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Some Results on Best Approximations without star-shaped

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بعض النتائج عن التقارب الأفضل بدون Star-shaped

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الخلاصة

في هذا البحث سنقدم مبرهنتين في فضاء بناخ، الأولى عن وجود النقطة الصامدة في تطبيقات افينية متعددة القيم ابدالية غير توسعية ومن ثم نستخدمها في برهان نظرية حول وجود التقارب الأفضل.

Abstract

In this paper, we give two theorems in Banach space. The first one is about the existence of a fixed point for generalized affine mapping, and a class of relatively nonexpansive commutative mappings and then we use it in the proof of theorem about existence of best approximation.

1. Introduction

Recent results have shown that many fixed points and best approximat-ion theorems previously established for Banach spaces have analogues with non-expansive mapping .In [5] the authors gave the best approximation theorem on a star-shaped set. The purpose of this paper is to show that the best approximation theorem can be obtained in Banach space for set-valued mapping without star - shaped assumption.

2. Preliminaries

To prove our results, we give the definitions:

Definition 2.1.[3]. Let X be a normed space and let C be a non empty subset of X. Let $x_0 \in X$. An element $y \in C$ is called a **best approximation** to x_0, if:

$$\|x_0 - y\| = d(x_0, C) = \inf\{\|x_0 - z\| : z \in C\}.$$

Let D be the set of best C- approximations to x_0 , and so

$$D = \{ y \in C : \|x_0 - y\| = d(x_0, C) \}$$

Definition 2.2.[8]. Let X be a normed space. A set C in X is said to be **convex**, if $\lambda x + (1 - \lambda)y \in C$, whenever $x, y \in C$ and $0 \leq \lambda \leq 1$.

A set C in X is said to be **star-shaped**, if there exists at least one point $q \in C$ such that the line segment $[x, q]$ joining x to q is contained in C for all $x \in C$ (that is $\lambda x + (1 - \lambda)q \in C$, for all $x \in C$ and $0 < \lambda < 1$). In this case, q is called the **star-center** of C .

Each convex set is star-shaped with respect to each of its points, but not conversely.

Definition 2.3.[7]. Let C be convex subset of normed linear space X. Then self-mapping T of C is said to be **affine** if:

$$T(\lambda x + (1 - \lambda)y) = T(\lambda x) + T((1-\lambda)y),$$

for all $x, y \in C$ and $\lambda \in (0, 1)$.
We give the definition providing the notion of affine with respect to a point , which is a generalization of an affine mapping , introduced by [8].

Definition 2.4.[7]. Let C be a nonempty, convex subset of normed linear space X, and let $q \in C$. A self-mapping T of C is said to be **affine with respect to q** if:

$$T(\lambda x + (1 - \lambda)q) = T(\lambda x) + T((1-\lambda)q),$$

for all $x \in C$ and $\lambda \in (0, 1)$.
The following example shows that an affine mapping with respect to a point need not be affine.

Example 2.1. [7] Let $X = \mathbb{R}$ and let $C = [0, 1]$. Define T on C by :

$$Tx = \begin{cases} 1, \text{ if } x \in [0,1), \\ 0, \text{ if } x = 1. \end{cases}$$

Then we have

$$\mathrm{T}(\lambda x + (1-\lambda)1/2) = \begin{cases} 1, \text{if } x \in [0,1) \text{and } \lambda \in (0,1), \\ 0, \text{if } x = 1 \text{and } \lambda = 1. \end{cases}$$

If $x \in [0, 1)$ and $\lambda \in (0, 1)$,then $T(x) = 1 = T(1/2)$.

$$T(\lambda x + (1 - \lambda)1/2) = 1 = \lambda T(x) + (1 - \lambda)T(1/2).$$

If $x = 1$ and $\lambda = 1$, then

$$T(\lambda x + (1 -\lambda)1/2) = 0 = \lambda T(x) + (1 -\lambda)(1/2).$$

Therefore T is an affine with respect to 1/2. If $x = 1$ and $\lambda = 1/2$, then

$$T(\lambda x + (1 -\lambda)1/2) = T(3/4) = 1 \neq 1/2 = \lambda T(1) + (1 -\lambda)T(1/2).$$

Hence , T is not affine.

Definition 2.5. [1] A map $T:C \to C$ ($C \subseteq X$) is said to be ***I*-contraction**, if there exists a self-map I on C and a real number $k\in (0, 1)$ such that

$$\| Tx - Ty \| \leq k \| Ix - Iy \|$$

for all $x, y \in C$.
If, in the above inequality, $k = 1$, then T is called ***I*-nonexpansive**.

Definition 2.6. [2] A map $T:C \to X$ is said to be demiclosed if and only if, whenever $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and $\{T x_n\}$ converges strongly to $y\in X$, then $Tx = y$.

Definition 2.7. [5] A pair of self-mapping (I, T) of a Banach space X is said to be **commutative** on C ($C \subset X$), if $ITx = TIx$ for all $x\in C$.

Throughout this paper, $F(T)$(resp.$F(I)$) denotes the set of fixed points of mapping T(resp.I).
We also use the following theorem, due to Hadzic[6]

Theorem 2.1. [6**]** Let $S, T : X \to X$ be continuous mappings and let $\wp$ be a family of self-mappings $A : X \to X$ such that :

1. $A(X) \subseteq S(X) \cap T(X)$, for all $A\in \wp$
2. A commutes with S and T, for all $A\in \wp$
3. $d(Ax, By) \leq q\, d(Sx, Ty)$,

for any $x, y \in X$ and for all $A, B\in \wp$ where $0 \leq q < 1$.Then, S, T and A have a unique, common fixed point in X for all $A\in \wp$.

3. Main Results

Throughout this section, X denotes a Banach space and C is a weakly compact subset of X. We have the following fixed point theorem for such a space:

Theorem 3.1. Let T and S be self – mappings of C and $\wp$ be a family of self – mappings $A : C \to C$ such that $A(X) \subseteq S(X) \cap T(X)$; A commutes with S and T. Let $p \in C$ such that $p \in F(T) \cap F(S)$; T and S are affine with respect to p and continuous in the weak topology. Let $k_n \in (0, 1)$ be a sequence with $\{k_n\} \to 1$ as n $\to \infty$, and for each n ≥ 1, for all $A \in \wp$ and for all $x \in C$, a mapping A_n defined by

$$A_n(x) = k_n Ax + (1 - k_n)T(p).$$

Then $F(T) \cap F(S) \cap F(A) \neq \emptyset$, for all $A \in \wp$ provided T- A and $S - A$ are demiclosed if T, S, A and $B \in \wp$ satisfy

$$\|Ax - By\| \leq \| Sx - Ty\| \ldots\ldots\ldots\ldots\ldots\ldots(3.1)$$

for all $x \neq y \in C$.

Proof

Since T is affine mapping with respect to $p, A(C) \subset T(C)$ and $p \in F(T)$, then

$$A_n(x) = k_n Ax + (1 - k_n)T(p)) \in \mathrm{T(C)}\ .$$

Hence $A_n(C) \subset T(C)$ for each n.

Since $p \in F(T)$, then
$A_n(x) = k_n Ax + (1 - k_n)T(p) = A_n(x) = k_n Ax + (1 - k_n)p.$
Since T is affine with respect to p and A and T are commutative, we have

$$\begin{aligned} A_nT(x) &= k_n TAx + (1 - k_n)\, p \\ &= k_n TAx + (1 - k_n)AT\, p \\ &= T(k_n Ax + (1 - k_n)\, p) \\ &= TA_n x, \end{aligned}$$

for each $x \in C$.Thus A_n and T are commutative for each n and $A_n(C) \subseteq C = T(C)$. Similarly, we can prove A_n and S are commutative for each n and $A_n(C) \subseteq C = S(C)$.Therefore $A(C) \subseteq S(C) \cap T(C)$. Similarly, we can define B_n for $B \in \wp$. Moreover,

$$\begin{aligned} \|A_n x - B_n y\| &= \| k_n Ax + (1 - k_n)p - k_n Bx + (1 - k_n)p\| \\ &= \| k_n Ax - k_n Bx \| \\ &= k_n \| Ax - Bx \| \\ &\leq \| Sx - Ty\|, \end{aligned}$$

for all x, $y \in C$. Furthermore, C is completed since the weak topology is Hausdorff and C is weakly compact . Therefore, maps $A \in \wp$ and T, S satisfy all the conditions of Theorem 2.1,which guarantees that $F(T) \cap F(S) \cap F(A) = \{x_n\}$ for each n and for all $A \in \wp$. Since C is weakly compact, there exist a subsequ-ence $\{x_m\}$ of $\{x_n\}$such that $x_m \xrightarrow{w} z \in C$ as m $\to \infty$. By the weak continuity of T, we have $z \in F(T)$. Also,

$$\begin{aligned}(T - A)x_m &= Tx_m - Ax_m \\ &= Tx_m - ((k_m)^{-1}Ax_m - ((k_m)^{-1} - 1)p) \\ &= x_m - (k_m)^{-1}x_m + ((k_m)^{-1} - 1)p) \\ &= (1 - (k_m)^{-1})x_m - (1 - (k_m)^{-1})p \\ &= (1 - (k_m)^{-1})(x_m - p)\end{aligned}$$

This implies that $(T - A)x_m \to 0$ strongly as m$\to \infty$, since $\{x_m\}$ is bounded and $k_m \to 1$ as m$\to \infty$.Now, demiclosedness of $T - A$ guarantees that $(T - A)z = 0$, that is, $Tz = Az$. Similarly, we can show $Az = Sz$, when $S - A$ is demiclosed. Hence,

$$F(T) \cap F(S) \cap F(A) \neq \emptyset$$

for all $A \in \wp$, this completes the proof. ■

We give an application to best approximation

Theorem3.2. Let T and S be self – mappings of C and let $\wp$ be a family of self – mappings $A : C \to C$. Let $A(\partial C) \subseteq C$ and $x_0 \in F(T) \cap F(S) \cap F(A)$. Let D be nonempty, weakly compact subset of X and $p \in D$ such that $p \in F(T) \cap F(S)$; T and S are affine with respect to p and continuous in the weak topology.
Let $k_n \in (0, 1)$ be a sequence with $\{k_n\} \to 1$ as n $\to \infty$, and for each n ≥ 1, for all $A \in \wp$ and for all $x \in C$, a mapping A_n defined by

$$A_n(x) = k_nAx + (1 - k_n)T(p).$$

Then $F(T) \cap F(S) \cap F(A) \neq \emptyset$, for all $A \in \wp$ provided T- A and $S - A$ are demiclosed if T, S ,A and $B \in \wp$ satisfy

$$\|Ax - By\| \leq \| Sx - Ty\| \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.1)$$

for all $x \neq y \in C$,

on D, and $T(D) = D = S(D)$. Let $k_n \in (0, 1)$ be a sequence with $\{k_n\} \to 1$ as n $\to \infty$ and for each n ≥ 1 , for all $A \in \wp$ and for all $x \in D$, a mapping A_n defind by

$$A_n(x) = k_nAx + (1 - k_n)T(p).$$

Then $D \cap F(T) \cap F(S) \cap F(A) \neq \emptyset$, for all $A \in \wp$,provided T- A and $S - A$ are demiclosed If $A \in \wp$ commutes with T and S on D and satisfy for all $x \in D \cup \{x_0\}$,

$$\|Ax - By\| \leq \| Sx - Ty\|.$$

Proof. Let $y \in D$, then $y \in \partial D$ and so $Ay \in C$, because $A(\partial C) \subseteq C$. Now, since $Tx_0 = Sx_0 = x_0 = Ax_0$, we have

$$\| Ay - x_0 \| = \| Ay - Bx_0 \| \leq \| Sy - Tx_0 \| = \| Sy - x_0 \| = \mathrm{d}(x_0, C).$$

This shows that $Ay \in D$. Consequently, $T(D) = S(D) = D = A(D)$ for all $A \in \wp$. Now, Theorem guarantees that

$$D \cap \mathrm{F}(T) \cap \mathrm{F}(S) \cap \mathrm{F}(A) \neq \emptyset,$$

for all $A \in \wp$.This completes the proof. ■

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