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A New Family of [0, 1] Truncated Nadarajah-Haghighi-G Properties with Real Data Applications

Khalaf H. Al-Habib *[A], Mundher A. Khaleel [A], Hazem Al-Mofleh [B]
[A] Collage of Computer Science and Mathematics, Tikrit University,
[B] Department of Mathematics, Tafila Technical University, Jordan.

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*Corresponding author:
Khalaf H. Al-Habib
Collage of Computer Science and Mathematics, Tikrit University

Abstract: In this paper, the [0,1] truncated Nadarajah Haghighi- G family is introduced, and some of its statistical properties such as expansion of its PDF are derived. In addition, quantile function, survival function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, odd functions, r^{th} moments, incomplete moments, moment generating function, order statistic, moments of order statistic, Inequality measures, mean residual life and entropy. The parameters of this family were estimated by maximum likelihood method.

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خصائص العائلة الجديدة
Nadarajah Haghighi المبتورة [٠،١] مع التطبيقات

حازم المفلح	منذر عبدالله خليل	خلف حسن خلف
قسم الرياضيات	كلية علوم الحاسوب والرياضيات	كلية علوم الحاسوب والرياضيات
جامعة الطفيلة التقنية/الأردن	جامعة تكريت	جامعة تكريت

المستخلص

في هذا البحث، تم تقديم عائلة (Nadarajah Haghighi-G) المبتورة [0, 1]، أضافة إلى ذلك اشتقاق بعض الخصائص الإحصائية مثل توسيع دالة الكثافة الاحتمالية. علاوة على ذلك، تم إيجاد دوال البقاء، ومعدل الخطر، ومعدل الخطر المعكوس، ومعدل الخطر التراكمي، والدالة الفردية، والعزوم، والعزوم الناقصة، والدالة المولدة للعزوم والإحصاءات المرتبة والإنتروبيا. كما تم تقدير معلمات العائلة الجديدة باستخدام طريقة الإمكان الأعظم.

الكلمات المفتاحية: البتر [0, 1]، العزوم، العائلة الجديدة، الإحصاءات المرتبة، التقدير.

1. Introduction

The statistical distributions are a vital part of our lives. It allows us to understand the world around us and make informed decisions. It also helps us in identify trends and opportunities. In the recent years, the modeling of lifetime data is an essential research topic. The researcher's studies on this subject have appeared, and their aim was introducing new statistical methodologies for dealing with lifetime phenomena. Several families of statistical distributions have been widely applied and used in variety of fields during the last several decades, including engineering, economic, medical sciences, demography, and so on. In this paper, a new class of generalized distributions (family) is proposed based on the $[0,1]$ Truncated, which it namely [0,1] Truncated Nadarajah-Haghighi-G ($[0,1]$ TNH-G) family of distributions. In additional, it includes generators of distributions as specific instances. The key advantage of the new distribution family is that it provides more flexibility distribution to the extremes of the density function, making it suited for analyzing data sets with a significant level of asymmetry and kurtosis. As a result, building new families of distributions has been investigated by adding extra shape parameter (s) to the baseline model. There has recently been considerable interest in creating new generators or generalized families of univariate continuous distributions that expand well-known distributions while also providing tremendous flexibility in modeling data in reality. The generated families generalized and extended most of the formal distributions. Some of the generators are Beta-G by Eugene et al. (2002), Exponential-G family introduced and studied by Cordeiro et al., (2013). The Weibull-G family introduced by Bourguignon et al., (2014), the generalized transmuted-G studied and introduced by Nofal et al., (2017). The Gompertz-G family was introduced by Alizadeh et al., (2017). The extended odd Fréchet-G family was introduced by Nasiru (2018). The Generalized odd Gamma-G family of distributions introduce by Hosseini et al., (2018), Marshall-Olkin alpha power family introduced by Nassar et al., (2019). Arshad et al. (2020) introduced the Gamma Kumaraswamy-G family of distributions. Khaleel et al., introduce the Marshall-Olkin Topp Leone-G family in (2020). The Marshall-Olkin-Weibull-H family introduced by Afify et al., (2022). The Odd Chen family introduced by Anzagra et al., (2022). Truncated distribution has been derived from that of a parent distribution, for example normal and exponential distribution by bounding the random variable form either below or above or both. Abid et al., (2017) using $[0,1]$ Truncated families of distributions by [0,1] Truncated Fréchet gamma and inverted gamma distribution they were discussed as a special case CDF, moments, mean, variance, skewness, kurtosis, median, characteristic function. Following the same method Khaleel et al., (2022) introduced [0,1] truncated inverse Weibull family. Khalaf and Khaleel in (2022) introduced and studied a new distribution named [0, 1] truncated exponentiated exponential Gompertz.

This paper introduces a new family named $[0,1]$ TNH-G. The aimed of study is derived some mathematical and statistical properties of this new family. Our idea is to generate a new family by using [0,1] Truncated-G method to define many new distributions for dealing with heavy tail data. The general motivations for $[0,1]$ TNH-G family are to generate distributions can handle both monotonic and non-monotonic hazard rate function (HRF) and introduce extreme tailed distributions for modeling lifetime data as well as skewness for symmetrical distributions.

The reset of the article is outlined as follows. In section 2, a useful $[0,1]$ TNH-G family of distributions is proposed. Section 3 introduces and discusses some important mathematical and statistical properties of the proposed family. In addition, to estimate the parameters of this new family, the MLE method is used and involved in section 4. A special sub-model from the [0,1] TNH-G family named [0,1] Truncated Nadarajah-Haghighi inverse Weibull distribution ([0,1] TNH-IW) is defined and discussed in section 5. Section 6 validated the estimates through simulation process of the [0,1] TNH-IW sub-model. In section 7, a real data set is used to illustrate the effectiveness of the [0,1] TNH-IW sub-model. The final conclusions are comprised in section 8.

2. The $[0,1]$ Truncated Nadarajah Haghighi-G family: The Nadarajah-Haghighi distribution introduced by Sarales Nadarajah and Firoozeh Haghighi as an extension of exponential distribution, the N-H distribution has the CDF and PDF as follows:

$$V(X) = 1 - e^{1-[1+bx]^{a}}, \quad a, b > 0 \ , x > 0 \tag{1}$$

$$v(x) = ab[1+bx]^{a-1} \, e^{1-[1+bx]^{a}} \quad a, b > 0 \ , x > 0 \tag{2}$$

A new class of continuous distribution was generated based on the interval [0,1] truncated cumulative distribution function V and G named $[0,1]TV\text{-}G$. Let $G(x)$ and $g(x)$ be any baseline CDF and PDF prepared for a random variable X. And let we have a continuous distribution $V(.)$ and $v(.)$ respectively, CDF and PDF are defined in eq (1) and eq (2). The proposed formula for the CDF to separate on the composing V with G would be.

$$F(x)_{[0,1]TV-G} = \frac{V\big(G(x)\big) - V(0)}{V(1) - V(0)} \tag{3}$$

Now, let $V(0) = 0$ then CDF in (3) can be rewritten as,

$$F(x)_{[0,1]TV-G} = \frac{V\big(G(x)\big)}{V(1)} \tag{4}$$

And it's associated PDF, $f(x) = \frac{d}{dx}(F(x))$, will be,

$$f(x)_{TV-G} = \frac{v\big(G(x)\big)g(x)}{V(1)} \tag{5}$$

A new generated family of [0,1] truncated based on Nadarajah-Haghighi distribution (NH) will introduce as follows. Let $V(.)$ and, $v(.)$ be the CDF and PDF of (NH) distribution respectively recall that eq (1) and eq (2) with two non-negative parameters $(a, b > 0)$. We have $V(0) = 0$,

So, let

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$$V(G(x)) = 1 - e^{1-(1+bG(x))^a}$$

$$V(1) = 1 - e^{1-(1+b)^a}$$

$$v(G(x)) = ab(1 + bG(x))^{a-1} e^{1-(1+bG(x))^a}$$

Then according to eq (3) and eq (4), the CDF and PDF for the new family of distribution named {[0,1] Truncated Nadarajah-Haghighi-G family} (symbolized by [0,1] TNH-G) will be

$$F(x)_{TNH-G} = \frac{1 - e^{1-(1+bG(x,\xi))^a}}{1 - e^{1-(1+b)^a}} \quad (6)$$

and

$$f(x)_{TNH-G} = \frac{ab(1 + bG(x,\xi))^{a-1} e^{1-(1+bG(x\xi,))^a} g(x,\xi)}{1 - e^{1-(1+b)^a}} \quad (7)$$

Where ξ is the vector of parameters for the baseline distribution.

The survival, hazard rate, reversed hazard rate, cumulative hazard rate, and odd functions that correspond to (6), (7)

$$S(x,a,b,\xi)_{[0,1]TNH-G} = \frac{e^{1-(1+bG(x,\xi))^a} - e^{1-(1+b)^a}}{1 - e^{1-(1+b)^a}}$$

$$h(x,a,b,\xi)_{[0,1]TNH-G} = \frac{ab[1 + bG(x,\xi)]^{a-1} e^{1-[1+bG(x,\xi)]^a} g(x)}{e^{1-[1+bG(x,\xi)]^a} - e^{1-(1+b)^a}}$$

$$r(x,a,b,\xi)_{[0,1]TNH-G} = \frac{ab(1+bG(x,\xi))^{a-1} e^{1-(1+bG(x,\xi))^a} g(x)}{1-e^{1-[1+bG(x,\xi)]^a}}$$

$$H(x,a,b,\xi)_{[0,1]TNH-G} = -\ln\left\{\frac{e^{1-[1+b\,G(x,\xi)]^a} - e^{1-[1+b]^a}}{1-e^{1-(1+bG(x))^a}}\right\}$$

$$O(x,a,b,\xi)_{[0,1]TNH-G} = \frac{1 - e^{1-(1+bG(x,\xi))^a}}{e^{1-(1+bG(x,\xi))^a} - e^{1-(1+b)^a}}$$

3. General Results: In this section, we derive general results of the new family [0, 1] Truncated Nadarajah-Haghighi-G family.

3-1. Quantile function: The quantile function of the $[0,1]\,TNH-G$ family of distributions can be obtained by inverting $u = F_{[0,1]TNH-G}(x)$ given in (6) as follows

$$Q(u) = G^{-1}\left(\frac{1}{b}\left(1 - \ln\{1 - u[1 - e^{1-[1+b]^a}]\}^{\frac{1}{a}}\right) - 1, \xi\right) \quad (8)$$

3-2. Mixture Representation: The mixture representation of the PDF is essential in the derivation of the statistical properties of $[0,1]$ TNH-G family of distributions. By take eq. (7) and reduce it

$$f(x)_{[0,1]TNH-G} = \frac{ab\left(1+bG(x,\xi)\right)^{a-1} e^{1-\left(1+bG(x,\xi)\right)^{a}}\ g(x,\xi)}{1-\ e^{1-(1+b)^{a}}}$$

By use the expansion of the exponential function (Faris and Khaleel,2022)

$$e^{u} = \sum_{j=0}^{\infty} \frac{u^{j}}{j!},$$

we get

$$e^{1-\left(1+bG(x,\xi)\right)^{a}} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(1-\left(1+bG(x,\xi)\right)^{a}\right)^{j}$$

$$f(x)_{[0,1]TNH-G} = \frac{ab\ g(x,\xi)\left(1+bG(x,\xi)\right)^{a-1}}{1-e^{1-(1+b)^{a}}} \sum_{j=0}^{\infty} \frac{1}{j!}\left(1-\left(1+bG(x,\xi)\right)^{a}\right)^{j}$$

Moreover, by using the generalized binomial theorem formula

$$(1-Z)^{a} = \sum_{k=0}^{\infty} \binom{a}{k} (-1)^{k} Z^{k} \text{ , we get,}$$

$$\left(1-\left(1+bG(x,\xi)\right)^{a}\right)^{j} = \sum_{k=0}^{\infty} \binom{j}{k} (-1)^{k} \left(1+bG(x,\xi)\right)^{aj}$$

then

$$f_{[0,1]TNH-G}(x,a,b,\xi) = \frac{ab\ g(x,\xi)}{1-\ e^{1-(1+b)^{a}}} \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{j}{k} \frac{(-1)^{k}}{j!} \left(1+bG(x,\xi)\right)^{aj+a-1}$$

Again, by using the generalized binomial theorem Abdullah et al., (2022)

$$\left(1+bG(x,\xi)\right)^{aj+a-1} = \sum_{m=0}^{\infty} \binom{aj+a-1}{m} b^{m} \left(G(x,\xi)\right)^{m(aj+a-1)}$$

then

$$f_{[0,1]TNH-G}(x,a,b,\xi) = \frac{ab\ g(x,\xi)}{1-e^{1-[1+b]^{a}}} \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \binom{j}{k} \frac{(-1)^{k}}{j!} \times \binom{aj+a-1}{m} b^{m}\ g(x,\xi) \left(G(x,\xi)\right)^{m(aj+a-1)}$$

$$= \Omega_{j,k,m}\ g(x,\xi) \left(G(x,\xi)\right)^{m(aj+a-1)} \quad (9)$$

where

$$\Omega_{j,k,m} = \frac{ab}{1 - e^{1-[1+b]^a}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{j}{k} \frac{(-1)^k}{j!} \binom{aj + a - 1}{m} b^m$$

3-3. Moments, Moments Generating Functions

Proposition 1: The r^{th} moment of the $[0,1]$ TNH-G family is given by:

$$\mu_r = \Omega_{j,k,m} \int_0^{\infty} x^r \, g(x,\xi) \left(G(x,\xi)\right)^{m(aj+a-1)} dx$$

Proof: the r^{th} moment of a random variable X defined as

$$\mu_r = \int_{-\infty}^{\infty} x^r f(x) \, dx$$

where $f(x)$ given in (9) so that

$$\mu_r = \int_{-\infty}^{\infty} x^r \Omega_{j,k,m} \, g(x,\xi) \left(G(x,\xi)\right)^{m(aj+a-1)} dx$$

$$= \Omega_{j,k,m} \int_0^{\infty} x^r g(x,\xi) \left(G(x,\xi)\right)^{m(aj+a-1)} dx$$

Instead of that, we can define the r^{th} moment in the quantile function as

$$\mu_r = \Omega_{j,k,m} \int_0^{1} u^k Q_G^r(u) du$$

Where $u = G(x) \; and \; Q_G(u)$ is the quantile of the baseline distribution.

3-4. Moment Generating Functions

Proposition 2: The moment generating functions (MGF) for $[0,1]$ TNH-G family of distributions is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r \, g(x,\xi) \left(G(x,\xi)\right)^{m(aj+a-1)} dx$$

Proof: the MGF of a random variable X is defined as follow

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r\ f(x)dx$$

where $f(x)$ was defined in (9) we get

$$M_X(t) = \Omega_{j,k,m} \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r g(x,\xi)\ (G(x,\xi))^{m(aj+a-1)} dx$$

$M_X(t)$ can be expressed in the terms of quantile function as

$$M_X(t) = \Omega_{j,k,m} \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 e^{tQ_G(x)} u^k du\ , 0 < u < 1$$

3-5. Incomplete Moments”

Proposition 3: The incomplete moments of [0,1] TNH-G family of distributions is defined as

$$M_r(y) = \Omega_{j,k,m} \int_{-\infty}^{y} x^r g(x,\xi)\ (G(x,\xi))^{m(aj+a-1)} dx$$

Proof: The incomplete moments of a random variable X defined by

$$M_r(y) = \int_{-\infty}^{y} x^r f(x)dx$$

where $f(x)$ given in (9) we have gotten

$$M_r(y) = \Omega_{j,k,m} \int_{-\infty}^{y} x^r g(x,\xi)\ (G(x,\xi))^{m(aj+a-1)} dx$$

In addition, the incomplete moments may be defined in the quantile function as.

$$M_r(y) = \Omega_{j,k,m} \int_{0}^{G(y)} u^k Q_G^r(u) du$$

3-6. Order statistic: Let $X_1, X_2, X_3, \dots, X_n$ have [0,1] TNH-G family with CDF, PDF defined in (6), (7) respectively and let $X_{1:n}, X_{2:n}, X_{3:n}, \dots, X_{n:n}$ be the order statistic obtained from this sample. Then the probability density function of r^{th} order statistic from [0,1] TNH-G is obtained by inserting (6), (7) in the following equation. Ahmed et al., (2020)

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!}\left(F(x)\right)^{p-1}\left(1-F(x)\right)^{n-p} f(x)$$

Expanding $\left(F(x)\right)^{p-1}$ in the defeintion of $f_{p:n}(x,\boldsymbol{\Phi})$ using binomial series expansion yields

$$\left(F(x)\right)^{p-1} = \sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}\left(1-F(x)\right)^i$$

Substituting it back into the expression of $f_{p:n}(x,\boldsymbol{\Phi})$ we get

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!}\sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}\left(1-F(x)\right)^{i+n-p} f(x)$$

$$= \frac{n!}{(p-1)!\,(n-p)!}\sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}\left(S(x)\right)^{i+n-p} f(x)$$

where $S(x)$ is the survival function of our family

$$S(x,a,b,\xi)_{[0,1]TNH-G} = \frac{e^{1-(1+bG(x,\xi))^a} - e^{1-(1+b)^a}}{e^{1-(1+b)^a}}$$

Then

$$\left(S(x)\right)^{i+n-p} = \left(\frac{e^{1-(1+bG(x,\xi))^a} - e^{1-(1+b)^a}}{1-e^{1-(1+b)^a}}\right)^{i+n-p}$$

$$= \frac{1}{\left(1-e^{1-(1+b)^a}\right)^{i+n-p}} e^{1-\left(1+bG(x,\xi)\right)^a}$$

$$\times\left(e^{1-(1+bG(x,\xi))^a} - e^{1-(1+b)^a}\right)^{i+n-p}$$

Employing a similar concept of expansion, the density function of [0,1] TNH-G family, a mixture representation of the PDF of the p^{th} order statistic is defined as

$$f_{X_{p:n}}(x) = \frac{n!}{(p-1)!\,(n-p)!}\Upsilon_{ikmjl}\left(G(x)\right)^{j} g(x) \tag{10}$$

where

$$\Upsilon_{ikmjl} = \sum_{i=0}^{p-1}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\binom{p-1}{i}\binom{al+am+a-1}{j}\binom{i+n-p}{k}$$

$$\times \frac{a(b)^{j+1}\,(-1)^{i+k+m+l}\,e^{k+1}\,(k)^m \left(e^{1-[1+b]^a}\right)^{i+n-p-k}}{(m!)(l!)(1-e^{1-[1+b]^a})^{i+n-p+1}}$$

3-7. Moments of Order statistic:

Proposition 4: The r^{th} moment of the p^{th} order statistic of [0,1] TNH-G family of distributions is given by

$$E\left(X_{p:n}^r\right) = \frac{n!}{(p-1)!\,(n-p)!} \Upsilon_{ikmjl} \int_0^\infty x^r \left(G(x)\right)^j g(x)dx$$

Proof: the r^{th} moment of the p^{th} order statistic, $E\left(X_{p:n}^r\right) of\ a$ random variable X is defined as follows

$$E\left(X_{p:n}^r\right) = \int_{-\infty}^{\infty} x^r f_{X_{p:n}}(x)dx$$

Now when $f_{X_{p:n}}(x)$ as in (10) we get that

$$E\left(X_{p:n}^r\right) = \int_0^\infty x^r \frac{n!}{(p-1)!\,(n-p)!} \Upsilon_{ikmjl} \left(G(x)\right)^j g(x)dx$$

Hence

$$E\left(X_{p:n}^r\right) = \frac{n!}{(p-1)!\,(n-p)!} \Upsilon_{ikmjl} \int_0^\infty x^r \left(G(x)\right)^j g(x)dx$$

3-8. Inequality Measure: Several fields like econometrics, insurance and reliability employ Lorenz and Bonferroni curves in the study of inequality measures like income and poverty.

3-8-1. Lorenz curve

Proposition 5: The Lorenz curve of $[0{,}1]$ TNH-G family of distributions defined by

$$L_F(y) = \frac{1}{\mu}\,\Omega_{j,k,m} \int_{-\infty}^{y} x\, g(x,\xi) \left(G(x,\xi)\right)^{m(aj+a-1)} dx$$

Proof: The Lorenz curve of a random variable X is defined as

$$L_F(y) = \frac{1}{\mu}\int_{-\infty}^{y} xf(x)dx$$

So that where $f(x)$ defined in (8) we get the Lorenz curve of [0,1] TNH-G family of distributions is

$$L_F(y) = \frac{1}{\mu}\int_{-\infty}^{y} x\,\Omega_{j,k,m} g(x,\xi)\left(G(x,\xi)\right)^{m(aj+a-1)}dx$$

the equation can be written as follows

$$= \frac{1}{\mu}\,\Omega_{j,k,m}\int_{-\infty}^{y} x\,g(x,\xi)\left(G(x,\xi)\right)^{m(aj+a-1)}dx \tag{11}$$

3-8-2. Bonferroni Curve

Proposition 6: The Bonferroni curve of [0,1] TNH-G family of distributions is

$$B_F(y) = \frac{1-e^{1-(1+b)^a}}{\mu\left(1-e^{1-(1+bG(y,\zeta))^a}\right)}\Omega_{j,k,m}\int_{-\infty}^{y} x\,g(x,\xi)\left(G(x,\xi)\right)^{m(aj+a-1)}dx$$

Proof: the Bonferroni Curve of a random variable X defined as $B_F(y) = \frac{L_F(y)}{F(y)}$ now where $L_F(y)$ as in (11) and $F(y)$ was defined in (6) with respect to y so that

$$B_F(y) = \frac{1-e^{1-(1+b)^a}}{\mu\left(1-e^{1-(1+bG(y,\zeta))^a}\right)}\Omega_{j,k,m}\int_{-\infty}^{y} x\,g(x,\xi)\left(G(x,\xi)\right)^{m(aj+a-1)}dx$$

3-9. Mean Residual Life

Proposition 7: The mean residual life of [0,1] TNH-G family of distributions is

$$\bar{M}(y) = \frac{1}{F(y)}\left(\mu - \Omega_{j,k,m}\int_{-\infty}^{y} x\,g(x,\xi)\left(G(x,\xi)\right)^{m(aj+a-1)}dx\right) - y$$

Proof: The mean residual life of a random variable X is defined as

$$\bar{M}(y) = E(X-y|X>y)$$

Thus

$$\bar{M}(y)=\frac{1}{F(y)}\left(\mu-\int_{-\infty}^{y}xf(x)dx\right)-y$$

Now substituted $f(x)$ as mixture density defined in (8) we get

$$\bar{M}(y)=\frac{1}{F(y)}\left(\mu-\Omega_{j,k,m}\int_{-\infty}^{y}x\,g(x,\xi)\left(G(x,\xi)\right)^{m(aj+a-1)}dx\right)-y$$

3-10. Rényi entropy: The entropy of a random variable X is a measure of variation of uncertainty.

Proposition 9: Rényi entropy for the $[0,1]$ TNH-G family of distributions is random variable X is defined by

$$I_R(c)=\frac{1}{1-c}\log\left\{\left(\Omega_{j,k,m}\right)^{c}\int_{-\infty}^{\infty}\left(g(x,\xi)\right)^{c}\left(G(x,\xi)\right)^{cm(aj+a-1)}dx\right\},c\neq 1,c>0,$$

Proof: Rényi entropy for the random variable X is defined by

$$I_R(c)=\frac{1}{1-c}\log\left\{\int_{-\infty}^{\infty}f^{c}(x)dx\right\},c\neq 1,c>0$$

Now when $f(x)$ which defined in (8) then

$$f^{c}(x)_{[0,1]TNH-G}=\left(\frac{ab\left(1+bG(x,\xi)\right)^{a-1}e^{1-\left(1+bG(x,\xi)\right)^{a}}g(x,\xi)}{1-e^{1-(1+b)^{a}}}\right)^{c}$$

Hence

$$I_R(c)=\frac{1}{1-c}\log\left\{\left(\Omega_{j,k,m}\right)^{c}\int_{-\infty}^{\infty}\left(g(x,\xi)\right)^{c}\left(G(x,\xi)\right)^{cm(aj+a-1)}dx\right\},c\neq 1,c>0,$$

3-10-1. Shannon entropy:

Proposition 10: The Shannon entropy of a $[0,1]$ TNH-G family is given by

$$\eta_x=E\left(-\log\left\{\Omega_{j,k,m}\,g(x,\xi)\left(G(x,\xi)\right)^{m(aj+a-1)}\right\}\right)$$

Proof: The Shannon entropy of a random variable X is defined as

$$\eta_x = E\{-\log f(x)\}$$

So that Shannon entropy of $[0,1]$ TNH-G family of distributions where $f(x)$ was defined in (8) is

$$\eta_x = E\left(-\log\left\{\Omega_{j,k,m}\, g(x,\xi)\,\left(G(x,\xi)\right)^{m(aj+a-1)}\right\}\right).$$

3-10-2. Delta Entropy

Proposition 11: The $\delta-$ entropy of $[0,1]$ TNH-G family of distributions

$$H(\delta) = \frac{1}{1-\delta}\log\left\{1-\left(\Omega_{j,k,m}\right)^c \int_{-\infty}^{\infty}\left(g(x,\xi)\right)^c \left(G(x,\xi)\right)^{cm(aj+a-1)}\,dx\right\}$$

Proof: since the $\delta-$ entropy of a random variable X is given by

$$H(\delta) = \frac{1}{1-\delta}\log\left\{1-\int_{-\infty}^{\infty} f^{\delta}(x)dx\right\}$$

Hence for $[0,1]$ TNH-G family of distributions is given by

$$H(\delta) = \frac{1}{1-\delta}\log\left\{1-\left(\Omega_{j,k,m}\right)^c \int_{-\infty}^{\infty}\left(g(x,\xi)\right)^c \left(G(x,\xi)\right)^{cm(aj+a-1)}\,dx\right\}$$

4. **Parameters Estimation:** The parameters of $[0\ 1]$ TNH-G family are estimated by using the maximum li;elihood method Abdal et al.,(2020). Suppose that $x_1\ x_2\ \ldots\ x_n$ is random sample of size n from the $[0\ 1]$ TNH-G family of distributions. Then the corresponding likelihood function is given by:

$$L(a,b,\xi) = \prod_{i=1}^{n} f(x_i,a,b,\xi)$$

$$= \prod_{i=1}^{n}\frac{ab(1+bG(x_i,\xi))^{a-1}\, e^{1-(1+bG(x_i,\xi))^a}\; g(x_i,\xi)}{1-\; e^{1-(1+b)^a}}$$

$$= \frac{(ab)^n\ \sum_{i=1}^{n} g(x_i,\xi)\ \sum_{i=1}^{n}(1+bG(x_i,\xi))^{a-1}\ e^{\sum_{i=1}^{n}1-(1+bG(x_i,\xi))^a}}{(1-\ e^{1-[1+b]^a})^n}$$

Now the log-likelihood function given as:

$$l = nlog(a) + nlog(b) + \sum_{i=1}^{n}\log\{g(x_i,\xi)\}$$
$$+(a-1)\sum_{i=1}^{n}\log\{1+bG(x_i,\xi)\} + \sum_{i=1}^{n}(1-[1+bG(x_i,\xi)]^a)$$
$$-n\log\{1-\ e^{1-[1+b]^a}\}$$

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Now we find the paretial deivitvie of l with respect to the parameter as follow:

$$l = n\log(a) + n\log(b) + \sum_{i=1}^{n} \log\left(g(x_i,\xi)\right) + (a-1)\sum_{i=1}^{n} \log\{1 + bG(x_i,\xi)\} + \sum_{i=1}^{n}(1 - [1 + bG(x_i,\xi)]^a) - n\log\{1 - e^{1-(1+b)^a}\}$$

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log\{1 + bG(x_i,\xi)\} + \frac{n(1+b)^a \log(1+b)\, e^{1-(1+b)^a}}{1 - e^{1-(1+b)^a}} - \sum_{i=1}^{n}(1 + bG(x_i,\xi))^a \log(1 + bG(x_i,\xi))$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \frac{(a-1)\, G(x_i,\xi)}{1 + bG(x_i,\xi)} - aG(x_i,\xi)\left(1 + bG(x_i,\xi)\right)^{a-1} - \frac{n\, a\, (1+b)^{a-1}\, e^{1-(1+b)^a}}{e^{1-(1+b)^a}}$$

$$\frac{\partial l}{\partial \xi} = \frac{\partial \sum_{i=1}^{n} \log\left(g(x_i,\xi)\right)}{\partial \xi} + (a-1)\frac{\partial \sum_{i=1}^{n} \log\{1 + bG(x_i,\xi)\}}{\partial \xi} + \frac{\partial \sum_{i=1}^{n}(1 - (1 + bG(x_i,\xi))^a)}{\partial \xi}$$

Equating the score functions to zero and numerically solving the system of equations using techniques such as Newton-Raphson method, gives the maximum likelihood estimates.

5. **A Special Model from $[0,1]$ TNH-G Family:** This section deals with a new special distribution, namely $[0{,}1]$ TNH-Inverse Weibull denoted by $[0{,}1]$ TNH-IW distribution.

5-1. The $[0,1]$ Truncated Nadarajah-Haghighi Inverse Weibull Distribution: Suppose that the baseline distribution is the inverse Weibull distribution with the following CDF and PDF, respectively.

$$G(x;\lambda,\beta) = e^{-\lambda x^{-\beta}} \quad (12)$$

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And

$$g(x;\lambda,\beta) = \lambda\beta x^{-(\beta+1)}\, e^{-\lambda x^{-\beta}};\ x > 0, \lambda, \beta\ > 0. \qquad (13)$$

Then the CDF of [0,1] TNH-IW distribution is obtained by substituting eq. (12) in eq. (6) as follows:

$$F(x;a,b,\lambda,\beta)_{[0,1]\,TNHIW} = \frac{1-e^{1-\left(1+b\left(e^{-\lambda x^{-\beta}}\right)\right)^{a}}}{1-e^{1-(1+b)^{a}}}\ ;\ x \geq 0, a, b,, \lambda, \beta\ > 0 \qquad (14)$$

In addition, the PDF can be obtained by substituting eqs. (12), (13) in eq. (7).

$$f(x;a,b,\lambda,\beta)_{[0,1]\,TNHIW} = \frac{ab\,\lambda\beta x^{-(\beta+1)}\left(1+b\left(e^{-\lambda x^{-\beta}}\right)\right)^{a-1}\ e^{1-\left(1+b\left(e^{-\lambda x^{-\beta}}\right)\right)^{a}}\ e^{-\lambda x^{-\beta}}}{1-e^{1-(1+b)^{a}}} \qquad (15)$$

The HRF of $[0,1]$ TNH-IW distribution is given by

$$h(x;a,b,\lambda,\beta)_{[0,1]\,TNHIW} = \frac{ab\lambda\beta x^{-(\beta+1)}\left(1+b\left(e^{-\lambda x^{-\beta}}\right)\right)^{a-1}\ e^{1-\left(1+b\left(e^{-\lambda x^{-\beta}}\right)\right)^{a}}\ e^{-\lambda x^{-\beta}}}{e^{1-\left(1+b\left(e^{-\lambda x^{-\beta}}\right)\right)^{a}} - e^{1-(1+b)^{a}}}.$$

Figure 1 shows the Shapes of PDF and hazard functions of $[0,1]$ TNH-IW distribution.

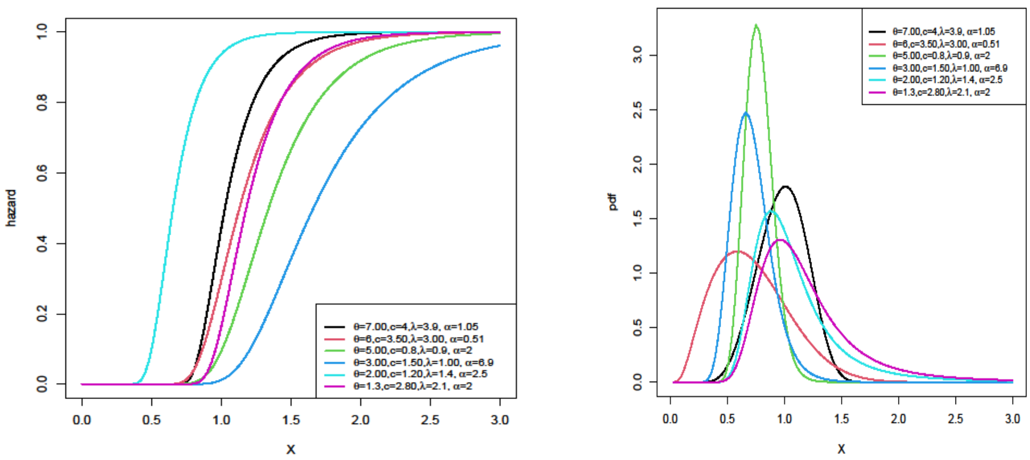

Figure (1): Shapes of PDF and hazard functions of $[0,1]$ TNH-IW distribution. This figure finds by author by using R program

6. Simulation Study: In this section we have conducted simulation study for $[0,1]$ TNH-IW distribution. We have generated samples of sizes

$n = \{50, 80, 120, 200, 300\}$ from the proposed model and parameters have been estimated by MLE method, the simulation study is in terms of the averages of the three quantities: absolute bias $|\text{Bais}(\boldsymbol{\theta})| = \frac{1}{\text{N}}\sum_{\text{i}=1}^{\text{N}} |\,\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}|$, mean square error (MSE), $MSE(\boldsymbol{\theta}) = \frac{1}{\text{N}}\sum_{\text{i}=1}^{\text{N}}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2$, and mean relative error (MRE), $MRE(\boldsymbol{\theta}) = \frac{1}{\text{N}}\sum_{\text{i}=1}^{\text{N}} |\,\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}|/\boldsymbol{\theta}$. All the computations are made by using *R* Statistical Software. Table 1 shows some simulation results for different values of $\boldsymbol{\theta}\ (a, b, \lambda, \beta)^T$. Based on the simulation criteria, from Table 1 one can be discovered that the maximum likelihood estimate strategy performs pretty well in estimating the $[0,1]$ TNH-IW distribution parameters.

7. Application: In this section, we fit the $[0,1]$TNH-IW distribution a real data set to demonstrate that the proposed distribution fits well when compared to competing distributions. *R* Statistical Software is used to calculate all the results. In order to obtain the best results, we used the following statistical criteria (-*l*, AIC, AIC, BIC, HQIC) for the proposed model compared to other models, such as beta inverse Weibull (BIW), Kumaraswamy inverse Weibull (KuIW), Exponential Generalized inverse Weibull (EGIW), Weibull inverse Weibull (WeIW), Gompertez inverse Weibull (GoIW), Marshal-Olkin inverse Weibull (MoIW) and inverse Weibull (IW). This data set is the employment of the failure rate dataset (103 hours) for the turbocharger for the engine type. The data set consists of 40 observations by Alobaidi et al., (2021), Hassan et al., (2021) and it is given as follows:

1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

According to the values shown in Tables 2 and 3 it is clear that the $[0,1]$TNH-IW distribution is superior the comparative distributions. The proposed expanded distribution provides an accurate representation because it has the lowest values according to the statistical and informational criteria, and the largest value of the p-value. It is clear that from Figures 2 and 3 the $[0,1]$ TNH-IW model provides the overall best fit and therefore could be chosen as the adequate model for explaining data.

Table 1: Bias, MSE and MRE of parameters of [0,1] TNH-IW distribution

$\boldsymbol{\theta} = (a = 0.75, b = 1.75, \lambda = 2.75, \beta = 1.40)^T$						
Est.	**Est. Par.**	$\boldsymbol{n=50}$	$\boldsymbol{n=80}$	$\boldsymbol{n=120}$	$\boldsymbol{n=200}$	$\boldsymbol{n=300}$
|Bias|	$\hat{a}$	0.52908	0.46918	0.41856	0.37869	0.33748
	$\hat{b}$	0.66939	0.65050	0.63931	0.59485	0.58769
	$\hat{\lambda}$	0.41075	0.37219	0.34554	0.31333	0.30438
	$\hat{\beta}$	0.83114	0.82258	0.81907	0.81681	0.81424
MSE	$\hat{a}$	0.46496	0.36732	0.29044	0.23835	0.18734
	$\hat{b}$	0.74693	0.66923	0.66362	0.55809	0.55312
	$\hat{\lambda}$	0.27060	0.21320	0.17740	0.13695	0.12506
	$\hat{\beta}$	0.70440	0.68663	0.67840	0.67256	0.66720
MRE	$\hat{a}$	0.88181	0.78197	0.69760	0.63116	0.56246
	$\hat{b}$	0.47813	0.46465	0.45665	0.42489	0.41978
	$\hat{\lambda}$	0.16765	0.15191	0.14103	0.12789	0.12424
	$\hat{\beta}$	1.38523	1.37097	1.36512	1.36134	1.35707
$\boldsymbol{\theta} = (a = 0.75, b = 0.50, \lambda = 0.40, \beta = 0.60)^T$						
Est.	Est. Par.	$\boldsymbol{n=50}$	$\boldsymbol{n=80}$	$\boldsymbol{n=120}$	$\boldsymbol{n=200}$	$\boldsymbol{n=300}$
|Bias|	$\hat{a}$	0.33564	0.32977	0.32930	0.31432	0.30503
	$\hat{b}$	0.97158	0.94561	0.93413	0.91817	0.91538
	$\hat{\lambda}$	2.03537	2.04078	2.04407	2.04608	2.04780
	$\hat{\beta}$	0.18191	0.15692	0.14249	0.12251	0.11018
MSE	$\hat{a}$	0.19489	0.18504	0.18815	0.16987	0.15870
	$\hat{b}$	1.10956	1.04976	1.01732	0.97077	0.96122
	$\hat{\lambda}$	4.14514	4.16624	4.17926	4.18714	4.19394
	$\hat{\beta}$	0.05903	0.04327	0.03637	0.02766	0.02308
MRE	$\hat{a}$	0.55940	0.54962	0.54883	0.52387	0.50839
	$\hat{b}$	0.69399	0.67544	0.66724	0.65583	0.65384
	$\hat{\lambda}$	0.83076	0.83297	0.83431	0.83514	0.83583
	$\hat{\beta}$	0.30319	0.26154	0.23749	0.20418	0.18363
$\boldsymbol{\theta} = (a = 1.50, b = 1.75, \lambda = 2.75, \beta = 2.45)^T$						
Est.	Est. Par.	$\boldsymbol{n=50}$	$\boldsymbol{n=80}$	$\boldsymbol{n=120}$	$\boldsymbol{n=200}$	$\boldsymbol{n=300}$
|Bias|	$\hat{a}$	0.88617	0.85938	0.85050	0.84884	0.85287
	$\hat{b}$	0.51764	0.49632	0.48027	0.46285	0.44859

	$\hat{\lambda}$	0.56162	0.48264	0.42878	0.37629	0.34425
	$\hat{\beta}$	1.87000	1.85971	1.85624	1.85675	1.85774
MSE	$\hat{a}$	1.03999	0.91795	0.85072	0.79700	0.77797
	$\hat{b}$	0.40302	0.36949	0.32247	0.27853	0.24824
	$\hat{\lambda}$	0.54095	0.39447	0.30641	0.22551	0.17815
	$\hat{\beta}$	3.56935	3.50734	3.47894	3.46910	3.46641
MRE	$\hat{a}$	1.47695	1.43230	1.41750	1.41473	1.42145
	$\hat{b}$	0.36974	0.35452	0.34305	0.33060	0.32042
	$\hat{\lambda}$	0.22923	0.19700	0.17501	0.15359	0.14051
	$\hat{\beta}$	3.11666	3.09952	3.09374	3.09459	3.09623

Table (1): Bias, MSE and MRE of parameters of [0,1] TNH-IW distribution (Continued)

$\boldsymbol{\theta} = (a = 1.50, b = 0.50, \lambda = 0.40, \beta = 1.40)^T$						
Est.	**Est. Par.**	$\boldsymbol{n = 50}$	$\boldsymbol{n = 80}$	$\boldsymbol{n = 120}$	$\boldsymbol{n = 200}$	$\boldsymbol{n = 300}$
|Bias|	$\hat{a}$	0.99922	0.99380	0.97838	0.96476	0.94529
	$\hat{b}$	0.93687	0.92891	0.90546	0.89628	0.89108
	$\hat{\lambda}$	2.03776	2.04107	2.04485	2.04762	2.04860
	$\hat{\beta}$	0.84751	0.84067	0.84451	0.83965	0.83784
MSE	$\hat{a}$	1.21107	1.18090	1.13642	1.08976	1.03320
	$\hat{b}$	1.02364	0.98353	0.92449	0.88486	0.85991
	$\hat{\lambda}$	4.15496	4.16752	4.18253	4.19345	4.19727
	$\hat{\beta}$	0.96017	0.89784	0.87178	0.82899	0.79828
MRE	$\hat{a}$	1.66537	1.65633	1.63063	1.60794	1.57548
	$\hat{b}$	0.66919	0.66351	0.64676	0.64020	0.63649
	$\hat{\lambda}$	0.83174	0.83309	0.83463	0.83576	0.83616
	$\hat{\beta}$	1.41252	1.40112	1.40751	1.39942	1.39640
$\boldsymbol{\theta} = (a = 4.00, b = 0.50, \lambda = 2.75, \beta = 1.40)^T$						
Est.	**Est. Par.**	$\boldsymbol{n = 50}$	$\boldsymbol{n = 80}$	$\boldsymbol{n = 120}$	$\boldsymbol{n = 200}$	$\boldsymbol{n = 300}$
|Bias|	$\hat{a}$	3.43707	3.47798	3.48808	3.53083	3.53740
	$\hat{b}$	0.91002	0.92533	0.93018	0.93086	0.92981
	$\hat{\lambda}$	0.57437	0.51238	0.47374	0.40910	0.38237
	$\hat{\beta}$	0.80889	0.79824	0.79347	0.79357	0.79387
MSE	$\hat{a}$	12.1286	12.3218	12.3402	12.5598	12.5762

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	$\hat{b}$	0.85541	0.87267	0.87779	0.87446	0.87044
	$\hat{\lambda}$	0.55279	0.43982	0.37401	0.26887	0.22437
	$\hat{\beta}$	0.68013	0.65351	0.64192	0.63756	0.63599
MRE	$\hat{a}$	5.72844	5.79663	5.81346	5.88472	5.89566
	$\hat{b}$	0.65001	0.66095	0.66441	0.66490	0.66415
	$\hat{\lambda}$	0.23444	0.20914	0.19336	0.16698	0.15607
	$\hat{\beta}$	1.3481	1.33040	1.32245	1.32262	1.32311
$\boldsymbol{\theta} = (a = 4.00, b = 3.00, \lambda = 1.60, \beta = 1.40)^T$						
Est.	**Est. Par.**	$n = 50$	$n = 80$	$n = 120$	$n = 200$	$n = 300$
|Bias|	$\hat{a}$	3.56264	3.53248	3.52988	3.50618	3.48457
	$\hat{b}$	1.72737	1.69151	1.66569	1.65021	1.63546
	$\hat{\lambda}$	0.83674	0.84914	0.85754	0.85842	0.85822
	$\hat{\beta}$	0.82166	0.82221	0.82455	0.82067	0.81737
MSE	$\hat{a}$	13.17458	12.83337	12.72352	12.48727	12.27564
	$\hat{b}$	3.17134	2.99493	2.86770	2.77645	2.70833
	$\hat{\lambda}$	0.72468	0.73628	0.74652	0.74407	0.74177
	$\hat{\beta}$	0.68887	0.68625	0.68848	0.67999	0.67311
MRE	$\hat{a}$	5.93773	5.88746	5.88314	5.84364	5.80762
	$\hat{b}$	1.23383	1.20822	1.18978	1.17872	1.16819}
	$\hat{\lambda}$	0.34153	0.34659	0.35002	0.35038	0.35030
	$\hat{\beta}$	1.36944	1.37035	1.37425	1.36779	1.36228

This table finds by author by using R program

Table (2): The K-S value with its corresponding p-value and W value of the data set

Model	**W**	**A**	**K-S**	***p*-value**
[0,1]TNH-IW	0.058	0.451	0.103	0.788
BeIW	0.155	1.073	0.137	0.436
KuIW	0.202	1.344	0.138	0.426
EGIW	0.187	1.259	0.170	0.196
WeIW	0.236	1.535	0.130	0.507
GoIW	0.102	0.738	0.163	0.235
MoIW	0.368	2.256	0.230	0.028
IW	0.606	3.480	0.243	0.017

This table finds by author by using R program

Table (3): Represented the values of statistically criteria (-LL, AIC, CAIC, BIC, HQIC)

Model	MLEs	$-l$	AIC	CAIC	BIC	HQIC
[0,1] TNH-IW	$\hat{a}$ = 51.61 $\hat{b}$ = 40.38 $\hat{\lambda}$ = 15.88 $\hat{\beta}$ = 0.35	81.29	170.59	171.73	177.34	173.03
BeIW	$\hat{a}$ = 0.305 $\hat{b}$ = 196.16 $\hat{\lambda}$ = 43.31 $\hat{\beta}$ = 0.95	85.37	178.75	179.90	185.51	181.20
KuIW	$\hat{a}$ = 3.91 $\hat{b}$ = 133.68 $\hat{\lambda}$ = 3.91 $\hat{\beta}$ = 0.59	87.27	182.55	183.69	189.31	184.99
EGIW	$\hat{a}$ = 70.21 $\hat{b}$ = 0.35 $\hat{\lambda}$ = 37.54 $\hat{\beta}$ = 1.03	86.79	181.59	182.74	188.35	184.04
WeIW	$\hat{a}$ = 5.37 $\hat{b}$ = 25.01 $\hat{\lambda}$ = 10.75 $\hat{\beta}$ = 14.39	88.80	185.60	186.74	192.35	188.04
GoIW	$\hat{a}$ = 0.01 $\hat{b}$ = 1.59 $\hat{\lambda}$ = 1.97 $\hat{\beta}$ = 1.82	83.78	177.48	178.62	184.24	179.92
MoIW	$\hat{a}$ = 8.55 $\hat{b}$ = 9.67 $\hat{\lambda}$ = 2.56	96.67	199.44	200.11	204.51	201.28
IW	$\hat{\lambda}$ = 20.06 $\hat{\beta}$ = 1.94	101.95	207.18	207.50	210.56	208.40

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This table finds by author by using R program

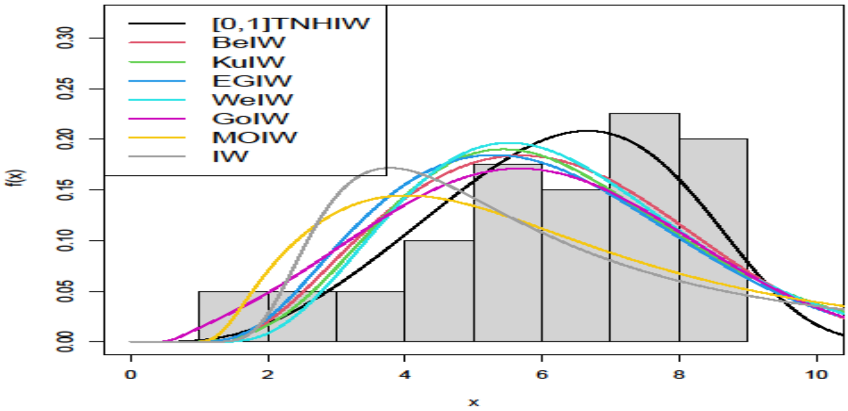

Figure (2): Estimated densities of model for data set. This figure finds by author by using R program.

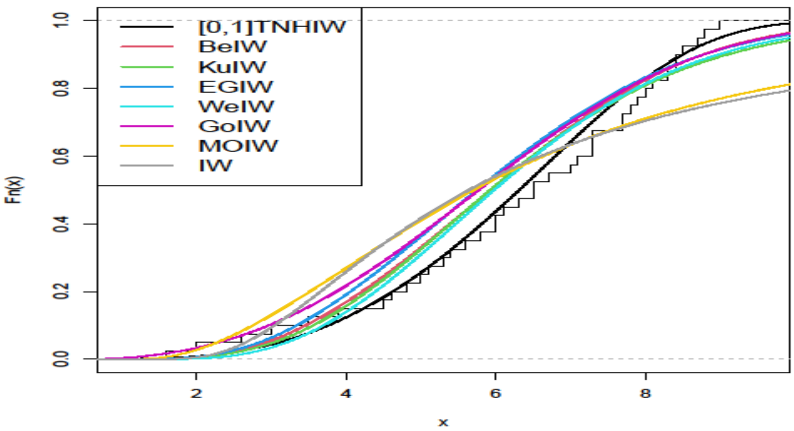

Figure (3): Estimated cumulative distribution function for data set. This figure finds by author by using R program

8. Conclusions: The flexibility of generalized models in modelling varying datasets remains a strong motivation for developing new families of distributions. The study developed a new family of distribution called the [0,1] TNH-G family. Statistical properties such as moments, order statistics, entropies of the new family are derived. [0,1] TNH-IW distribution introduces as a special model of new family. Finally, we estimate parameters of the new family by MLE.

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