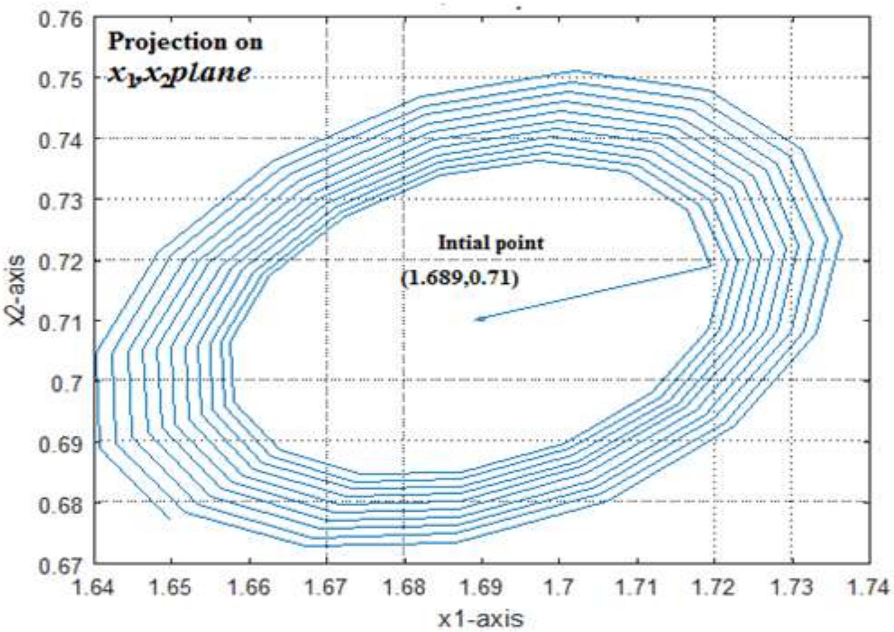

Figure (6a) With initial point (1.689,0.7101,3,0), the trajectory converges to a stable limit cycle around $E_{62} = (1.6899,0.7101,3,0)$, where E_{62} is as an equilibrium point of the system (2.3) with the parameters given in Table (3). Projection on $x_1 x_2 plane$.

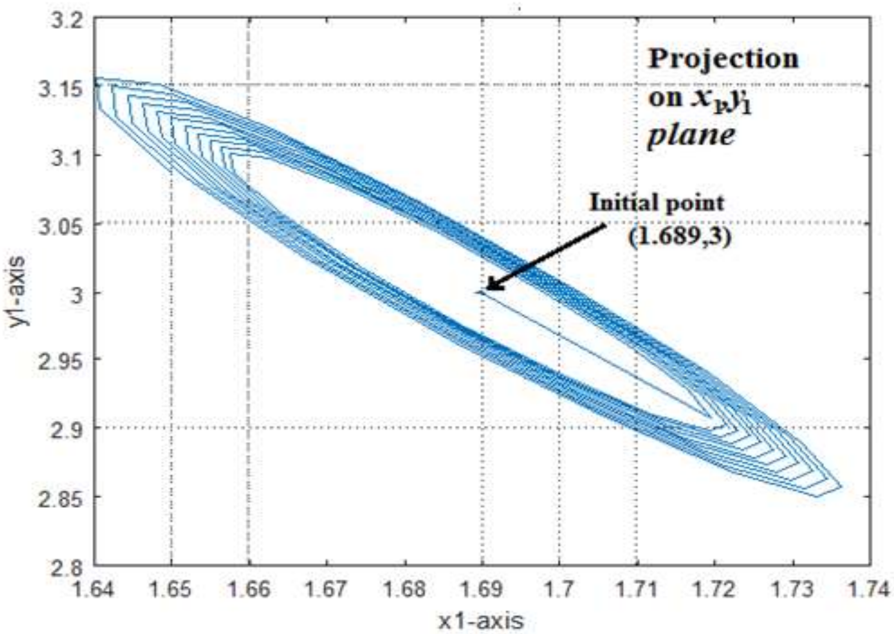

Figure (6b) With initial point (1.689,0.7101,3,0), the trajectory converges to a stable limit cycle around $E_{62} = (1.6899,0.7101,3,0)$, where E_{62} is as an equilibrium point of the system (2.3) with the parameters given in Table (3). Projection on $x_1 y_1 plane$.

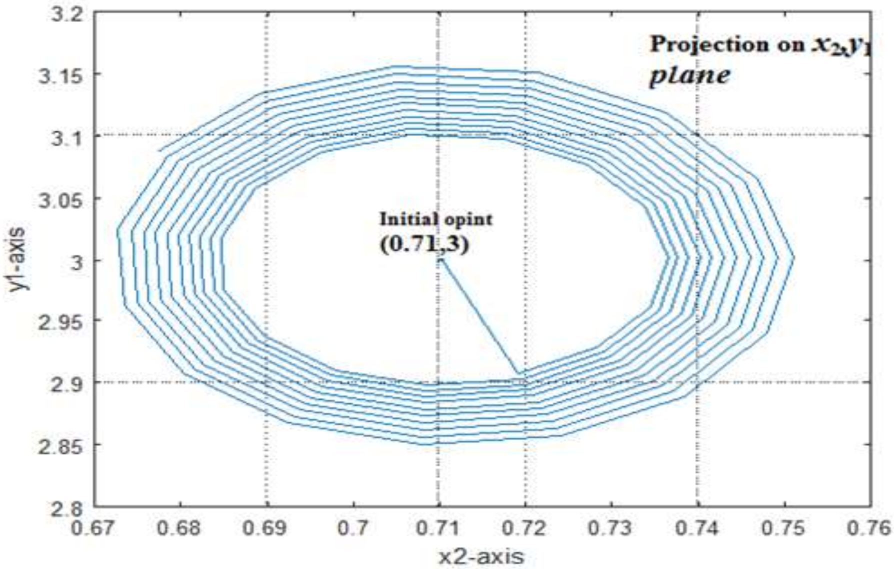

Figure (6c) With initial point (1.689,0.7101,3,0), the trajectory converges to a stable limit cycle around $E_{62} = (1.6899, 0.7101, 3, 0)$, where E_{62} is as an equilibrium point of the system (2.3) with the parameters given in Table (3). Projection on $x_2 y_1 plane$.

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دراسة نموذج غير خطي لنوعين من الفرائس ونوعين من المفترسات

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ملخص

يتناول بحثنا هذا اقتراح وتحليل نموذج الفرسية والمفترس. يتضمن نموذجنا فريستين ومفترسين ، حيث تعيش الفريستين في موئلين وتمتلك ميزة الدفاع الجماعي. واحد من الحيوانات المفترسة لديها ميل للتحول بين الموئلين. درسنا محدودية الحلول المجبة والاستقرار المحلي لجميع النقاط الثابتة الممكنة للنموذج في حالة مؤشر التحويل $n = 1$، وعلاوة على ذلك، تم تعريف دالة ليابونوف مناسبة لايجاد حوض التجاذب للنقطة الثابتة الداخلية. و اخيرا فمنا بدراسة عددية لبعض الامثلة لنقاط ثابتة مستقرة محليا واخرى غير مستقرة وكذلك وجود دارة غائية.