When compact sets are g-closed

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Abstract: This paper is devoted to introduce new concepts which are called K(gc), gK(gc), L(gc), gL(gc) and locally L(gc)-spaces. Several various theorems about these concepts are proved. Further more properties are stated as well as the relationships between these concepts and LC-spaces are investigated.

Key words: g-closed, KC-spaces and LC-spaces.

1-Introduction: It is known that compact subset of a Hausdorff space is closed, this motivates the author [7] to introduce the concept of KC-space, these are the spaces in which every compact subset is closed. Lindelof spaces have always played a highly expressive role in topology. They were introduced by Alexandroff and Urysohn back in 1929. In 1979 the authors [5] introduce a new concept namely LC-spaces, these are the spaces whose lindelof sets are closed. The aim of this paper is to continue the study of KC-spaces (LC-spaces).

2-Preliminaries: The basic definitions that needed in this work are recalled. In this work, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated, a topological space is denoted by (X, τ) (or simply by X). For a subset A of X, the closure and the interior of A in X are denoted by cl(A) and Int(A) respectively. A space X is said to be K_2- space if cl(A) is compact, whenever A is compact set in X[6]. Also a subset F of a space X is g-closed if cl(F)$\subset$U, whenever U is open and containing F[4], X is said to be gT_1 if for every two distinct points x and y in X, there exist two g-open sets U and V such that x$\in$ U and y$\notin$ U, also x$\notin$ V and y$\in$ V [3], and gT_2 if for every two distinct points x and y in X, there exist two disjoint g-open sets U and V containing x and y respectively [3]. A space X is said to be g-regular if whenever F is g-closed in X and x$\in$ X with x$\notin$ F, then there are two disjoint g-open sets U and V containing x and F respectively [3]. A space X is said to be gT_3 if whenever it is gT_1 and g-regular [3] and X is said to be g-compact if for every g-open cover of X has a finite subcover[2]. A function f from a space X into a space Y is said to be g^{**}-continuous if $f^{-1}(U)$ is g-open, whenever U is g-open subset of a space Y. Also f is said to be g^{**}-closed if f(F) is g-closed, whenever F is g-closed [3].

3-Weak forms of KC-spaces:

The author in [7] introduce the concept KC-spaces; in the present paper we introduce a generalization of KC-spaces namely K(gc) and gK(gc), also we study the

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Properties and facts about these concepts and the relationships between this concepts and KC-space.

Definition 3.1 A space X is said to be K(gc)-space if every compact set in X is g -closed. So every KC-space is K(gc), but the converse is not true in general.

Example 3.1: Let $X \neq \phi$ and Γ be the indiscrete topology on X. Then (X, Γ) i s K(gc) but not KC-space. Since if B is a nonempty proper set in X. Clearly B is compact but not closed. Also it is g-closed, since the only open set which contains B is the whole space and cl(B) = X.

Definition 3.2 A space X is said to be gK(gc)-space if every g-compact set in X is g -closed. So every K(gc)-space is gK(gc), but the converse is not true in general.

Definition 3.3 A space X is said to be gK_2 if g-cl(A) is compact, whenever A is compact set in X.

Theorem 3.1: Every K(gc)-space is gK_2.
Proof: Let K be compact set in K(gc)-space X, then it is g-closed, that is, $Cl_g(K) = K$, which implies to $Cl_g(K)$ is also compact.

Definition 3.4 A space X is said to be locally g-compact if for each point in X has a neighbourhood base which is consisting of g-compact sets. So every locally compact space is locally g-compact, but the converse is not true in general.

Lemma 3.1[1]: A space X is gT_1 if and only if every singleton set is g-closed.

Theorem 3.2 Every K(gc)-space is gT_1.

Proof: Suppose X is K(gc)-space and $x \in X$, since {x} is finite, then it is compact in X, which is K(gc)-space, then it is g-closed. So by lemma 3.1 X is gT_1.

Theorem 3.3 Every gT_3-space is gT_2.

Proof: Let x and **y** be two distinct points in X, so {x} is g-closed, since X is gT_1 and $y \notin \{x\}$, but X is g-regular, then there exist two disjoint g-open sets U and V such that $x \in \{x\} \subset U$ and $y \in V$. Therefore X is gT_2-space.

Definition 3.5: A set M is said to be g-neighbourhood of a point $x \in X$ if there exists a g-open set U such that $x \in U \subset X$. Clearly every neighbourhood is g-neighbourhood but the converse may be not true.

Example 3.2: Let $X \neq \phi$ and Γ be the indiscrete topology on X. Then in (X, Γ) the one point set {x} is g-neighbourhood but not neighbourhood.

Theorem 3.4 The following are equivalent for a space X:

1) X is g-regular
2) If U is g-open in X and $x \in X$ with $x \in U$, then there is a g-open set V containing x such that $g\text{-}cl(V) \subset U$.
3) Each $x \in X$ has ag-neighbourhood base consisting of g-closed sets.

Proof: (1)$\rightarrow$ (2) Suppose X is g-regular, U is g-open in X and x$\in$ U, then X-U is a g -closed set in X not containing x, so disjoint g-open sets V and W can be found with x$\in$ V and X-U$\subset$W. Then X-W is a g-closed set contained in U and containing V, so g-cl(V)$\subset$U. (2) $\rightarrow$(3) if (2) applies, then every g-open set U containing x contains a g-closed neighbourhood (namely g-cl(V)) of X, so the g-closed neighbourhoods of x form a neighbourhood base. (3) $\rightarrow$(1) suppose (3) applies and A is a g-closed set in X not containing x. Then X-A is a g-neighbourhood of x, so there is a g-closed neighbourhood B of x with B$\subset$X-A. Then g-Int(B) and X-B are disjoint g-open sets containing x and A respectively, where g-Int(B) the set of all g-interior points. Thus X is g-regular.

Theorem 3.5: Every T_2-space is K(gc)space.

Theorem 3.6 If X is locally g-compact and K(gc)-space, then X is gT_2-space.

Proof: Given X is locally g-compact, then every x$\in$ X has a neighbourhood base consisting of g-compact sets, but X is K(gc), then these compact sets are g-closed and hence x has neighbourhood base consisting of g-closed sets, then by theorem 3.4, X is g-regular space and by theorem 3.2 X is gT_1, then it is gT_3-space, that is, X is gT_2.

Theorem 3.7: Every g-compact set in gT_2-space is g-closed.

Proof: Let A be a g-compact set in a gT_2 –space X. If p$\in$ X-A, so for each q$\in$ A, there are two disjoint g-open sets U and V containing q and p respectively. The collection {U(q):q $\in$ A} is a g-open cover of A which is g-compact, then there is finite subcover of A, that is, A$\subset$ $\bigcup_{i=1}^{n}$ U(q_i). Put V_1=$\bigcap_{i=1}^{n}$ V_{qi}(p) and U_1=$\bigcup_{i=1}^{n}$ U(q_i). Then V_1 is a g-open set containing p. We claim that $U_1\cap V_1=\phi$, so let x$\in$ U_1, then x$\in$ U(q_i) for some i, so x$\notin$ V_{qi}(p), hence x$\notin$ V_1. Thus $U_1\cap V_1=\phi$. Also A$\subset U_1$, that is, A$\cap V_1$= ϕ which implies $V_1\subset$X-A. Therefore A is g-closed.

Corollary 3.1: Every gT_2-space is gK(gc)-space.

Theorem 3.8: The g^{**}-continuous image of g-compact set is g-compact.

Proof: Let f be g^{**}-continuous function from a space X into a space Y and suppose B is g-compact set in X. To show that B is also g-compact, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be g-open cover of f(B), that is, f(B)= $\bigcup_{\alpha\in\Lambda} U_\alpha$. So B$\subset$ f^{-1}f(B)= $f^{-1}(\bigcup_{\alpha\in\Lambda} U_\alpha)=\bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha)$, then {$f^{-1}(U_\alpha)$}is a g-open cover of B, which is g-compact, then B$\subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha i})$. But f(B) $\subseteq$f$\bigcup_{i=1}^{n} f^{-1}(U_{\alpha i})=\bigcup_{i=1}^{n}$f $f^{-1}(U_{\alpha i})\subset\bigcup_{i=1}^{n} U_{\alpha i}$. Therefore f(B) is g-compact set.

Theorem 3.9: Every continuous function from compact into a K(gc)-space is g-closed function.

Proof: Let A be closed set in X, which is compact, then A is compact. But f is continuous, then f(A) is compact in Y, which is K(gc)-space, then f(A) is g-closed. Therefore f is g-closed.

Lemma 3.2[1]: Every g-closed subset of g-compact space is g-compact.

Theorem 3.10: Every g^{**}-continuous function from g-compact into K(gc)-space is g^{**}-closed function.

Proof: Let f be g^{**}-continuous function from g-compact X into K(gc)-space Y. Also let B be g-closed set in X. So by lemma 3.2 B is g-compact also by theorem 3.8 f(B) is g-compact, which implies it is compact in Y, which is K(gc), then f(B) is g-closed. Therefore f is g^{**}-closed.

Corollary 3.2: Every g^{**}-continuous function from g-compact space into gK(gc) -space is g^{**}-closed.

Remark 3.2: The continuous image of K(gc)-space is not necessarily K(gc).

Example 3.3: Consider I_R: (R, Γ_u) $\rightarrow$(R, Γ), where I_R is the identity function, Γ_u and Γ are usual and cofinite topologies respectively. Clearly (R, Γ_u) is K(gc)-space.. Since every compact set in R is closed and bounded, this implies it g-closed. But I_R(R) = R and (R,Γ) K(gc)-space. Since if given [0, 1], which is compact and U=R-{5}, so U$\in$ Γ, then [0, 1] $\subset$ U, but cl([0, 1])=R$\not\subset$U. So (R, Γ) is not K(gc).

Theorem 3.11: Let f be g^{**}-continuous injective function from X into a gK(gc) – space Y, then X is also gK(gc).

Proof: Let W be any g-compact subset of X, then by theorem 3.7 f(W) is g-compact set in Y, which is gK(gc), then f(W) is g-closed also f is g^{**}-continuous, so f^{-1}(f(W))=W. Therefore X is gK(gc)-space.

Theorem 3.12: The property of space being K(gc) is a hereditary property.

Proof: Let Y be a subspace of K(gc)-space X and A be any compact subset of Y, then A is compact in X, which is K(gc), then A is g-closed in X. But A= A$\cap$X, then A is g-closed in Y. Therefore Y is also K(gc).

Theorem 3.13: Let f be a homeomorphism function from a space X into a space Y, if U is g-open set in X, then f(U) is also g-open.

Proof: Let F be any closed subset of f(U), so f^{-1}(F)$\subset$ f^{-1}f(U)=U, but U is g-closed, then f^{-1}(F)$\subset$Int(U), which implies F= f(f^{-1}(F))$\subset$f(Int(U))=Int(f(U)). Therefore f(U) is also g-open.

Corollary 3.3: Let f be a homeomorphism function from a space X into a space Y, if U is g-closed set in X, then f(U) is also g-closed.

Corollary 3.4: Let f be a homeomorphism function from a space X into a space Y, if M is g-compact set in X, then f(M) is also g-compact.

Theorem 3.14: The property of space being K(gc) is a topological property.

Proof: Let f be a homeomorphism function from a K(gc)-space X into a space Y and B be compact set in Y, then f^{-1}(B) is compact in X, which is K(gc), then f^{-1}(B) is g-closed and by corollary 3.3 f(f^{-1}(B))=B is g-closed set in Y.

Corollary 3.5: The property of space being gK(gc) is a topological property.

4. Further type of LC-spaces:

In 1979 the authors [5] introduce a new concept namely LC-spaces, these are the spaces in which every lindelof sets are closed. In the present paper we introduce a new concept namely L(gc)-spaces which is a weak form of LC-spaces.

Definition 4.1 A space X is said to be L(gc)-space if every lindelof set is g-closed. So every LC-space is L(gc) but the converse is not true in general.

Example 4.1: Let R with the indiscrete topology Γ. Clearly (R, Γ) is L(gc), since for every Lindelof set difference from R and ϕ is g-closed but not closed.

Theorem 4.1 Every L(gc)-space is gT_1.

Theorem 4.2 Every locally g-compact L(gc) is gT_2.

Proof: Let X be a locally g-compact and L(gc)-space, then X is K(gc). So by theorem 3.6 X is gT_2-space.

Theorem 4.3 The property of space being L(gc) is a hereditary property.

Proof: The proof is similar to theorem 3.12.

Theorem 4.4: If X is L(gc) and $T_{\frac{1}{2}}$ -space, then every compact set in X is finite.

Proof: Let A be compact set in X. If A is finite, then the proof is finished, if A is infinite, then either A is countable or uncountable. Suppose A is countable and U is any set in A, then U is countable, so U is lindelof in A, which implies it is lindelof in X, which is L(gc), then U is g-closed in X. But X is $T_{\frac{1}{2}}$, and then U is closed in X. But U $\cap$ A=U, then U is closed in A, that is, A is discrete but A is compact, then A is finite, which is a contradiction. If A is uncountable, then there exists a subset K of A is countable and so K is lindelof in A, so it is lindelof in X, which is L(gc) and $T_{\frac{1}{2}}$ -space, then K is closed. Put K= {a_1, a_2 . . .}. Let U_1= K^c, now $a_1 \in U_2$=A-{a_1, a_2,…}

and $a_2 \in$ A-{a_3, a_4…}…, then $\{U_i\}_{i=1}^{\infty}$ is an open cover of A, which has no finite subcover, which is a contradiction. Then A is finite.

Definition 4.3: A space X is said to be g-lindelof if for every g-open cover of X has a countable subcovre. Clearly every g-lindelof-space is lindelof but the converse may be not true.

Example 4.2: Let R with the indiscrete topology Γ. Clearly every subset of R is lindelof, since the only open cover of any set is just R. But (R, Γ) is not g-lindelof, since if given Q^c=R-Q , then it is not g-lindelof, since {{x}}:x$\in$ Q^c } is a cover of Q^c consisting of g-open sets, which can not be reduce to a countable subcover.

Theorem 4.5: The g^{**}-continuous image of g-lindelof set is also g-lindelof.

Proof: Let f be g^{**}-continuous function from a space X into a space Y and let K be g-lindelof set in X. To show that f(K) is also g-lindelof, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a g-open cover of f(K), that is, f(K)$\subseteq \underset{\alpha\in\Lambda}{\cup}\{U_\alpha\}$, then K$\subset f^{-1}$f(K)$\subseteq f^{-1} \underset{\alpha\in\Lambda}{\cup}\{U_\alpha\} = \underset{\alpha\in\Lambda}{\cup}\{f^{-1}U_\alpha\}$, which is also g-open cover of K, but K is g-lindelof, then it is has a countable subcover, that is, K$\subseteq \overset{\infty}{\underset{i\equiv1}{\cup}}\{f^{-1}U_{\alpha i}\}$, which implies to f(K)$\subseteq \overset{\infty}{\underset{i\equiv1}{\cup}}\{U_{\alpha i}\}$. Therefore f(K) is g-lindelof.

Theorem 4.6: The property of space being g-lidelof is a topological property.

Proof: Let f be a homeomorphism function from a g-lindelof space X into a space Y. Suppose $\{U_\alpha\}_{\alpha\in\Lambda}$ be g-open cover of Y, that is, Y=$\underset{\alpha\in\Lambda}{\cup}\{U_\alpha\}$, then X= f^{-1}**(Y)** = $f^{-1} \underset{\alpha\in\Lambda}{\cup}\{U_\alpha\}$. So by theorem 3.13 $\{f^{-1}U_\alpha\}$ is g-open cover of X, which is g-lindelof, then X=$\overset{\infty}{\underset{i\equiv1}{\cup}}\{f^{-1}U_{\alpha i}\}$, which implies to

Y=f(X)=f($\overset{\infty}{\underset{i\equiv1}{\cup}}\{f^{-1}U_{\alpha i}\}$)=$\overset{\infty}{\underset{i\equiv1}{\cup}}f\{f^{-1}U_{\alpha i}\}$=$\overset{\infty}{\underset{i\equiv1}{\cup}}\{U_{\alpha i}\}$. Therefore Y is also g-lindelof.

Definition 4.3: A space X is said to be gL(gc)-space if every g-lindelof set in X is g-closed. So every LC-space is gL(gc) and every L(gc)-space is gL(gc) but the converses are not true in general.

Theorem 4.7: Let f be a homeomorphism function from a space X into a space Y if X is gL(gc)-space, then Y is also is gL(gc).

Proof: Let B be a g-lindelof set in Y, then f^{-1}(B) is g-lindelof in X, which is gL(gc)-space, then it is g-closed, but f is a homeomorphism. So by theorem 3.13 f(f^{-1}(B))=B is g-closed in Y. Therefore Y is also gL(gc).

Definition 4.4: A space X is said to be locally L(gc)-space if every point in X has L(gc)-neighbourhood. So every L(gc)-space is locally L(gc).

Lemma 4.1[3]: If (Y, Γ_Y) is a g-closed subspace of a space (X, Γ_X), then if B is g-closed in Y, then it is g-closed in X.

Theorem 4.8: A space X is an L(gc)-space if and only if each point has closed neighbourhoood which is an L(gc)-subspace.

Proof: If X is L(gc)-space, then for each x$\in$ X, X itself is a closed neighbourhood of x, which is L(gc). Conversely, Let L be a lindelof set in X and a point x$\in$ X such that x$\notin$ L. Choose a closed neighbourhood W_x of x, which is L(gc)-subspace, then $W_x \cap$L is closed in L, which is lindelof, then $W_x \cap$L is lidelof in W_x, but W_x is L(gc)-subspace, then $W_x \cap$L is g-closed in W_x, which is closed so it is g-closed. So by lemma 4.1 $W_x \cap$L is g-closed in X. Then W_x-($W_x \cap$L) =W_x-L is a g-open set containing x and disjoint with L. Therefore L is g-closed set in X.

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