GENERALIZED JORDAN LEFT DERIVATION ON SOME GAMMA RING

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ABSTRACT

In this paper we define a Generalized Jordan left derivation on Γ-ring and show that the existence of a non-zero generalized Jordan left derivation D on a completely prime Γ-ring implies D is a generalized left derivation .Furthermore we show that every generalized Jordan left derivation on Γ-ring has a commutator left non-zero divisor is a generalized left derivation on Γ-ring.

Key wards : Γ-ring, prime Γ-ring, prime Γ-ring , left derivation, ,Jordan left derivation, generalized Jordan left derivation .

1-INTRODUCTION

An additive mapping d:R→R is called a left derivation (resp.,Jordan left derivation)if d(xy)=xd(y)+yd(x)(resp.,d(x^2)=2xd(x)) holds for all x,y$\in$R.Clearly, every left derivation is a Jordan left derivation,Thus ,it is natural to ask that :whether every Jordan left derivation on a ring is a left derivation? In [1] authors answered the above question in case the underlying ring R is 2-torsion free and prime and in [5] Rajaa c.Shaheen,answered the above question in case the underlying ring R is a 2-torsion free and has a commutator left non-zero divisor and define the concept of generalized Jordan left derivation and Generalized left derivation as follows:

Let R be a ring,and let δ:R→R be an additive map,if there is a left derivation (resp.,Jordan left derivation)d: R→R such that δ(xy)=x δ(y)+yd(x) (resp., δ(x^2)=x δ(x)+xd(x) for all x,y$\in$R,then δ is called a generalized left derivation and d is called the relating left derivation(resp., then δ is called a generalized Jordan left derivation and d is called the relating Jordan left derivation).and study the same problem .In this paper we define a generalized Jordan left derivation and we show that the existence of a non-zero generalized Jordan left derivation D:M→M on a completely prime Γ-ring M which satisfy the condition xα yβ z=xβ yα z for all x,y,z$\in$M and $\alpha,\beta\in\Gamma$ implies D is a generalized left derivation and *we show that every generalized Jordan left derivation on Γ-ring has a commutator left non-zero divisor is a generalized left derivation.*

Let M and Γ be additive abelian groups, M is called a Γ-ring if for any x,y,z $\in$M and α, $\beta\in\Gamma$,the following conditions are satisfied

(1) $x\alpha y \in M$
(2)$(x+y)\alpha z=x\alpha z+y\alpha z$
$x(\alpha+\beta)z=x\alpha z+x\beta z$
$x\alpha(y+z)=x\alpha y+x\alpha z$
(3)$(x\alpha y)\beta z=x\alpha(y\beta z)$

The notion of Γ-ring was introduced by Nobusawa[4] and generalized by Barnes[2],many properties of Γ-ring were obtained by many researcheres.

M is called a 2-torsion free if 2x=0 implies x=0 for all $x\in M$.A Γ-ring M is called prime if $a\Gamma M\Gamma b=0$ implies a=0 or b=0 and M is called completely prime if $a\Gamma b=0$ implies a=0 or b=0($a,b\in M$),Since $a\Gamma b\Gamma a\Gamma b\subset a\Gamma M\Gamma b$,then every completely prime Γ-ring is prime. In [3] Y.Ceven defined a Jordan left derivation as follows

Definition 1.1 **:-Let M be a Γ-ring and let** ***$d:M\rightarrow M$ be an additive map.d is called a Left derivation if for any $a,b\in M$ and $\alpha\in\Gamma$,***

$d(a\alpha b)=a\alpha d(b)+b\alpha d(a)$,

d is called a Jordan left derivation if for any $a\in M$ and $\alpha\in\Gamma$,

$d(a\alpha a)=2a\alpha d(a)$.

In this paper ,we generalized the above definition by giving the following definition

Definition 1.2:- Let M be a Γ-ring and let $D:M\rightarrow M$ ***be an additive map.Then D is called a Generalized left derivation if there exist a left derivation*** $d:M\rightarrow M$ such that

$D(a\alpha b)=a\alpha D(b)+b\alpha d(a)$,for all $a,b\in M$ ***and $\alpha\in\Gamma$,***

Definition 1.3:- Let M be a Γ-ring and let $D:M\rightarrow M$ be an additive map.Then D is ***called a Generalized Jordan left derivation if there exist a Jordan left derivation*** $d:M\rightarrow M$ such that

$D(a\alpha a)=a\alpha D(a)+a\alpha d(a)$,for all $a\in M$ ***and $\alpha\in\Gamma$,***

2. RESULT

Lemma 2.1:- Let M be a Γ-ring, $D:M\rightarrow M$ ***be a Generalized Jordan left derivation and*** $d:M\rightarrow M$ ***be the relating Jordan left derivation then the following statements hold:***

(i) ***$D(a\alpha b+b\alpha a)=a\alpha D(b)+b\alpha D(a)+a\alpha d(b)+b\alpha d(a)$ for all*** $a,b\in M$ ***and $\alpha\in\Gamma$,***

Especially if M is 2-torsion free and $a\alpha b\beta c=a\beta b\alpha c$, for all $a,b,c\in M$ ***and $\alpha,\beta\in\Gamma$, then***

(ii) ***$D(a\alpha b\beta a)=a\alpha b\beta D(a)+2a\alpha b\beta d(a)+a\alpha a\beta d(b)-b\alpha a\beta d(a)$***

(iii) ***$D(a\alpha b\beta c+c\alpha b\beta a)=a\alpha b\beta D(c)+c\alpha b\beta D(a)+2a\alpha b\beta d(c)+2c\alpha b\beta d(a)+a\alpha c\beta d(b)+c\alpha a\beta d(b)-b\alpha a\beta d(c)-b\alpha c\beta d(a)$.***

Proof:-(i)Since D is a Generalized Jordan left derivation then

$D(a\alpha a)=a\alpha D(a)+a\alpha d(a)$,for all $a\in M$ ***and $\alpha\in\Gamma$,……………(1)***

By linearizing (1),we get for all $a,b\in M$ ***and $\alpha\in\Gamma$,***

$D(a\alpha b+b\alpha a)=a\alpha D(b)+b\alpha D(a)+a\alpha d(b)+b\alpha d(a)$,…………….(2)

(ii)In (2) replace b by $a\beta b+b\beta a$, $\beta \in \Gamma$

$W= D(a\alpha (a\beta b+b\beta a) +(a\beta b+b\beta a) \alpha a)$

$=a\alpha D(a\beta b+b\beta a)+(a\beta b+b\beta a) \alpha D(a)+$

$a\alpha d(a\beta b+b\beta a)+(a\beta b+b\beta a) \alpha d(a)$

$=a\alpha a\beta D(b)+a \alpha b \beta D(a)+a \alpha a \beta d(b)+a \alpha b \beta d(a)+a \beta b \alpha D(a)+b \beta a \alpha$

$D(a)+2a \alpha a \beta d(b)+2a \alpha b \beta d(a)+a \beta b \alpha d(a)+b \beta a \alpha d(a)$

on the other hand,

$W= D(a\alpha (a\beta b+b\beta a) +(a\beta b+b\beta a) \alpha a)$

$=D(a \alpha a \beta b+2a \alpha b \beta a+b \beta a \alpha a)$

$= D(a \alpha a \beta b+ b \beta a \alpha a)+2D(a \alpha b \beta a)$

$= a \alpha a \beta D(b)+b \beta D(a \alpha a)+ a \alpha a \beta d(b)+b \beta d(a \alpha a) +2D(a \alpha b \beta a)$

$=a \alpha a\beta D(b)+b\beta a\alpha D(a)+b\beta a\alpha d(a)+ a \alpha a\beta d(b)+2b\beta a\alpha d(a)+2D(a\alpha b\beta a)$

then by comparing these two expression of W and by using the fact of 2-torsion free ring ,we get

$D(a\alpha b\beta a)=a\alpha b\beta D(a)+2a\alpha b\beta d(a)+a\alpha a\beta d(b)-b\alpha a\beta d(a)$………..(3

(iii)by linearizing (3) we find that

$D(a\alpha b\beta c+c\alpha b\beta a)=a\alpha b\beta D(c)+c\alpha b\beta D(a)+2a\alpha b\beta d(c)+$

$2c\alpha b\beta d(a)+a\alpha c\beta d(b)+c\alpha a\beta d(b)-b\alpha a\beta d(c)-b\alpha c\beta d(a)$…….(4)

Now we shall give the following lemma which is necessary to prove [lemma 2.3]

Lemma 2.2:-let M be a 2-torsion free Γ-ring and $d:M\rightarrow M$ is a Jordan left derivation on M and $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha , \beta \in \Gamma$ then

$(a \alpha a \alpha b-2a \alpha b \alpha a+b \alpha a \alpha a) \beta d(b)=0.$

Proof:- From [3 ,Lemma 2.2,i],we have

$(a \alpha a \alpha b-2a \alpha b \alpha a+b \alpha a \alpha a) \beta d(a)=0$…………….(5)

by replacing a by a+b in (5),we find that

$((a+b) \alpha (a+b)\alpha b-2(a+b)\alpha b\alpha (a+b)+b\alpha (a+b)\alpha (a+b)) \beta d(a+b)=0$

$(a\alpha a\alpha b+b\alpha a\alpha b+a\alpha b\alpha b+b\alpha b\alpha b-2a\alpha b\alpha a-2a\alpha b\alpha b-2b\alpha b\alpha a-2b\alpha b \alpha b+b \alpha a \alpha a+b \alpha b \alpha b+b \alpha a \alpha b+b \alpha b \alpha a) \beta (d(a)+d(b))=0$

$(a \alpha a \alpha b+2b \alpha a \alpha b-2a \alpha b \alpha a-a \alpha b \alpha b-b \alpha b \alpha a+b \alpha a \alpha a) \beta (d(a)+d(b))=0$

$((a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)-(b\alpha b\alpha a-2b\alpha a\alpha b+a\alpha b\alpha b)) \beta d(a)+$

$((a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)-(b\alpha b\alpha a-2b\alpha a\alpha b+a\alpha b\alpha b)) \beta d(b)=0$

and since $(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(a)=0$, we get

$-(b\alpha b\alpha a-2b\alpha a\alpha b+a\alpha b\alpha b)\beta d(a)+(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0$ …………..(6)

since

$0=d([a,b]\beta[a,b])$

$=d(a\beta(b\alpha a\alpha b)+(b\alpha a\alpha b)\beta a)-d(a\alpha(b\alpha b)\alpha a)-d(b\alpha(a\alpha a)\alpha b)$

$=2(a\beta d(b\alpha a\alpha b)+(b\alpha a\alpha b)\beta d(a))-a\alpha a\beta d(b\alpha b)-3a\alpha b\alpha b\beta d(a)+b\alpha b$

$\alpha a\beta d(a)-b\alpha b\beta d(a\alpha a)-3b\alpha a\alpha a\beta d(b)+a\alpha a\alpha b\beta d(b)$

$=-3(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)-(a\alpha b\alpha b-2b\alpha a\alpha b+b\alpha b\alpha a)\beta d(a)$

and hence

$(a\alpha b\alpha b-2b\alpha a\alpha b+b\alpha b\alpha a\beta d(a)+3(a\alpha a\alpha b-a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0$

………………………(7)

from(6) and (7) ,we get

$4(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0$

and since M is 2-torsion free Γ-ring,then

$(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0$◙

***Lemma2.3**:- Let M be a 2-torsion free Γ-ring which satisfy the condition $a\alpha b\beta c=a\beta b\alpha c$, for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$, then $D:M\to M$ be a Generalized Jordan left derivation and $d:M\to M$ be the relating Jordan left derivation then*

$[a,b]\beta(D(b\alpha a)-b\alpha D(a)-a\alpha d(b))=0$

***Proof** :- In (4) replace c by $[a,b]=a\alpha b-b\alpha a$*

$Y=D(a\alpha b\beta[a,b]+[a,b]\alpha b\beta a)$

$=a\alpha b\beta D([a,b])+[a,b]\alpha b\beta D(a)+2a\alpha b\beta d([a,b])+2[a,b]\alpha b\beta d(a)+a\alpha[a,b]\beta d(b)+$

$[a,b]\alpha a\beta d(b)-b\alpha a\beta d([a,b])-b\alpha[a,b]\beta d(a)$

$=a\alpha b\beta D(a\alpha b)-a\alpha b\beta D(b\alpha a)+[a,b]\alpha b\beta D(a)+2a\alpha b\beta d([a,b])+$

$2[a,b]\alpha b\beta d(a)+a\alpha[a,b]\beta d(b)+[a,b]\alpha a\beta d(b)-b\alpha a\beta d([a,b])-b\alpha[a,b]\beta d(a)$

since $[a,b]\beta d([a,b])=0$ from [3,lemma 2.2,(iii)]

$Y=a\alpha b\beta D(a\alpha b)-a\alpha b\beta D(b\alpha a)+[a,b]\alpha b\beta D(a)+a\alpha b\beta d([a,b])+2[a,b]\alpha b\beta d(a)-$

$b\alpha[a,b]\beta d(a)+a\alpha[a,b]\beta d(b)+[a,b]\alpha a\beta d(b)$

On the other hand,

$Y=D(a\alpha b\beta[a,b]+[a,b]\alpha b\beta a)$

$=D(a\alpha b\beta a\alpha b-a\alpha b\beta b\alpha a+a\alpha b\beta b\alpha a-b\alpha a\beta b\alpha a)$

$=D(a\alpha b\beta a\alpha b)-D(b\alpha a\beta b\alpha a)$

$= a\alpha b\beta D(a\alpha b)+ a\alpha b\beta d(a\alpha b)- b\alpha a\beta D(b\alpha a)+ b\alpha a\beta d(b\alpha a)$

Then by comparing these two expression of Y and since from [3,Lemma2.2],we have

$[a,b]\beta d(a\alpha b)=a\alpha [a,b]\beta d(b)+b\alpha [a,b]\beta d(a)$……………….(8)

then we get

$-[a,b]\beta (D(b\alpha a)-b\alpha D(a)-a\alpha d(b))+a\alpha b\beta d([a,b])+2[a,b]\alpha b\beta d(a)-b\alpha [a,b]$
$\beta d(a)+[a,b]\beta d(a\alpha b)-b\alpha [a,b]\beta d(a)-a\alpha b\beta d(a\alpha b)+b\alpha a\beta d(b\alpha a)=0$

then

$-[a,b]\beta (D(b\alpha a)-b\alpha D(a)-a\alpha d(b))+a\alpha b\beta d([a,b])+2[a,b]\alpha b\beta d(a)-2b\alpha [a,b]$
$\beta d(a)+ a\alpha b\beta d(a\alpha b) -b\alpha a\beta d(a\alpha b)- a\alpha b\beta d(a\alpha b) +b\alpha a\beta d(b\alpha a)=0$

$-[a,b]\beta (D(b\alpha a)-b\alpha D(a)-a\alpha d(b))+[a,b]\beta d([a,b])+2[a,b]\alpha b\beta d(a)$
$-2b\alpha [a,b]\beta d(a)=0$

by [3,Lemma2.2,iii],we have

$[a,b]\beta d([a,b])=0$ ***for all*** $a,b\in M$ ***and*** $\alpha,\beta\in\Gamma$.

Then

$-[a,b]\beta (D(b\alpha a)-b\alpha D(a)-a\alpha d(b))=2(-[a,b]\alpha b+b\alpha [a,b])\beta d(a)$

$=2(b\alpha [a,b]-[a,b]\alpha b)\beta d(a)$

since

$(b\alpha [a,b]-[a,b]\alpha b)\beta d(a)=[a,b]\beta d(a\alpha b)-a\alpha [a,b]\beta d(b)-[a,b]\alpha b\beta d(a)$

$=[a,b]\beta (d(a\alpha b)-b\alpha d(a))- a\alpha [a,b]\beta d(b)$

$= [a,b]\beta a\alpha d(b)-a\alpha [a,b]\beta d(b)$

$=([a,b]\alpha a-a\alpha [a,b])\beta d(b)$

then

$-[a,b]\beta (D(b\alpha a)-b\alpha D(a)-a\alpha d(b))=-2(a\alpha [a,b]- [a,b]\alpha a)\beta d(b)$

$=-2(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)$…..(9)

so by [Lemma 2.2],we get

$[a,b]\beta (D(b\alpha a)-b\alpha D(a)-a\alpha d(b))=0$………………..(10)

<u>Theorem2.4</u>:- let M be a 2-torsion free Γ***-ring has a commuator left non-Zero divisor and*** $a\alpha b\beta c=a\beta b\alpha c$ ***for all*** $a,b,c\in M$ ***and*** $\alpha,\beta\in\Gamma$ ***and*** $D:M\to M$ ***is a Generalized Jordan left derivation on M*** and $d:M\to M$ ***is the relating Jordan left derivation Then D is a Generalized left derivation on M.***

<u>Proof</u>:- if we suppose that

G(b,a)= *D(b α a)-b α D(a)-a α d(b)*

Then (10) becomes

[a,b] β G(b,a)=0 for all a,b$\in$M and α , $\beta \in \Gamma$ …………(11)

Since M has a commutator left non-Zero divisor then $\exists$ *x,y $\in$M and α , $\beta \in \Gamma$ such that*

[x,y] β c=0 implies that c=0

It is easy to see that from (11),

[x,y] β G(y,x)=0 and so G(y,x)=0

In (11) replace a by a+x,then we get

[x,b] β G(b,a)+ [a,b] β G(b,x)=0……………………(12)

replace b by b+y in (12),then we get

[x,y] β G(b,a)=0 $\Rightarrow$ G(b,a)=0

i.e D(b α a)=b α D(a)+a α d(b)

$\Rightarrow$ *D is a Generalized left derivation on M.*

Theorem 2.5:- *let M be a 2-torsion free completely prime Γ-ring and aα bβ c=aβ bα c for all a,b,c$\in$M and α , $\beta \in \Gamma$ and D:M→M is a Generalized Jordan left derivation on M .Then D is a Generalized left derivation on M.*

Proof*:- Since[a,b] β G(b,a)=0 for all a,b$\in$M and α , $\beta \in \Gamma$*

And since M is completely prime Γ-ring,then either[a,b]=0 or G(b,a)=0

If [a,b]=0 for all a,b$\in$M and $\alpha \in \Gamma$ $\Rightarrow$M is commutative and so D is a Generalized left derivation on M .if G(b,a)=0 then D is a Generalized left derivation on M.

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