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Solving Some Kinds of the Three Order Partial Differential Equations with Homogeneous Degree

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Abstract
In this paper, wefind the solution for some kinds of nonlinear third order partial differential equations of homogeneous degree which have the general form

$$A\,Z_{xxx}^2 + B\,Z_{yyy}^2 + C\,Z_{xxy}^2 + D\,Z_{xyy}^2 + E\,Z_{xx}^2 + F\,Z_{yy}^2 + G\,Z_{xy}^2 + H\,Z_x^2 + I\,Z_y^2 + J\,Z^2 = 0,$$

where*A*,*B*,*C*,*D*,*E*and*F* are linear functions of the dependent variable *Z*and partial derivatives ofthe dependent variable with respect to the independent variables x and y,and that by using some of the assumptions .

الخلاصـــــة
في هذا البحثتم إيجاد الحل لبعض أنواع المعادلات التفاضلية الجزئية اللاخطية من الرتبة الثالثة والمتجانسة الدرجةوالتي صيغتها العامة:

$$A\,Z_{xxx}^2 + B\,Z_{yyy}^2 + C\,Z_{xxy}^2 + D\,Z_{xyy}^2 + E\,Z_{xx}^2 + F\,Z_{yy}^2 + G\,Z_{xy}^2 + H\,Z_x^2 + I\,Z_y^2 + J\,Z^2 = 0,$$

حيث $A,B,C,\ ,\ D\ ,E,F,I\ ,\ H\ \ ,G$

دوال خطية للمتغير المعتمدZواشتقاقا ته الجزئية بالنسبة للمتغيرين xو y ، وذلك باستخدام بعض الفرضيات .

Mathematics Subject Classification:35GXX

1.Introduction
The researcher Kudaer [8], studied the linear second order ordinarydifferential equations,which have the form $y'' + P(x)\,y' + Q(x)\,y = 0$, and used the assumption $y(x) = e^{\int Z(x)\,dx}$ to find the general solution of it , and the solution depends on the forms of *P(x)* and *Q(x)* .

The researcher Abd Al-Sada [1], studied the linear second order partialdifferential equations,with constant coefficients of the form

$$A\,Z_{xx} + B\,Z_{yy} + C\,Z_{xy} + D\,Z_x + E\,Z_y + F\,Z = 0,$$

Where A , B , C , D , E and F are arbitrary constants , and used the assumption $Z(x,y)=e^{\int u(x)\,dx+\int v(y)\,dy}$ to find the complete solution of it.

The researcher Hanoon [4], studied the linear second orderpartialdifferential equations, withvariable coefficients of the form

$$A(x,y)\,Z_{xx}+B(x,y)\,Z_{yy}+C(x,y)\,Z_{xy}+D(x,y)\,Z_x+E(x,y)\,Z_y+F(x,y)\,Z=0,$$

where A , B , C , D , E and F are functions of x or $y$or both x and y . To solve this kind of equations and used the assumptions

$$Z(x,y)=e^{\int \frac{u(x)}{x}\,dx+\int v(y)\,dy},\ Z(x,y)=e^{\int u(x)\,dx+\int \frac{v(y)}{y}\,dy} \text{ and } Z(x,y)=e^{\int \frac{u(x)}{x}\,dx+\int \frac{v(y)}{y}\,dy},$$

these assumptions represent the complete solution of the above equation .

The researcher Mohsin [12], studied the nonlinear second order partial differential equations, of homogeneous degree which have the general form

$$A\,Z_{xx}+B\,Z_{xy}+C\,Z_{yy}+D\,Z_x+E\,Z_y+F\,Z=0,$$

where$A$,B,C,D,$E$and$F$ are linear functions of dependent variable $Z$and partial derivatives of dependent variable with respect to the independent variables x and y,she using the following assumptions

$$Z(x,y)=e^{\int u(x)dx+\int v(y)dy},\ Z(x,y)=e^{\int \frac{u(x)}{x}dx+\int v(y)dy},\ Z(x,y)=e^{\int u(x)dx+\int \frac{v(y)}{y}dy}$$

and $Z(x,y)=e^{\int \frac{u(x)}{x}dx+\int \frac{v(y)}{y}dy}$ to find the complete solutions of the above kind of equations.

The researcher Ketap [7],studied the linear third order partial differential equations, with constant coefficients of the form

$$A\,Z_{xxx}+B\,Z_{yyy}+C\,Z_{xxy}+D\,Z_{xyy}+E\,Z_{xx}+F\,Z_{yy}+G\,Z_{xy}+H\,Z_x+I\,Z_y+J\,Z=0,$$

where A, ... I and J are arbitrary constants , and used the assumption $Z(x,y)=e^{\int u(x)\,dx+\int v(y)\,dy}$ to find the complete solution of it.

The researcherHaoer [5], studied the nonlinear second order partial differential equations, of homogeneous degree which have the general form

$A\,Z_{xx}^2+B\,Z_{xy}^2+C\,Z_{yy}^2+D\,Z_x^2+E\,Z_y^2+F\,Z^2=0$, where A,B,C,D,$EandF$ are linear functions of dependent variable $Z$and partial derivatives of dependent variable with respect to the independent variables x and y,he using the following assumptions

$$Z(x,y)=e^{\int u(x)dx+\int v(y)dy},\ Z(x,y)=e^{\int \frac{u(x)}{x}dx+\int v(y)dy},\ Z(x,y)=e^{\int u(x)dx+\int \frac{v(y)}{y}dy}$$

and $Z(x,y)=e^{\int \frac{u(x)}{x}dx+\int \frac{v(y)}{y}dy}$ to find the complete solutions of the above kind of equations.

In this paper, we solve the nonlinear third order partial differential equations of homogeneous degree of three which have the general form

$$A\,Z_{xxx}^2+B\,Z_{yyy}^2+C\,Z_{xxy}^2+D\,Z_{xyy}^2+E\,Z_{xx}^2+F\,Z_{yy}^2+G\,Z_{xy}^2+H\,Z_x^2+I\,Z_y^2+J\,Z^2=0,$$

where$A$,B,C,D,$E$and$F$ are linear functions of dependent variable $Z$and partial derivatives of dependent variable with respect to the independent variables x and y.

By using the assumptions $Z(x,y)=e^{\int u(x)dx+\int v(y)dy}$, $Z(x,y)=e^{\int \frac{u(x)}{x}dx+\int v(y)dy}$ and $Z(x,y)=e^{\int u(x)dx+\int \frac{v(y)}{y}dy}$ we get the complete solutions of the above equation.

2. The Complete Solution of Nonlinear the Third Order partial Differential Equations of homogeneous degree of three

The aim of this paper is to solve the nonlinear third order of partial differential equations, of homogeneous degree of three which have the general form

$$A Z_{xxx}^2 + B Z_{yyy}^2 + C Z_{xxy}^2 + D Z_{xyy}^2 + E Z_{xx}^2 + F Z_{yy}^2 + G Z_{xy}^2 + H Z_x^2 + I Z_y^2 + J Z^2 = 0,$$

where A,B,C,D,E and F are linear functions of dependent variable Z and partial derivatives of dependent variable with respect to the independent variables x and y.

So, for this purpose we will search forfunctions $u(x)$ and $v(y)$ such that the assumptions

$Z(x,y)=e^{\int u(x)dx+\int v(y)dy}$, $Z(x,y)=e^{\int \frac{u(x)}{x}dx+\int v(y)dy}$ and $Z(x,y)=e^{\int u(x)dx+\int \frac{v(y)}{y}dy}$

give the complete solution to the above equation and thisequation may be classifiedto many cases, but those which distinguishing the followingcases are considered :

Case(1) **:**

a) $\mathrm{A}\, Z^2\, Z_{xxx} = 0$

b) $\mathrm{B}\, \mathrm{Z}^2\, Z_{yyy} = 0$.

c) $\mathrm{C}\, Z\, Z_{xxy}^2 + \mathrm{D}\, Z\, Z_{xyy}^2 = 0$

Case(2) **:**

a) $\mathrm{A}\ Z Z_{xxx}^2 + \mathrm{C}\, y\, Z\, Z_x\, Z_{xxy} + \mathrm{D}\, y^2 Z_{xx}^2 Z_{xyy} = 0$

b) $\mathrm{B}\, Z_y^2 Z_{yyy} + \mathrm{E}\, x^4 Z_y\, Z_{xx}^2 + \mathrm{F}\, x\, Z\, Z_{xx} Z_{yy} = 0$

Case(٣) **:**

a) $\mathrm{A}\, Z_{xx}^2\, Z_{xxx} + \mathrm{C}\, y\, Z\, Z_{xx} Z_{xxy} + \mathrm{D}\, y^2 Z_x^2 Z_{xyy} + \mathrm{E}\, y\, Z\, Z_y Z_{xx} + \mathrm{F}\, y^2 Z_x^2 Z_{yy} +$

$\mathrm{G}\, y\, Z^2 Z_{xy} + \mathrm{H}\, Z_x^3 + \mathrm{I}\, y\, Z^2 Z_y + \mathrm{J}\, Z^3 = 0$

b) $\mathrm{B}\, Z\, Z_{yyy}^2 + \mathrm{C}\, x^2\, Z_y^2\, Z_{xxy} + \mathrm{D}\, x Z\, Z_{yy} Z_{xyy} + \mathrm{E}\, x^2\, Z_y^2\, Z_{xx} + \mathrm{F}\, Z\, Z_{yy}^2 +$

$\mathrm{G}\, x Z_y^2 Z_{xy} + \mathrm{H}\, x Z^2\, Z_x + \mathrm{I}\, Z_y^3 + \mathrm{J}\, Z^3 = 0$

Where A, … , I and J are real constants.

Now , these cases will be solved as follows :

Case(1)-a-: By using the assumption

$Z(x,y)=e^{\int u(x)dx+\int v(y)dy}$, we get

$Z_x = u(x)\, e^{\int u(x)dx+\int v(y)dy} \Rightarrow Z_{xx} = (u'(x)+u^2(x))\, e^{\int u(x)dx+\int v(y)dy}$

$\Rightarrow Z_{xxx} = (u''(x)+3u(x)\,u'(x)+u^3(x))\, e^{\int u(x)dx+\int v(y)dy}$

So, the equation A $Z^2\, Z_{xxx}=0$, will be transformed to the form

$[\mathrm{A}(u''(x)+3u(x)\,u'(x)+u^3(x))]e^{3[\int u(x)dx+\int v(y)dy]}=0$,

and since $e^{3[\int u(x)dx+\int v(y)dy]}\neq 0$

So, $\mathrm{A}[u''(x)+3u(x)\,u'(x)+u^3(x)]=0 \quad \Rightarrow u''(x)+3u(x)\,u'(x)+u^3(x)=0$

This equation is called beloved equation[10],[14].The beloved equation possessesboth Left painleve Series (LPS) and Right Painleve Series (RPS) [9] , it can be solved by Riccati transformation [11],[13]. And also itcan be solved by using nonlocal symmetry [6] .

In this paper, we find the general solution of the beloved equation by using the reduction of order and the suitable substitution$u^2(x)=t\,p(x)$where $p(x)=\dfrac{du}{dx}$ and t ,$p(x)>0$

Let $u'(x)=p(x)\Rightarrow u''(x)=p(x)\dfrac{dp}{du}\Rightarrow p(x)\dfrac{dp}{du}+3u(x)p(x)+u^3(x)=0$

Suppose $u^2(x)=p(x)t \Rightarrow 2u(x)du = p(x)dt + tdp \Rightarrow$

$$du=\frac{pdt+tdp}{2t^{\frac{1}{2}}p^{\frac{1}{2}}}\Rightarrow pdp+(3p^{\frac{3}{2}}t^{\frac{1}{2}}+p^{\frac{3}{2}}t^{\frac{3}{2}})\frac{tdp+pdt}{2t^{\frac{1}{2}}p^{\frac{1}{2}}}=0\Rightarrow$$

$$(2+3t+t^2)dp+(3+t)pdt=0\Rightarrow\frac{dp}{p}+\frac{(3+t)dt}{2+3t+t^2}=0\Rightarrow$$

$$\int\frac{dp}{p}+\int\frac{(3+t)dt}{(t+1)(t+2)}=\int 0dt\Rightarrow \ln p+\ln(t+1)^2-\ln(t+2)=\ln c_1 \;;\; \mathbf{c_{1>0}} \;\; t>-2$$

$$\Rightarrow \ln\frac{p(t+1)^2}{t+2}=\ln c_1\Rightarrow p(t^2+2t+1)=(t+2)c_1$$

Since $t=\dfrac{u^2(x)}{p}$

$$\Rightarrow p(\frac{u^4(x)}{p^2}+\frac{2u^2(x)}{p}+1)-c_1\frac{u^2(x)}{p}-2c_1=0\Rightarrow$$

$$u^4(x)+2u^2(x)p+p^2-c_1u^2(x)-2c_1p=0\Rightarrow$$

$$P^2+(2u^2(x)-2c_1)p+u^4(x)-c_1u^2(x)=0\Rightarrow$$

$$p=c_1-u^2(x)\pm\sqrt{c_1^2-c_1u^2(x)}\Rightarrow$$

i) If $p=c_1-u^2(x)+\sqrt{c_1^2-c_1u^2(x)}$

Now,let $u(x)=\sqrt{c_1}\sin\sigma \Rightarrow \frac{du}{dx}=p=\sqrt{c_1}\cos\sigma\frac{d\sigma}{dx}\Rightarrow$

$$\sqrt{c_1}\cos\sigma\frac{d\sigma}{dx}=c_1-c_1\sin^2\sigma+\sqrt{c_1^2-c_1^2\sin^2\sigma}\Rightarrow$$

$$\frac{1}{\sqrt{c_1}}\left(\frac{\cos\sigma}{1-\sin^2\sigma+\cos\sigma}\right)d\sigma=dx\Rightarrow\frac{1}{\sqrt{c_1}}\int\frac{d\sigma}{1+\cos\sigma}=\int dx\Rightarrow$$

$$\frac{1}{\sqrt{c_1}}(\csc\sigma-\cot\sigma)=x+c_2\Rightarrow\frac{1}{\sqrt{c_1}}\left(\frac{1-\cos\sigma}{\sin\sigma}\right)=x+c_2\Rightarrow$$

$$\frac{1}{\sqrt{c_1}}\left(\frac{1-\sqrt{1-\sin^2\sigma}}{\sin\sigma}\right)=x+c_2$$

Since $\sin\sigma=\frac{u(x)}{\sqrt{c_1}}\Rightarrow\frac{1}{\sqrt{c_1}}\left(\frac{1-\sqrt{1-\frac{u^2(x)}{c_1}}}{\frac{u(x)}{\sqrt{c_1}}}\right)=x+c_2\Rightarrow$

$$1-\sqrt{1-\frac{u^2(x)}{c_1}}=(x+c_2)u(x)\Rightarrow1-(x+c_2)u(x)=\sqrt{1-\frac{u^2(x)}{c_1}}\Rightarrow$$

$$1-2(x+c_2)u(x)+(x+c_2)^2u^2(x)=1-\frac{u^2(x)}{c_1}\Rightarrow$$

$$(x+c_2)^2u^2(x)-2(x+c_2)u(x)+\frac{u^2(x)}{c_1}=0\Rightarrow$$

$$u(x)[(x+c_2)^2u(x)-2(x+c_2)+\frac{u(x)}{c_1}]=0$$

Either $u(x)=0$ is trivial solution

Or $(x+c_2)^2u(x)-2(x+c_2)+\frac{u(x)}{c_1}=0\Rightarrow$

$$[(x+c_2)^2+\frac{1}{c_1}]u(x)=2(x+c_2)\Rightarrow$$

$u(x)=\dfrac{2(x+c_2)}{(x+c_2)^2+\frac{1}{c_1}}$, this is the general solution of the beloved equation.

Then

$$Z(x,y)=e^{\int[\frac{2(x+c_2)}{(x+c_2)^2+\frac{1}{c_1}}]dx+\int v(y)dy}=e^{\ln((x+c_2)^2+\frac{1}{c_1})+h(y)+a_1}$$

So the complete solution is given by:

$$Z(x,y)=A_1Y_1(y)[(x+c_2)^2+c_3]\quad;\ A_1=e^{a_1},c_3=\frac{1}{c_1},Y_1(y)=e^{h(y)}$$

where A_1 , c_2 and c_3 are arbitrary constants and $Y_1(y)$ is an arbitrary function of y .
Domain : $-\infty<x<\infty$, $-\infty<y<\infty$

ii) If $p = c_1 - u^2(x) - \sqrt{c_1^2 - c_1 u^2(x)}$

By the same method, we get the same solution as in above form.

Case(1)-b-: By using the assumption

$Z(x,y) = e^{\int u(x)dx + \int v(y)dy}$, we get

$Z_y = v(y)\, e^{\int u(x)dx + \int v(y)dy} \Rightarrow Z_{yy} = (v'(y) + v^2(y))\, e^{\int u(x)dx + \int v(y)dy}$

$\Rightarrow Z_{yyy} = (v''(y) + 3v(y)\, v'(y) + v^3(y))\, e^{\int u(x)dx + \int v(y)dy}$

So, the equation $\mathrm{B}\, Z^2\, Z_{yyy} = 0$, will be transformed to the form

$[\mathrm{B}(v''(y) + 3v(y)\, v'(y) + v^3(y))]\, e^{3[\int u(x)dx + \int v(y)dy]} = 0$,

Since $e^{3[\int u(x)dx + \int v(y)dy]} \neq 0$

So, $\mathrm{B}[v''(y) + 3v(y)\, v'(y) + v^3(y)] = 0 \quad \Rightarrow v''(y) + 3v(y)\, v'(y) + v^3(y) = 0$,

then by the same method as in case –a-, we get the complete solution which is given by :

$Z(x,y) = \mathrm{A}_2 X_1(x)[(y + c_2)^2 + c_3]$; $\mathrm{A}_2 = e^{a_1}, c_3 = \frac{1}{c_1}, X_1(x) = e^{h(x)}$

where A_2 , c_2 and c_3 are arbitrary constants and $X_1(x)$ is an arbitrary function of x .

Domain : $-\infty < x < \infty$, $-\infty < y < \infty$.

Case(1)-c-: By using the assumption

$Z(x,y) = e^{\int u(x)dx + \int v(y)dy}$, we get

$Z_{xxy} = v(y)\,(u'(x) + u^2(x))\, e^{\int u(x)dx + \int v(y)dy}$

$Z_{xy} = u(x)\, v(y)\, \mathrm{e}^{\int u(x)\, dx + \int v(y)\, dy} \Rightarrow Z_{xyy} = (u(x)\,(v'(y) + v^2(y))\, \mathrm{e}^{\int u(x)\, dx + \int v(y)\, dy}$

Then the equation $\mathrm{C}\, Z\, Z_{xxy}^2 + \mathrm{D}\, Z\, Z_{xyy}^2 = 0$, will be transformed to the form

$[C(v^2(y)(u'(x) + u^2(x))^2) + D(u^2(x)(v'(y) + v^2(y))^2]e^{3[\int u(x)dx + \int v(y)dy]} = 0$

Since $e^{3[\int u(x)dx + \int v(y)dy]} \neq 0$

$\Rightarrow C(v^2(y)(u'(x) + u^2(x))^2 + D(u^2(x)(v'(y) + v^2(y))^2 = 0$

$\Rightarrow C(\frac{u'(x) + u^2(x)}{u(x)})^2 + D(\frac{v'(y) + v^2(y)}{v(y)})^2 = 0$.

This equation is variable separable equation [3] .

Let $C(\frac{u'(x) + u^2(x)}{u(x)})^2 = -D(\frac{v'(y) + v^2(y)}{v(y)})^2 = \lambda^2$

$\Rightarrow u'(x) + u^2(x) \pm \frac{\lambda}{\sqrt{C}} u(x) = 0 \ldots(1)$.

And $v'(y) + v^2(y) \pm \frac{\lambda i}{\sqrt{D}} v(y) = 0 \ldots(2)$.

The equation (1) is Bernoulli'sequation[2], it can be solved as follows :

$$u(x)=\frac{e^{\pm\frac{\lambda}{\sqrt{C}}x}}{\int e^{\pm\frac{\lambda}{\sqrt{C}}x}\,dx}$$

And equation (2) is also Bernoulli'sequation [2], it can be solved as follows :

$$v(y)=\frac{e^{\pm\frac{\lambda i}{\sqrt{D}}y}}{\int e^{\pm\frac{\lambda i}{\sqrt{D}}y}\,dy}$$

So,
$$Z(x,y)=e^{\int\frac{e^{\pm\frac{\lambda}{\sqrt{C}}x}}{\int e^{\pm\frac{\lambda}{\sqrt{C}}x}dx}dx+\int\frac{e^{\pm\frac{\lambda i}{\sqrt{D}}y}}{\int e^{\pm\frac{\lambda i}{\sqrt{D}}y}dy}dy}$$

$$=e^{\ln(\int e^{\pm\frac{\lambda}{\sqrt{C}}x}dx)+\ln(\int e^{\pm\frac{\lambda i}{\sqrt{D}}y}dy)+c_1}$$

$$=(\pm\frac{\sqrt{C}}{\lambda}e^{\pm\frac{\lambda}{\sqrt{C}}x})(\pm\frac{\sqrt{D}}{\lambda i}e^{\pm\frac{\lambda i}{\sqrt{D}}y})c_2\ \ ;\ c_2=e^{c_1},\ \lambda\neq 0$$

$$=K\,e^{\pm\frac{\lambda}{\sqrt{C}}x\pm\frac{\lambda i}{\sqrt{D}}y}\qquad ;\ K=(\pm\frac{\sqrt{C}}{\lambda})(\pm\frac{\sqrt{D}}{\lambda i})c_2\ \text{ and }\ C,D\geq 0$$

So ,thecomplete solution is given by :

$$\left[Z(x,y)-K\,e^{-\frac{\lambda}{\sqrt{C}}x-\frac{\lambda i}{\sqrt{D}}y}\right]\left[Z(x,y)-K\,e^{\frac{\lambda}{\sqrt{C}}x+\frac{\lambda i}{\sqrt{D}}y}\right]\left[Z(x,y)-K\,e^{\frac{\lambda}{\sqrt{C}}x-\frac{\lambda i}{\sqrt{D}}y}\right]=0$$

where K , and λ are arbitrary constants.
Domain :$-\infty<x<\infty$, $-\infty<y<\infty$.

Example : To solve the partial differential equation:

$$4\,Z\,Z_{xxy}^2+9\,Z\,Z_{xyy}^2=0\ \text{ where }\ C=4\ ,\ D=9.$$

This equation is similar to the equation in case(1)-c-,then by using the form

$$Z(x,y)=K\,e^{\pm\frac{\lambda}{\sqrt{C}}x\pm\frac{\lambda i}{\sqrt{D}}y}\qquad ;\ K=(\pm\frac{\sqrt{C}}{\lambda})(\pm\frac{\sqrt{D}}{\lambda i})c_2\ \text{ and }\ C,D\geq 0$$

We get the complete solution to the above P.D.E which has the form

$$Z(x,y)=\mathrm{K}\,e^{\pm\frac{\lambda}{2}x\pm\frac{\lambda i}{3}y}\quad ;\mathrm{K}=(\pm\frac{2}{\lambda})(\pm\frac{3}{\lambda i})\mathrm{c}_2 \quad\Rightarrow$$

$$\left[Z(x,y)-(\pm\frac{2}{\lambda})(\pm\frac{3}{\lambda i})\mathrm{c}_2\ e^{-\frac{\lambda}{2}x-\frac{\lambda i}{3}y}\right]\left[Z(x,y)-(\pm\frac{2}{\lambda})(\pm\frac{3}{\lambda i})\mathrm{c}_2\ e^{\frac{\lambda}{2}x+\frac{\lambda i}{3}y}\right]$$

$$\left[Z(x,y)-(\pm\frac{2}{\lambda})(\pm\frac{3}{\lambda i})\mathrm{c}_2\ e^{\frac{\lambda}{2}x-\frac{\lambda i}{3}y}\right]=0$$

where c_2 , and λ are arbitrary constants.
Domain : $-\infty<x<\infty$, $-\infty<y<\infty$.

<u>Case(2)-a-</u>: By using the assumption

$$Z(x,y)=e^{\int u(x)dx+\int\frac{v(y)}{y}dy}\text{ , we get}$$

$$Z_x=u(x)\,e^{\int u(x)dx+\int\frac{v(y)}{y}dy}\Rightarrow Z_{xx}=(u'(x)+u^2(x))\,e^{\int u(x)dx+\int\frac{v(y)}{y}dy}$$

$$\Rightarrow Z_{xxx}=(u''(x)+3u(x)\,u'(x)+u^3(x))\,e^{\int u(x)dx+\int\frac{v(y)}{y}dy}$$

$$Z_{xxy}=(\frac{v(y)}{y}(u'(x)+u^2(x)))e^{\int u(x)dx+\int\frac{v(y)}{y}dy}$$

$$Z_{xy}=\frac{v(y)}{y}u(x)e^{\int u(x)dx+\int\frac{v(y)}{y}dy}$$

$$\Rightarrow Z_{xyy}=u(x)\left(\frac{yv'(y)+v^2(y)-v(y)}{y^2}\right)e^{\int u(x)dx+\int\frac{v(y)}{y}dy}$$

So, the equation $\mathrm{A}\ ZZ_{xxx}^2+\mathrm{C}\,y\,Z\,Z_x\,Z_{xxy}+\mathrm{D}\,y^2Z_{xx}^2Z_{xyy}=0$, will be transformed to the form

$$\left[\mathrm{A}\left(u''(x)+3u(x)\,u'(x)+u^3(x)\right)^2+\mathrm{C}y\left(\frac{v(y)}{y}u(x)(u'(x)+u^2(x))\right)+\right.$$

$$\left.\mathrm{D}\,y^2\left(u(x)(u'(x)+u^2(x))^2\left(\frac{yv'(y)+v^2(y)-v(y)}{y^2}\right)\right)\right]e^{3[\int u(x)dx+\int\frac{v(y)}{y}dy]}=0$$

And since $e^{3[\int u(x)dx+\int\frac{v(y)}{y}dy]}\neq 0$

$$\text{So }\mathrm{A}\left(u''(x)+3u(x)\,u'(x)+u^3(x)\right)^2+\mathrm{C}\left(v(y)\,u(x)\,(u'(x)+u^2(x))\right)+$$

$$\mathrm{D}\left(u(x)\,(u'(x)+u^2(x))^2(yv'(y)+v^2(y)-v(y))\right)=0 \qquad ..(3)$$

Here we cannot separate the variables in this equation , so we suppose that $u(x)=\lambda$ where λ is an arbitrary constant, then the last equation becomes

$$yv'(y)+v^2(y)+\left(\frac{C}{D\lambda^2}-1\right)v(y)+\frac{A\lambda}{D}=0$$

Let $A_1=\frac{C}{D\lambda^2}-1$ and $A_2=\frac{A\lambda}{D}$

Then the last equation becomes:

$$yv'(y)+v^2(y)+A_1v(y)+A_2=0 \ldots(4).$$

This equation is variable separable equation [3], andit can be solved as follows :

$$\frac{dv}{\left(v(y)+\frac{A_1}{2}\right)^2+d^2}+\frac{dy}{y}=0 \qquad ; \qquad d^2=A_2-\frac{A_1^2}{4}$$

$i)$ If $A_2\neq\frac{A_1^2}{4}$, we get $\frac{1}{d}\tan^{-1}\left(\frac{v(y)+\frac{A_1}{2}}{d}\right)=-\ln(c\,y)$; $c\neq 0$

$$\Rightarrow \quad v(y)=-d\tan(d\ln c\,y)-\frac{A_1}{2}$$

$ii)$ If $A_2=\frac{A_1^2}{4}$, we get

$$\frac{dv}{\left(v(y)+\frac{A_1}{2}\right)^2}+\frac{dy}{y}=0 \qquad \Rightarrow \qquad \frac{-1}{v(y)+\frac{A_1}{2}}=-\ln c\,y \quad ; \ c\neq 0$$

$$\Rightarrow \ v(y)=\frac{1}{\ln c\,y}-\frac{A_1}{2}$$

Then the complete solution of the equation(3), is given by :

$i)$ If $A_2\neq\frac{A_1^2}{4}$, weget

$$Z(x,y)=e^{\int \lambda\, dx+\int \frac{(-d\tan(d\ln c\, y)-\frac{A_1}{2})}{y}dy} \quad ; \quad c\neq 0$$

$$=e^{\lambda x-\frac{A_1}{2}\ln y+\ln(\cos(d\ln c\, y))+g}$$

$$=K\, y^{-\frac{A_1}{2}} e^{\lambda x}(\cos(d\ln c\, y)) \quad ; \ K=e^{g}$$

$$=K\, y^{\frac{1}{2}-\frac{C}{2D\lambda^2}} e^{\lambda x}\left(\cos\left(\sqrt{\frac{A\lambda}{D}-\frac{(\frac{C}{D\lambda^2}-1)^2}{4}}\ \ln(cy)\right)\right.\ ,\ (\frac{A\lambda}{D}-\frac{(\frac{C}{D\lambda^2}-1)^2}{4})\geq 0,\ c\neq 0$$

Where K,λ and c are arbitrary constants.

Domain : $-\infty<x<\infty$, $y>0$.

ii) If $A_2=\frac{A_1^2}{4}$, we get

$$Z(x,y)=e^{\int \lambda\, dx+\int \frac{(\frac{1}{\ln c\, y}-\frac{A_1}{2})}{y}dy} \quad ; c\neq 0$$

$$=e^{\lambda x-\frac{A_1}{2}\ln y+\ln(\ln c\, y)+g}$$

$$=K\, y^{-\frac{A_1}{2}} e^{\lambda x}\ln(c\, y) \quad ; \ K=e^{g}$$

$$=K\, y^{\frac{1}{2}-\frac{C}{2D\lambda^2}} e^{\lambda x}\ln(cy)$$

Where K,λ and c are arbitrary constants.
Domain : $-\infty<x<\infty$, $y>0$.

Case(2)-b-: By using the assumption

$Z(x,y)=e^{\int \frac{u(x)}{x}dx+\int v(y)dy}$, we get

$$Z_y=v(y)\, e^{\int \frac{u(x)}{x}dx+\int v(y)dy} \Rightarrow Z_{yy}=(v'(y)+v^2(y))\, e^{\int \frac{u(x)}{x}dx+\int v(y)dy}$$

$$\Rightarrow Z_{yyy}=(v''(y)+3v(y)\, v'(y)+v^3(y))\, e^{\int \frac{u(x)}{x}dx+\int v(y)dy}$$

$$Z_x=\frac{u(x)}{x}\ e^{\int \frac{u(x)}{x}dx+\int v(y)dy} \Rightarrow Z_{xx}=(\frac{xu'(x)+u^2(x)-u(x)}{x^2})\, e^{\int \frac{u(x)}{x}dx+\int v(y)dy}$$

So, the equation $\mathrm{B}\, Z_y^2 Z_{yyy} + \mathrm{E}\, x^4 Z_y\, Z_{xx}^2 + \mathrm{F}\, x^2\, Z\, Z_{xx} Z_{yy} = 0$, will be transformed to the form

$$\left[\mathrm{B}\left(v^2(y)(v''(y)+3v(y)\,v'(y)+v^3(y))\right)+\mathrm{E}x^4\left(v(y)\left(\frac{xu'(x)+u^2(x)-u(x)}{x^2}\right)^2\right)+\right.$$

$$\left.\mathrm{F}x^2\left(\left(\frac{xu'(x)+u^2(x)-u(x)}{x^2}\right)(v'(y)+v^2(y))\right)\right] e^{3[\int \frac{u(x)}{x}dx+\int v(y)dy]} = 0$$

Since $e^{3[\int \frac{u(x)}{x}dx+\int v(y)dy]} \neq 0$

So $\mathrm{B}\left(v^2(y)(v''(y)+3v(y)\,v'(y)+v^3(y))\right)+\mathrm{E}\left(v(y)(xu'(x)+u^2(x)-u(x))^2\right)$

$+\,\mathrm{F}\left((xu'(x)+u^2(x)-u(x))(v'(y)+v^2(y))\right)=0$ ...(5)

Here wecannot separate the variables in this equation, so we suppose that $v(y)=\lambda$ where λ is an arbitrary constant, then the last equation becomes

$$\mathrm{B}\lambda^5+\mathrm{E}\,\lambda(xu'(x)+u^2(x)-u(x))^2+\mathrm{F}\,\lambda^2(xu'(x)+u^2(x)-u(x))=0$$

This equation can be solved as follows :

$$xu'(x)+u^2(x)-u(x)=\frac{-\mathrm{F}\lambda^2\pm\sqrt{\mathrm{F}^2\lambda^4-4\mathrm{EB}\lambda^6}}{2\mathrm{E}\lambda}\quad ;(\mathrm{F}^2\lambda^4-4\mathrm{EB}\lambda^6)\geq 0$$

$$xu'(x)+u^2(x)-u(x)-\mathrm{A}_1=0\quad ;\quad \mathrm{A}_1=\frac{-\mathrm{F}\lambda^2\pm\sqrt{\mathrm{F}^2\lambda^4-4\mathrm{EB}\lambda^6}}{2\mathrm{E}\lambda}$$

This equation is variable separable equation [3], it can besolved as follows :

$$\frac{du}{\left(u(x)-\frac{1}{2}\right)^2-\mathrm{d}^2}+\frac{\mathrm{d}x}{x}=0\quad ;\quad \mathrm{d}=\sqrt{\mathrm{A}_1+\frac{1}{4}}$$

i) If $\mathrm{A}_1\neq-\frac{1}{4}$, we get $-\frac{1}{\mathrm{d}}\tanh^{-1}\left(\frac{u(x)-\frac{1}{2}}{\mathrm{d}}\right)=-\ln(\mathrm{c}\,y)\ ;\ \mathrm{c}\neq 0$

$$\Rightarrow\quad u(x)=\mathrm{d}\tanh(\mathrm{d}\ln \mathrm{c}\,x)+\frac{1}{2}$$

ii) If $\mathrm{A}_1=-\frac{1}{4}$, we get

$$\frac{du}{\left(u(x)-\frac{1}{2}\right)^2}+\frac{\mathrm{d}x}{x}=0\quad\Rightarrow\quad\frac{-1}{u(x)-\frac{1}{2}}=-\ln \mathrm{c}\,x\quad ;\ \mathrm{c}\neq 0$$

$$\Rightarrow u(x)=\frac{1}{\ln \mathrm{c}\,x}+\frac{1}{2}\quad ;\quad \mathrm{c}\neq 0$$

Then the complete solution of the equation(5), is given by :

$i)$ If $A_1 \neq -\frac{1}{4}$, we get

So ,thecomplete solution is given by :

$$Z(x,y)=e^{\int \frac{(d\tanh(d\ln c\,x)+\frac{1}{2})}{x}dx+\int \lambda dy} \quad ;c\,x>0$$

$$= e^{\ln[\cosh(d\ln(cx))]+\frac{1}{2}\ln x+\lambda y+g}$$

$$= K\,x^{\frac{1}{2}}e^{\lambda y}(\cosh(d\ln cx)) \quad ;\ K=e^{g}\ ,c\,x>0$$

$$= K\,x^{\frac{1}{2}}e^{\lambda y}\left(\cosh\left(\sqrt{\frac{-F\lambda^2 \pm \sqrt{F^2\lambda^4-4EB\lambda^6}}{2E\lambda}+\frac{1}{4}}\ \ln(cx)\right)\right) \quad ;c\,x>0\ ,\lambda\neq 0,$$

$$\frac{-F\lambda^2 \pm \sqrt{F^2\lambda^4-4EB\lambda^6}}{2E\lambda}\geq 0$$

So ,thecomplete solution is given by :

$$\left[Z(x,y)-K\,x^{\frac{1}{2}}e^{\lambda y}\left(\cosh\left(\sqrt{\frac{-2F\lambda^2+E\lambda+2\sqrt{F^2\lambda^4-4EB\lambda^6}}{4E\lambda}}\ \ln(cx)\right)\right)\right]$$

$$\left[Z(x,y)-K\,x^{\frac{1}{2}}e^{\lambda y}\left(\cosh\left(\sqrt{\frac{-2F\lambda^2+E\lambda-2\sqrt{F^2\lambda^4-4EB\lambda^6}}{4E\lambda}}\ \ln(cx)\right)\right)\right]=0 \quad ;c\,x>0\ ,\lambda\neq 0,$$

$$\frac{-2F\lambda^2+E\lambda-2\sqrt{F^2\lambda^4-4EB\lambda^6}}{4E\lambda}\geq 0$$

Where K, λ and c are arbitrary constants.

Domain :$x>0$, $-\infty<y<\infty$.

$ii)$ If $A_1 = -\frac{1}{4}$, we get

$$Z(x,y)=e^{\int \frac{(\frac{1}{\ln c\,x}+\frac{1}{2})}{x}dx+\int \lambda dy} \quad ;c\,x>0$$

$$= K\,x^{\frac{1}{2}}e^{\lambda y}\ln(c\,x) \quad ;\ K=e^{g}\ ,c\,x>0$$

$$= K\,x^{\frac{1}{2}}e^{\lambda y}\ln(c\,x) \quad ;\ c\,x>0$$

Where K, λ and c are arbitrary constants.

Domain :$x>0$, $-\infty<y<\infty$.

Example : To solve the partial differential equation:

$$2Z_y^2 Z_{yyy} + 5x^4 Z_y Z_{xx}^2 + x^2 Z Z_{xx} Z_{yy} = 0 \text{ where } B=2,\ E=5,\ F=1.$$

This equation is similar to the equation in case(2)-b-, and if $A_1 \neq -\frac{1}{4}$

where $A_1 = \frac{-\lambda^2 \pm \sqrt{\lambda^4 - 40\lambda^6}}{10\lambda}$ thenby using the form

$$\left[Z(x,y) - K x^{\frac{1}{2}} e^{\lambda y}(\cosh\left(\sqrt{\frac{-2F\lambda^2 + E\lambda + 2\sqrt{F^2\lambda^4 - 4EB\lambda^6}}{4E\lambda}}\ \ln(cx)\right)\right]$$

$$\left[Z(x,y) - K x^{\frac{1}{2}} e^{\lambda y}(\cosh\left(\sqrt{\frac{-2F\lambda^2 + E\lambda - 2\sqrt{F^2\lambda^4 - 4EB\lambda^6}}{4E\lambda}}\ \ln(cx)\right)\right] = 0 \ ;c\,x > 0\ ,\lambda \neq 0,$$

$$\frac{-2F\lambda^2 + E\lambda - 2\sqrt{F^2\lambda^4 - 4EB\lambda^6}}{4E\lambda} \geq 0$$

So ,thecomplete solution is given by :

$$\left[Z(x,y) - K x^{\frac{1}{2}} e^{\lambda y}(\cosh\left(\sqrt{\frac{-2\lambda^2 + 5\lambda + 2\sqrt{\lambda^4 - 40\lambda^6}}{20\lambda}}\ \ln(cx)\right)\right]$$

$$\left[Z(x,y) - K x^{\frac{1}{2}} e^{\lambda y}(\cosh\left(\sqrt{\frac{-2\lambda^2 + 5\lambda - 2\sqrt{\lambda^4 - 40\lambda^6}}{20\lambda}}\ \ln(cx)\right)\right] = 0 \ ;c\,x > 0\ ,\lambda \neq 0,$$

$$\frac{-2\lambda^2 + 5\lambda - 2\sqrt{\lambda^4 - 40\lambda^6}}{20\lambda} \geq 0$$

Where $K = e^g$, λ and c are arbitrary constants.
Domain :$x > 0$, $-\infty < y < \infty$.

Now if $A_1 = -\frac{1}{4}$, sothe complete solution is given by :

$$Z(x,y) = K x^{\frac{1}{2}} e^{\lambda y} \ln(c\,x) \qquad ;\ c\,x > 0$$

Where $K = e^g$, λ and c are arbitrary constants.
Domain :$x > 0$, $-\infty < y < \infty$.

Case(3)-a-: By using the assumption

$Z(x,y) = e^{\int u(x)dx + \int \frac{v(y)}{y}dy}$, we get

$$Z_y = \frac{v(y)}{y} e^{\int u(x)dx + \int \frac{v(y)}{y}dy} \Rightarrow Z_{yy} = \left(\frac{yv'(y) + v^2(y) - v(y)}{y^2}\right) e^{\int u(x)dx + \int \frac{v(y)}{y}dy}$$

Andusing $Z_x, Z_{xx}, Z_{xxx}, Z_{xy}, Z_{xxy}$ and Z_{xyy} form the case (2)-a-, then the equation

$$\mathrm{A}\, Z_{xx}^2 Z_{xxx} + \mathrm{C}\, y\, Z\, Z_{xx} Z_{xxy} + \mathrm{D}\, y^2 Z_x^2 Z_{xyy} + \mathrm{E}\, y\, Z\, Z_y Z_{xx} + \mathrm{F}\, y^2 Z_x^2 Z_{yy} +$$
$$\mathrm{G}\, y\, Z^2 Z_{xy} + \mathrm{H}\, Z_x^3 + \mathrm{I}\, y\, Z^2 Z_y + \mathrm{J}\, Z^3 = 0$$

will be transformed to the form

$$\Bigg[\mathrm{A}\Big((u'(x)+u^2(x))^2(u''(x)+3u(x)\,u'(x)+u^3(x))\Big)+\mathrm{C}y\left(\frac{v(y)}{y}(u'(x)+u^2(x))^2\right)+$$
$$\mathrm{D}y^2\left(u^3(x)\,(\frac{yv'(y)+v^2(y)-v(y)}{y^2})\right)+\mathrm{E}y\left(\frac{v(y)}{y}(u'(x)+u^2(x))\right)+$$
$$\mathrm{F}y^2\left(u^2(x)\,(\frac{yv'(y)+v^2(y)-v(y)}{y^2})\right)+\mathrm{G}y(u(x)\frac{v(y)}{y})+\mathrm{H}u^3(x)+\mathrm{I}y\frac{v(y)}{y}+\mathrm{J}\Bigg]e^{3[\int u(x)dx+\int \frac{v(y)}{y}dy]}=0$$

And since $e^{3[\int u(x)dx+\int \frac{v(y)}{y}dy]} \neq 0$

So
$$\mathrm{A}\Big((u'(x)+u^2(x))^2(u''(x)+3u(x)\,u'(x)+u^3(x))\Big)+\mathrm{C}\Big((v(y)(u'(x)+u^2(x))^2\Big)+$$
$$\mathrm{D}\Big(u^3(x)\,(yv'(y)+v^2(y)-v(y))\Big)+\mathrm{E}\Big(v(y)(u'(x)+u^2(x))\Big)+$$
$$\mathrm{F}\Big(u^2(x)\,(yv'(y)+v^2(y)-v(y))\Big)+\mathrm{G}v(y)u(x)+\mathrm{H}u^3(x)+\mathrm{I}v(y)+\mathrm{J}=0 \qquad \ldots(6)$$

Here we cannot separate the variables in this equation , so we suppose that $u(x)=\lambda$ where λ is an arbitrary constant, then the last equation becomes

$$yv'(y)+v^2(y)+\frac{\mathrm{C}\lambda^4-\mathrm{D}\lambda^3+(\mathrm{E}-\mathrm{F})\lambda^2+\mathrm{G}\lambda+\mathrm{I}}{\mathrm{D}\lambda^3+\mathrm{F}\lambda^2}v(y)+\frac{\mathrm{A}\lambda^7+\mathrm{H}\lambda^3+\mathrm{J}}{\mathrm{D}\lambda^3+\mathrm{F}\lambda^2}=0$$

Let $\mathrm{A}_1=\dfrac{\mathrm{C}\lambda^4-\mathrm{D}\lambda^3+(\mathrm{E}-\mathrm{F})\lambda^2+\mathrm{G}\lambda+\mathrm{I}}{\mathrm{D}\lambda^3+\mathrm{F}\lambda^2}$ and $\mathrm{A}_2=\dfrac{\mathrm{A}\lambda^7+\mathrm{H}\lambda^3+\mathrm{J}}{\mathrm{D}\lambda^3+\mathrm{F}\lambda^2}$

Then the last equation becomes:

$$yv'(y)+v^2(y)+\mathrm{A}_1v(y)+\mathrm{A}_2=0 \qquad \ldots(7)$$

This equation is variable separable equation [3],it can be solved as follows

$$\frac{dv}{(v(y)+\frac{A_1}{2})^2+d^2}+\frac{dy}{y}=0 \quad ; d=\sqrt{A_2-\frac{A_1^2}{4}}$$

$i)$ if $A_2 \neq \frac{A_1^2}{4}$

$$\frac{1}{d}\tan^{-1}\left(\frac{v(y)+\frac{A_1}{2}}{d}\right)=-\ln(cy) \quad ; \ cy>0$$

$$\Rightarrow v(y)=-d\tan(d\ln(cy))-\frac{A_1}{2} \qquad ; \ cy>0$$

$ii)$ if $A_2 = \frac{A_1^2}{4}$

$$\frac{dv}{(v(y)+\frac{A_1}{2})^2}+\frac{dy}{y}=0 \ \Rightarrow -\frac{1}{v(y)+\frac{A_1}{2}}=-\ln(cy) \ ; \ cy>0$$

$$\Rightarrow v(y)=\frac{1}{\ln(cy)}-\frac{A_1}{2} \qquad ; \ cy>0$$

Then the complete solution of the equation(6), is given by :

$i)$ if $A_2 \neq \frac{A_1^2}{4}$

$$Z(x,y)=e^{\int \lambda dx+\int \frac{(-d\tan(d\ln(cy))-\frac{A_1}{2})}{y}dy} \qquad ; cy>0$$

$$=e^{\lambda x-\frac{A_1}{2}\ln y+\ln\left|\cos(d\ln(cy))\right|+g}$$

$$=K\, y^{-\frac{A_1}{2}} e^{\lambda x}(\cos(d\ln cy)) \quad ; \ K=e^{g} \text{ and } cy>0$$

$$=K\, y^{-\frac{C\lambda^4-D\lambda^3+(E-F)\lambda^2+G\lambda+I}{2(D\lambda^3+F\lambda^2)}} e^{\lambda x}$$

$$\cos\left(\sqrt{\frac{A\lambda^7+H\lambda^3+J}{D\lambda^3+F\lambda^2}-\frac{(C\lambda^4-D\lambda^3+(E-F)\lambda^2+G\lambda+I)^2}{4(D\lambda^3+F\lambda^2)^2}}\ \ln(cy)\right) \quad ; \ cy>0,$$

$$\left(\frac{A\lambda^7+H\lambda^3+J}{D\lambda^3+F\lambda^2}-\frac{(C\lambda^4-D\lambda^3+(E-F)\lambda^2+G\lambda+I)^2}{4(D\lambda^3+F\lambda^2)^2}\right)\geq 0$$

Where K, λ and c are arbitrary constants.
Domain : $-\infty < x < \infty$, $y > 0$.

$ii)$ if $A_2 = \dfrac{A_1^2}{4}$

$$Z(x,y) = e^{\int \lambda dx + \int \frac{\frac{1}{\ln(cy)} - \frac{A_1}{2})}{y} dy} \qquad ; \; c\,y > 0$$

$$= e^{\lambda x - \frac{A_1}{2} \ln y + \ln \left| \ln(c\,y) \right| + g}$$

$$= K\, y^{-\frac{A_1}{2}} e^{\lambda x} \ln(c\,y) \qquad ; \; K = e^{g} \text{ and } cy > 0$$

$$= K\, y^{-\frac{C\lambda^4 - D\lambda^3 + (E-F)\lambda^2 + G\lambda + I}{2(D\lambda^3 + F\lambda^2)}} e^{\lambda x} \ln(cy) \qquad ; \; cy > 0$$

Where K, λ and c are arbitrary constants.
Domain : $-\infty < x < \infty$, $y > 0$.

<u>Case(3)-b-</u>:: By using the assumption

$$Z(x,y) = e^{\int \frac{u(x)}{x} dx + \int v(y) dy}, \text{ we get}$$

$$Z_{xxy} = v(y)\left(\frac{xu'(x) + u^2(x) - u(x)}{x^2}\right) e^{\int \frac{u(x)}{x} dx + \int v(y) dy}$$

$$Z_{xy} = \left(\frac{u(x)}{x}\right) v(y)\, e^{\int \frac{u(x)}{x} dx + \int v(y) dy}$$

$$\Rightarrow Z_{xyy} = \left(\frac{u(x)}{x}\right)(v'(y) + v^2(y))\, e^{\int \frac{u(x)}{x} dx + \int v(y) dy}$$

And using Z_x, Z_{xx}, Z_y, Z_{yy} and Z_{yyy} form the case (2)-b-, then the equation

$$B\,Z\,Z_{yyy}^2 + C\,x^2\,Z_y^2\,Z_{xxy} + D\,xZ\,Z_{yy} Z_{xyy} + E\,x^2\,Z_y^2\,Z_{xx} + F\,Z\,Z_{yy}^2 +$$

$$G\,xZ_y^2 Z_{xy} + H\,x\,Z^2\,Z_x + I\,Z_y^3 + J\,Z^3 = 0$$

Will be transformed to the form

$$\Bigg[\mathrm{B}(v''(y)+3v(y)\,v'(y)+v^3(y))^2+\mathrm{C}x^2\left(v^3(y)(\frac{xu'(x)+u^2(x)-u(x)}{x^2})\right)+$$

$$\mathrm{D}x\left((\frac{u(x)}{x})(v'(y)+v^2(y))^2\right)+\mathrm{E}x^2\left(v^2(y)(\frac{xu'(x)+u^2(x)-u(x)}{x^2})\right)+$$

$$\mathrm{F}(v'(y)+v^2(y))^2+\mathrm{G}x\left(v^3(y)\frac{u(x)}{x}\right)+\mathrm{H}x(\frac{u(x)}{x})+\mathrm{I}v^3(y)+\mathrm{J}\Bigg]e^{3[\int\frac{u(x)}{x}dx+\int v(y)dy]}=0$$

Since $e^{3[\int\frac{u(x)}{x}dx+\int v(y)dy]}\neq 0$

So, $$\mathrm{B}(v''(y)+3v(y)\,v'(y)+v^3(y))^2+\mathrm{C}\left(v^3(y)(xu'(x)+u^2(x)-u(x))\right)+$$
$$\mathrm{D}\left(u(x)(v'(y)+v^2(y))^2\right)+\mathrm{E}\left(v^2(y)(xu'(x)+u^2(x)-u(x))\right)+$$
$$\mathrm{F}(v'(y)+v^2(y))^2+\mathrm{G}v^3(y)u(x)+\mathrm{H}u(x)+\mathrm{I}v^3(y)+\mathrm{J}=0 \qquad ...(8)$$

Here we cannot separate the variables in this equation , so we suppose that $v(y)=\lambda$ where λ is an arbitrary constant, then the last equation becomes

$$xu'(x)+u^2(x)+\frac{(\mathrm{G}-\mathrm{C})\lambda^3+(\mathrm{D}\lambda^2-\mathrm{E})\lambda^2+\mathrm{H}}{\mathrm{C}\lambda^3+\mathrm{E}\lambda^2}u(x)+\frac{\mathrm{B}\lambda^6+\mathrm{F}\lambda^4+\mathrm{I}\lambda^3+\mathrm{J}}{\mathrm{C}\lambda^3+\mathrm{E}\lambda^2}=0$$

Let $\mathrm{B}_1=\frac{(\mathrm{G}-\mathrm{C})\lambda^3+(\mathrm{D}\lambda^2-\mathrm{E})\lambda^2+\mathrm{H}}{\mathrm{C}\lambda^3+\mathrm{E}\lambda^2}$ and $\mathrm{B}_2=\frac{\mathrm{B}\lambda^6+\mathrm{F}\lambda^4+\mathrm{I}\lambda^3+\mathrm{J}}{\mathrm{C}\lambda^3+\mathrm{E}\lambda^2}$.

Then the last equation becomes:

$$xu'(x)+u^2(x)+\mathrm{B}_1u(x)+\mathrm{B}_2=0 \quad \ldots(9)$$

This equation is similar to the equation (7) , and by the same method :

$ii)$ if $\mathrm{B}_2=\frac{\mathrm{B}_1^2}{4}$

$$v(y)=-\mathrm{b}\tan(\mathrm{b}\ln(\mathrm{c}x))-\frac{\mathrm{B}_1}{2} \quad ; \ \mathrm{b}=\sqrt{\mathrm{B}_2-\frac{\mathrm{B}_1^2}{4}} \quad \text{and } \mathrm{c}x>0$$

$ii)$ if $\mathrm{B}_2=\frac{\mathrm{B}_1^2}{4}$

$$v(y)=\frac{1}{\ln(cx)}-\frac{B_1}{2} \qquad ; \ cx>0$$

Then the complete solution of the equation(٨), is given by :

$i)$ if $B_2 \neq \frac{B_1^2}{4}$

$$Z(x,y)=e^{\int\frac{(-b\tan(b\ln(cx))-\frac{B_1}{2})}{x}dx+\int\lambda\,dy} \qquad ; \ cx>0$$

$$=e^{\lambda y-\frac{B_1}{2}\ln x+\ln\left|\cos(b\ln(c\,x))\right|+g}$$

$$=K\,x^{-\frac{B_1}{2}}e^{\lambda y}(\cos(b\ln c\,x)) \qquad ; \ K=e^{g} \ \text{and} \ cx>0$$

$$=K\,x^{-\frac{(G-C)\lambda^3+(D\lambda^2-E)\lambda^2+H}{2(C\lambda^3+E\lambda^2)}}e^{\lambda y}$$

$$\cos\left(\sqrt{\frac{B\lambda^6+F\lambda^4+I\lambda^3+J}{C\lambda^3+E\lambda^2}-\frac{((G-C)\lambda^3+(D\lambda^2-E)\lambda^2+H)^2}{4(C\lambda^3+E\lambda^2)^2}}\ \ln(c\,x)\right) \qquad ; \ cx>0,$$

$$\left(\frac{B\lambda^6+F\lambda^4+I\lambda^3+J}{C\lambda^3+E\lambda^2}-\frac{((G-C)\lambda^3+(D\lambda^2-E)\lambda^2+H)^2}{4(C\lambda^3+E\lambda^2)^2}\right)\geq 0$$

Where K, λ and c are arbitrary constants.

Domain :$x>0$, $-\infty<y<\infty$.

$ii)$ if $B_2=\frac{B_1^2}{4}$

$$Z(x,y)=e^{\int\frac{\frac{1}{\ln(cx)}-\frac{B_1}{2})}{x}dx+\int\lambda\,dy} \qquad ; \ cx>0$$

$$=e^{\lambda y-\frac{B_1}{2}\ln x+\ln\left|\ln(c\,x)\right|+g}$$

$$=K\,x^{-\frac{B_1}{2}}e^{\lambda y}\ln(cx) \qquad ; \ K=e^{g} \ \text{and} \ cx>0$$

$$=K\,x^{-\frac{(G-C)\lambda^3+(D\lambda^2-E)\lambda^2+H}{2(C\lambda^3+E\lambda^2)}}e^{\lambda y}\ln(cx) \qquad ; \ cx>0$$

Where K, λ and c are arbitrary constants.

Domain :$x>0$, $-\infty<y<\infty$.

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