Strong Convergence Theorems of Ishikawa Iteration Process with Errors in Banach Space

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Abstract

The purpose of this paper is to study the strong convergence of the sequence of the Ishikawa iteration methods with errors to fixed points and solutions of local strongly accretive and local strongly pseudo-contractive operators in the framework of Banach spaces. Our main results is to improve and extend some results about Ishikawa iteration for type of contractive, announced by many others.

مبرهنات التقارب القوي لتكرار إشيكاوا ذات الخطأ في فضاء بناخ

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الخلاصة

الغرض من هذا البحث هو دراسة التقارب القوي لمتتابعة طرق التكرار لاشيكاوا ذات الخطأ للنقاط الصامدة وحلول للتطبيقات المتزايدة القوية المحلية والتطبيقات الانكماشية الكاذبة القوية المحلية في اطار فضاءات بناخ. إحدى نتائجنا هي تحسين وتوسيع بعض النتائج حول تكرار إشيكاوا لانماط من الانكماشية المعلنة عند آخرين كثيرين.

1. Introduction

Let X be an arbitrary Banach space with norm $\|\cdot\|$ and the dual space X*. The normalized duality mapping $J:X \longrightarrow 2^X$ is defined by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}$$

where $\langle \cdot,\cdot \rangle$ denotes the generalized duality pairing. It is known that if X is uniformly smooth, then J is single valued and uniformly continuous on any bounded subset of X, we shall denote single-valued normalized duality mapping by i.

Let $T:D(T) \subseteq X \longrightarrow X$ be an operator, where D(T) and R(T) denote the domain and range of T, respectively, and I denote the identity mapping on X.

We recall the following two iterative processes to Ishikawa and Mann, [1], [2]:

i- Let K be a nonempty convex subset of X, and $T:K \longrightarrow K$ be a mapping, for any given $x_0 \in K$ the sequence $\langle x_n \rangle$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n \qquad (n \geq 0)$$

is called Ishikawa iteration sequence, where $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ are two real sequences in [0,1] satisfying some conditions.

ii- In particular, if $\beta_n = 0$ for all $n \geq 0$ in (i), then $\langle x_n \rangle$ defined by

$x_0 \in K$, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$, $n \geq 0$

is called the Mann iteration sequence.

Recently Liu in [3] introduced the following iteration method which he called Ishikawa (Mann) iteration method with errors.

For a nonempty subset K of X and a mapping $T:K \longrightarrow X$, the sequence $\langle x_n \rangle$ defined for arbitrary x_0 in K by

$y_n = (1 - \beta_n)x_n + \beta_n Tx_n + v_n$,

$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n + u_n$ for all n = 0, 1,2,…,

where $\langle u_n \rangle$ and $\langle v_n \rangle$ are two summable sequences in X (i.e., $\sum_{n=0}^{\infty} \|u_n\| < \infty$ and $\sum_{n=0}^{\infty} \|v_n\| < \infty$), $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ are two real sequences in [0,1], satisfying suitable conditions, is called the Ishikawa iterates with errors. If $\beta_n = 0$ and $v_n = 0$ for all n, then the sequence $\langle x_n \rangle$ is called the Mann iterates with errors.

The purpose of this paper is to define the Ishikawa iterates with errors to fixed points and solutions of local strongly accretive and local strongly pseudo-contractive operators equations. Our main results improve and extend the corresponding results recently obtained by [3] .[4] and[5].

1.1 Definition: [5]

Let X be a real Banach space and $T:D(T) \subseteq X \longrightarrow X$ be an operator, then

1. T is said to be local strongly pseudo contractive if for each $x \in D(T)$ there exists $t_x > 1$ such that for all $y \in D(T)$ and $r > 0$
$$\|x - y\| \le \|(1 + r)(x - y) - rt_x(Tx - Ty)\| \qquad \ldots(1)$$
2. T is called local strongly accretive if for given $x \in D(T)$ there exists $k_x \in (0,1)$ such that for each $y \in D(T)$ there is $j(x - y) \in J(x - y)$ satisfying
$$\langle Tx - Ty, j(x - y)\rangle \ \ge\ k_x \|x - y\|^2 \qquad \ldots(2)$$
3. T is called strongly pseudo contractive (respectively, strongly accretive) if it is local strongly pseudo contractive (respectively, local strongly accretive) and $t_x \equiv t$ (respectively, $k_x \equiv k$) is independent of $x \in D(T)$.
4. T is said to be accretive for if given $x, y \in D(T)$ there is $j(x - y) \in J(x - y)$ satisfying
$\langle Tx - Ty, j(x - y)\rangle \ \ge\ 0$.

1.2 Remarks:

1. Each strongly pseudo contractive operator is local strongly pseudo contractive.

2. Each strongly accretive operator is local strongly accretive.

3. T is local strongly pseudo contractive if and only if I – T is local strongly accretive and $k_x = 1 - \frac{1}{t}$, where t_x and k_x are the constants appearing in (1) and (2) respectively, see [5].

The following lemma plays an important role in proving our main results.

1.1 Lemma: [1], [2]

Let X be a Banach space. Then for all $x, y \in X$ and $j(x + y) \in J(x + y)$,
$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

2. Main Results

Now, we state and prove the following theorems:

2.1 Theorem:

Let X be a uniformly smooth Banach space and $T:X \longrightarrow X$ be a local strongly accretive mapping, i.e.
$$\langle Tx - Ty, j(x - y)\rangle \ge k_x \|x - y\|^2 \qquad \ldots(3)$$
where $k_x \in (0,1)$, suppose that there exists a solution of the equation $Tx = f$ for some $f \in X$. For $f \in X$ define $S: X \longrightarrow X$ by $Sx = f + x - Tx$ for all $x \in X$, and

suppose that R(S) is bounded. Let $x_0 \in K$ the Ishikawa iteration sequence $<x_n>$ with errors is defined by

$y_n = (1 - \beta_n)x_n + \beta n S x_n + b_n v_n$ …(4)

$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + a_n u_n$ for all n = 0, 1,2,… . …(5)

where $<\alpha_n>$, $<\beta_n>$, $<a_n>$ and $<b_n>$ are sequences in [0,1] satisfying

$\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$ …(6)

$\sum_{n=0}^{\infty} \alpha_n = \infty$ …(7)

$a_n \le \alpha_n^{1+c}$, c > 0, $b_n \le \beta_n$ …(8)

and $<u_n>$ and $<v_n>$ are two bounded sequence in X, then $<x_n>$ converges strongly to the unique solution of the equation Tx = f.

proof: Let Tw = f, so that w is a fixed point of S. Since T is local strongly accretive, it follows from definition of S that

$<Sx - Sy, j(x - y)> \le \|x - y\|^2 - k_x \|x - y\|^2$ …(9)

Setting y = w, we have

$<Sx - Sw, j(x - w)> \le \|x - w\|^2 - k_x \|x - w\|^2$ …(10)

If z is a fixed point of S, then (10) with x = z implies w = z.

We prove that $<x_n>$ and $<y_n>$ are bounded. Let

$A = \sup\{\|Sx_n - w\| + \|Sy_n - w\| : n \ge 0\} + \|x_0 - w\|$

$B = \sup\{\|u_n\| + \|v_n\| : n \ge 0\}$

M = A + B

From (5) and (8), we get

$$\|x_{n+1} - w\| \le (1 - \alpha_n)\|x_n - w\| + \alpha_n \|Sy_n - w\| + a_n \|u_n\|$$
$$\le (1 - \alpha_n)\|x_n - w\| + \alpha_n A + \alpha_n B$$

Hence

$\|x_{n+1} - w\| \le (1 - \alpha_n)\|x_n - w\| + \alpha_n M$ …(11)

Now, from (4) and (8), we have

$$\|y_n - w\| \le (1 - \beta_n)\|x_n - w\| + \beta_n \|Sx_n - w\| + b_n \|v_n\|$$
$$\le (1 - \beta_n)\|x_n - w\| + \beta_n A + \beta_n B$$

Then

$\|y_n - w\| \le (1 - \beta_n)\|x_n - w\| + \beta_n M$ …(12)

$\|x_n - w\| \le M$ …(13)

Now, we show by induction that

for all n ≥ 0. For n = 0 we have $\|x_0 - w\| \le A \le M$, by definition of A and M.

Assume now that $\|x_n - w\| \le M$ for some $n \ge 0$. Then by (11), we have

$$\|x_{n+1} - w\| \le (1-\alpha_n)\|x_n - w\| + \alpha_n M$$
$$\le (1-\alpha_n)M + \alpha_n M = M.$$

Therefore, by induction we conclude that (13) holds substituting (13) into (12), we get

$$\|y_n - w\| \le M \qquad \ldots(14)$$

From (12), we have

$$\|y_n - w\|^2 \le (1-\beta_n)^2\|x_n - w\|^2 + 2\beta_n(1-\beta_n)M\|x_n - w\| + \beta_n^2 M^2$$

Since $1 - \beta_n \le 1$ and $\|x_n - w\| \le M$, we get

$$\|y_n - w\|^2 \le \|x_n - w\|^2 + 2\beta_n M^2 \qquad \ldots(15)$$

Using lemma (1.1), we get

$$\|x_{n+1} - w\|^2 \le \|(1-\alpha_n)(x_n - w) + a_n u_n + \alpha_n(Sy_n - w)\|^2$$
$$\le \|(1-\alpha_n)(x_n - w) + a_n u_n\|^2 + 2\alpha_n < Sy_n - w, j(x_{n+1} - w) >$$
$$\le (1-\alpha_n)^2\|x_n - w\|^2 + 2(1-\alpha_n)a_n\|x_n - w\|\|u_n\| + a_n^2\|u_n\|^2 +$$
$$2\alpha_n < Sy_n - w, j(y_n - w) > + 2\alpha_n < Sy_n - w, j(x_{n+1} - w) - j(y_n - w) >$$

Hence, using (9) and definition of M, we get

$$\|x_{n+1} - w\|^2 \le \|x_n - w\|^2 - 2\alpha_n\|x_n - w\|^2 + \alpha_n^2\|x_n - w\|^2 + 2(1-\alpha_n)a_n M^2 + a_n^2 M^2 +$$
$$2\alpha_n\|y_n - w\|^2 - 2\alpha_n k_x\|y_n - w\| + 2\alpha_n c_n$$

where

$$c_n = < Sy_n - w, j(x_{n+1} - w) - j(y_n - w) > \qquad \ldots(16)$$

By (13) and (15) and using that $a_n \le \alpha_n \alpha_n^c$, we obtain

$$\|x_{n+1} - w\|^2 \le \|x_n - w\|^2 - 2\alpha_n\|x_n - w\|^2 + \alpha_n^2 M^2 + 2\alpha_n\alpha_n^c M^2 + 2\alpha_n\|x_n - w\|^2 +$$
$$4\alpha_n\beta_n M^2 - 2\alpha_n k_x\|y_n - w\|^2 + 2\alpha_n c_n$$

and hence

$$\|x_{n+1} - w\|^2 \le \|x_n - w\|^2 - 2\alpha_n k_x\|y_n - w\|^2 + \alpha_n\lambda_n \qquad \ldots(17)$$

where $\lambda_n = (\alpha_n + 2\alpha_n^c + 4\beta_n)M^2 + 2c_n$.

First we show that $c_n \longrightarrow 0$ as $n \longrightarrow \infty$, observe that from (4) and (5), we have

$$\|x_{n+1} - y_n\| \le \|(\beta_n - \alpha_n)(x_n - w) + \alpha_n(Sy_n - w) - \beta_n(Sx_n - w) + a_n u_n - b_n v_n\|$$
$$\le (\beta_n - \alpha_n)\|x_n - w\| + \alpha_n\|Sy_n - w\| + \beta_n\|Sx_n - w\| + \alpha_n\|u_n\| + \beta_n\|v_n\|$$

and hence, by (13) and definition of M.

$$\|x_{n+1} - y_n\|^2 \le (3\beta_n + \alpha_n)M \qquad \ldots(18)$$

Therefore $\|x_{n+1} - w - (y_n - w)\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Since $<x_{n+1} - w>$, $< y_n - w>$ and $<Sy_n - w>$ are bounded and j is uniformly continuous on any bounded subset of X, we have

$j(x_{n+1}-w)-j(y_n-w) \longrightarrow 0$ as $n \longrightarrow \infty$, $c_n = < Sx_n - w, j(x_{n+1} - w) - j(y_n - w) > \longrightarrow 0$ as $n \longrightarrow \infty$. Thus

$$\lim_{n\to\infty} \lambda_n = 0,$$

$$\inf\{\|y_n - w\|^2 : n \ge 0\} = S \ge 0.$$

We prove that S = 0. Assume the contrary, i.e., S > 0. Then $\|y_n - w\|^2 \ge S > 0$ for all $n \ge 0$.

Hence

$k_x(\|y_n - w\|^2) \ge k_x(S) > 0$ where $k_x \in (0,1)$.

Thus from (17)

$$\|x_{n+1} - w\|^2 \le \|x_n - w\|^2 - \alpha_n k_x(S) - \alpha_n [k_x(S) - \lambda_n] \quad \ldots(19)$$

for all $n \ge 0$. Since $\lim_{n\to\infty} \lambda_n = 0$, there exists a positive integer n_0 such that $\lambda_n \le k_x(S)$ for all $n \ge n_0$.

Therefore, from (19), we have

$$\|x_{n+1} - w\|^2 \le \|x_n - w\|^2 - \alpha_n k_x(S),$$

or

$\alpha_n k_x(S) \le \|x_n - w\|^2 - \|x_{n+1} - w\|^2$ for all $n \ge n_0$.

Hence

$$k_x(S) \cdot \sum_{j=n_0}^{n} \alpha_j = \|x_{n_0} - w\|^2 - \|x_{n+1} - w\|^2 \le \|x_{n_0} - w\|^2,$$

which implies $\sum_{n=0}^{\infty} \alpha_n < \infty$, contradicting (7). Therefore, S = 0.

From definition of S, there exists a subsequence of $< \|y_n - w\| >$, which we will denote by $< \|y_i - w\| >$, such that

$$\lim_{j\to\infty} \|y_j - w\| = 0 \quad \ldots(20)$$

Observe that from (4) for all $n \ge 0$, we have

$$\begin{aligned}\|x_n - w\| &\le \|y_n - w + \beta_n(x_n - w) - \beta_n(Sx_n - w) + b_n v_n\| \\ &\le \|y_n - w\| + \beta_n\|x_n - w\| + \beta_n\|Sx_n - w\| + b_n\|v_n\|.\end{aligned}$$

Since $b_n \le \beta_n$, by definition of A, B and M we get

$$\|x_n - w\| \le \|y_n - w\| + 3\beta_n M, \text{ for all } n \ge 0 \quad \ldots(21)$$

Thus by (6), (21) and (20), we have

$$\lim_{j\to\infty} \|x_j - w\| = 0 \quad \ldots(22)$$

Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty}\lambda_n = 0$, there exists a positive integer N_0 such that

$\alpha_n \le \frac{\varepsilon}{3M}, \beta_n \le \frac{\varepsilon}{3M}, \lambda_n \le k_x\left(\frac{\varepsilon}{3}\right)^2$ for all $n \ge N_0$.

From (22), there exists $k \ge N_0$ such that

$$\|x_k - w\| < \varepsilon \qquad \ldots(23)$$

We prove by induction that

$$\|x_{k+n} - w\| < \varepsilon \text{ for all } n \ge 0 \qquad \ldots(24)$$

For n = 0 we see that (24) holds by (23).

Suppose that (24) holds for some $n \ge 0$ and that $\|x_{k+n+1} - w\| \ge \varepsilon$. Then by (18), we get

$$\begin{aligned}\varepsilon \le \|x_{k+n+1} - w\| &= \|y_{k+n} - w + x_{k+n+1} - y_{k+n}\| \\ &\le \|y_{k+n} - w\| + \|x_{k+n+1} - y_{k+n}\| \\ &\le \|y_{k+n} - w\| + (\alpha_{k+n} + 3\beta_{k+n})M \\ &\le \|y_{k+n} - w\| + \frac{2\varepsilon}{3}\end{aligned}$$

Hence

$$\|y_{k+n} - w\| \ge \frac{\varepsilon}{3}$$

From (17), we get

$$\begin{aligned}\varepsilon^2 \le \|x_{k+n+1} - w\|^2 &\le \|x_{k+n} - w\|^2 - 2\alpha_{k+n}k_x\left(\frac{\varepsilon}{3}\right)^2 + \alpha_{k+n}k_x\left(\frac{\varepsilon}{3}\right)^2 \\ &\le \|x_{k+n} - w\|^2 < \varepsilon^2,\end{aligned}$$

which is a contradiction. Thus we proved (24). Since ε is arbitrary, from (24), we have $\lim_{n\to\infty}\|x_n - w\| = 0$.

2.1 Remark:

If in theorem (2.1), $\beta_n = 0$, $b_n = 0$, then we obtain a result that deals with the Mann iterative process with errors.

Now, we state the Ishikawa and Mann iterative process with errors for the local strongly pseudo contractive operators.

2.2 Theorem:

Let X be a uniformly smooth Banach space, let K be a non empty bounded closed convex subset of X and $T:K \longrightarrow K$ be a local strongly pseudo contractive mapping. Let w be a fixed point of T and let for $x_0 \in K$ the Ishikawa iteration sequence $\langle x_n \rangle$ be defined by

$y_n = \bar{\beta}_n x_n + \beta_n T x_n + b_n v_n$

$x_{n+1} = \bar{\alpha}_n \; x_n + \alpha_n T y_n + a_n u_n,\ n \geq 0$

where $<u_n>$, $<v_n> \subset K$, $<\alpha_n>$, $<\beta_n>$, $<a_n>$, $<b_n>$ are sequences as in theorem (2.1) and

$\bar{\alpha}_n = 1 - \alpha_n - a_n$,

$\bar{\beta}_n = 1 - \beta_n - b_n$.

Then $<x_n>$ convereges strongly to the unique fixed point of T.

proof: Obviously $<x_n>$ and $<y_n>$ are both contained in K and therefore, bounded. Since T is local strongly pseudo-contractive, then (I – T) is local strongly accretive-further, since T local strongly pseudo-contractive with y = w and T = S, we get (10) the proof of theorem (2.1) follows.

2.3 Remark:

Theorems (2.1)and (2.2) are extends and improves the main result of Liu [3,Theorem 1] in the following ways:

1. The assumption that $<u_n>$ and $<v_n>$ are two summable sequences is replaced by the assumption that $<u_n>$ and $<v_n>$ are two bounded sequences.
2. T need not to be Lipschitz.
3. The assumption that T is a strongly accretive mapping is replaced by the assumption that T is local strongly accretive and local strongly pseudo-contractive respectively.

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