Approximation of Functions by Some Types of Beta-Operators

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Abstract:

In this paper, we study an approximation of continuous functions by using some types of Beta- operators (modified Beta-operator and modified mulit Beta-operator) defined on the some normed space.

الخلاصة:

في هذا البحث تم دراسة تقريب الدول المستمرة باستخدام بعض انواع مؤثرات بيتا(مؤثر بيتا المطور ومؤثر بيتا المطور المتعدد) والمعرفة على بعض الفضاءات المعيارية.

1. Introduction:

In 1889 Karl Weierstrass, proved the fundamental theorem in the approximation theory which is called "Weierstrass approximation theorem ", S.N.Bernstein in 1912[3] used a sequence of positive linear operators called Bernstein polynomial and several papers are generalization of Bernstein polynomials in the interval $[0,\infty)$ like korovkin [1].

In this work, we introduce a new sequence of positive linear operators $\beta_{n_1,n_2,\ldots,n_m}(f;x_1,x_2,\ldots,x_m)$ of modified mulit-Beta operators to approximate a function of m independent variables.

2. Definitions and Notations:

Let $f:[0,\infty)\to R$ be any function and the function $\omega_\alpha:[0,\infty)\to R^+$ is defined by $\omega_\alpha(x)=\overset{\alpha x}{e},\alpha\ge 1$ and recall that the modified Beta operator $\beta_n:L_{p,\alpha}\longrightarrow L_{p,\alpha}$ is an operator defined by $\beta_n(f;x)=\frac{1}{n}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!(n-1)!}x^k(1+x)^k f\left(\frac{k}{n}\right)$ $n\in N$ and $x\in[0,\infty)$, [3]. It is clear that β_n is a positive linear operator.

The following proposition gives some properties for the operator β_n .

Proposition (2.1):-

For $x\in[0,\infty)$ and $n\in N$, then the following statements holds:

(1) $\beta_n(f;x)=1$, where $f(x)=1$, $\forall x\in[0,\infty)$.

(2) $\beta_n(f;x)=\frac{(n+1)x}{n}$, where $f(x)=x$, $\forall x\in[0,\infty)$.

(3) $\beta_n(f;x)=\frac{(n+1)(n+2)x^2+(n+1)x}{n^2}$, where $f(x)=x^2$, $\forall x\in[0,\infty)$.

Proof:-

(1) Let $f(x)=1$, then $\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx = \frac{1}{\alpha p} < \infty$. Therefore; $f \in L_{p,\alpha}$. By using [3], one can have $\beta_n(f;x)=1$.

(2) Let $f(x)=x$, then $\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx = \frac{1}{(\alpha p)^{p+1}} < \infty$. Therefore; $f \in L_{p,\alpha}$. By using [3], one can have $\beta_n(f;x) = \frac{(n+1)x}{n}$.

(3) Let $f(x)=x^2$, Then $\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx = \frac{2}{(\alpha p)^{2p+1}} < \infty$. Therefore; $f \in L_{p,\alpha}$. By using [3], one can have $\beta_n(f;x) = \frac{(n+1)(n+2)x^2+(n+1)x}{n^2}$.

Next, we prove that $\beta_n f$ is convergent to f as $n \to \infty$. But before that we need the following lemma.

Lemma (2.2), [2]:-

Let L_n be a uniformly bounded sequence of positive linear operators from $L_{p,\alpha}$ into itself satisfying the condition $\lim_{n\to\infty}\|L_n(f)-f\|_{p,\alpha}=0$, where $f(x)=1,\ x,\ x^2$ then for every $f \in L_{p,\alpha}$, $\lim_{n\to\infty}\|L_n(f)-f\|_{p,\alpha}=0$.

3. Approximation of Functions of One Variable by modified Beta Operator:

Here, we approximate any continuous function defined on $[0,\infty)$ by the modified Beta operator β_n.

Lemma (3.1):-

For each $1 \le p < \infty$, $L_{p,\alpha} = \left\{ f \,\middle|\, f:[0,\infty) \longrightarrow R \text{ is a continuous function such that } \int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx < \infty \right\}$ is a normed space where $\omega_\alpha(x) = e^{\alpha x}$, α is a positive real number.

Proof:-

It is easy to check $0 \in L_{p,\alpha}$. Therefore $L_{p,\alpha} \neq \phi$. Define + and . on $L_{p,\alpha}$ by $(f+g)(x) = f(x)+g(x)\ \ \forall f,g \in L_{p,\alpha}$ and $(c.f)(x) = cf(x)\ \ \forall f \in L_{p,\alpha}$ and $c \in$ □.

Then by using [4, p. 236], one can have:

$\int_0^\infty \left|\frac{(f+g)(x)}{\omega_\alpha(x)}\right|^p dx \le 2^p \int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx + 2^p \int_0^\infty \left|\frac{g(x)}{\omega_\alpha(x)}\right|^p dx < \infty.$ Thus $f+g \in L_{p,\alpha}$. Moreover; $\int_0^\infty \left|\frac{(cf)(x)}{\omega_\alpha(x)}\right|^p dx = |c|^p \int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx < \infty.$ Then $c.f \in L_{p,\alpha}$.

The other conditions for $L_{p,\alpha}$ to be a vector space is easy to be verified, thus we omitted them.

Define $\|.\|_{p,\alpha} : L_{p,\alpha} \longrightarrow R^+ \cup \{0\}$ by $\|f\|_{p,\alpha} = \left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}}$.

We prove $\|.\|_{p,\alpha}$ is a norm on $L_{p,\alpha}$. To do this, we must prove the following conditions:

(i) If $f=0$ then $\|f\|_{p,\alpha} = \left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}} = 0$. Conversely if $\|f\|_{p,\alpha} = 0$, then $\left|\frac{f(x)}{\omega_\alpha(x)}\right|^p = 0 \quad \forall x \in [0,\infty)$ and hence $f(x)=0,\ \forall x \in [0,\infty)$. Therefore; $f=0$.

(ii) Let $f, g \in L_{p,\alpha}$ then

$$\|f+g\|_{p,\alpha} = \left(\int_0^\infty \left|\frac{f(x)+g(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}}$$

$$\le \left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}} + \left(\int_0^\infty \left|\frac{g(x)}{w_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}}$$

$$= \|f\|_{p,\alpha} + \|g\|_{p,\alpha}.$$

(iv) Let $\lambda \in$ □ and $f \in L_{p,\alpha}$ then

$$\|\lambda\, f\|_{p,\alpha} = \left(\int_0^\infty \left|\frac{(\lambda f)(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}}$$

$$= |\lambda| \left(\int_0^\infty \left|\frac{f(x)}{\omega_\alpha(x)}\right|^p dx \right)^{\frac{1}{p}}$$

$$= |\lambda| \ \|f\|_{p,\alpha}$$

Therefore; $L_{p,\alpha}$ is a normed space.

Now, we are in the position that we can give the following theorem.

Theorem (3.2):-

Let $f \in L_{p,\alpha}$ then $\beta_n f \longrightarrow f$ as $n \to \infty$.

Proof:-

From [2], β_n is a uniformly bounded sequence of positive linear operators.

Let $f(x)=1, \quad \forall x \in [0,\infty)$, then from proposition (2.1) one can have: $\lim_{n\to\infty} \|\beta_n f - f\|_{p,\alpha} = 0.$

Also, for $f(x)=x \quad \forall x \in [0,\infty)$, one can get:

$$\lim_{n\to\infty} \|\beta_n f - f\|_{p,\alpha} = \lim_{n\to\infty} \frac{1}{n}\left(\int_0^\infty \left| \frac{x}{\omega_\alpha(x)} \right|^p dx \right)^{\frac{1}{p}}$$

$$= 0$$

Moreover; for $f(x)=x^2 \quad \forall x \in [0,\infty)$, one can have:

$$\lim_{n\to\infty} \|\beta_n f - f\|_{p,\alpha} = \lim_{n\to\infty} \left(\int_0^\infty \left| \frac{\frac{(3n+2)x^2 + (n+1)x}{n^2}}{\omega_\alpha(x)} \right|^p dx \right)^{\frac{1}{p}}$$

Then $\lim_{n\to\infty} \|\beta_n f - f\|_{p,\alpha} \le \left(\int_0^\infty \left| \frac{x^2}{\omega_\alpha(x)} \right|^p dx \right)^{\frac{1}{p}} \lim_{n\to\infty} \frac{3n+2}{n^2} + \left(\int_0^\infty \left| \frac{x}{\omega_\alpha(x)} \right|^p dx \right)^{\frac{1}{p}} \lim_{n\to\infty} \frac{(n+1)}{n^2} = 0$

Thus, $\lim_{n\to\infty} \|\beta_n f - f\|_{p,\alpha} = 0.$

Then by using lemma (2.2), one can get desired result.

<u>4. Approximation of Functions of Multiple Variables by Modified Multi-Bata-Operator:</u>

Here, we generalized the results that are given in the pervious section to be valid for the modified multi-Beta operator and we approximate any continuous function of m independent variables on $[0,\infty)^m$ by this operators.

For any $(x_1, x_2, \dots x_m) \in [0,\infty)^m$ *and* $n_1, n_2, \dots n_m \in N$ we define the modified multi–Beta operator $\beta_{n_1,n_2,\dots,n_m} : L_{q,\alpha} \longrightarrow L_{q,\alpha}$ by:

$$\beta_{n_1,n_2,\dots,n_m}(f; x_1, x_2, \dots, x_m) = \frac{1}{\prod_{i=1}^m n_i} \sum_{k_m=0}^\infty \sum_{k_{m-1}=0}^\infty \dots \sum_{k_1=0}^\infty \prod_{i=1}^m \frac{(n_i + k_i)}{k_i!(n_i - 1)!} (x_i)^{k_i} (1+x_i)^{n_i - k_i - 1} f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}, \dots, \frac{k_m}{n_m} \right).$$

<u>Lemma (4.1):-</u>

For each $1 \le q < \infty$, $L_{q,\alpha} = \left\{ f \middle| f : [0,\infty)^m \longrightarrow R \right.$ is a continuous function such that $\left. \int_0^\infty \int_0^\infty \dots \int_0^\infty \left| \frac{f(x_1, x_2, \dots, x_m)}{\omega_\alpha(x_1, x_2, \dots, x_m)} \right|^q dx_1 dx_2 \dots dx_m < \infty \right\}$ is a normed space, where $\omega_\alpha(x_1, x_2, \dots, x_m) = e^{\alpha \sum_{i=1}^m x_i}$, α is a positive real number.

<u>Proof:-</u>

It is easy to check that $L_{q,\alpha}$ is a vector space.

Define $\|\,.\,\|_{q,\alpha} : L_{q,\alpha} \longrightarrow R^+ \cup \{0\}$ by:

$$\|f\|_{q,\alpha}=\left(\int_0^\infty\int_0^\infty\dots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}.$$

Then we prove $\|\,.\,\|_{q,\alpha}$ is a norm on $L_{q,\alpha}$. To do this, we must prove the following conditions:

(i) If $f=0$ then $\|f\|_{q,\alpha}=\left(\int_0^\infty\int_0^\infty\dots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}=0$. Conversely let $\|f\|_{q,\alpha}=0$ then $\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q=0$ and hence $f(x_1,x_2,\dots,x_m)=0 \quad \forall x_i\geq 0, i=1,2,\dots,m$. Therefore $f=0$.

(ii) Let $f,g\in L_{q,\alpha}$ then

$$\|f+g\|_{q,\alpha}\leq\left(\int_0^\infty\int_0^\infty\dots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}+\left(\int_0^\infty\int_0^\infty\dots\int_0^\infty\left|\frac{g(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}$$

$$=\|f\|_{q,\alpha}+\|g\|_{q,\alpha}$$

(iii) Let $\lambda\in\Box$ and $f\in L_{q,\alpha}$ then

$$\|\lambda f\|_{q,\alpha}=\left(\int_0^\infty\int_0^\infty\dots\int_0^\infty\left|\frac{(\lambda f)(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}$$

$$=|\lambda|\left(\int_0^\infty\int_0^\infty\dots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m\right)^{\frac{1}{q}}$$

$$=|\lambda|\ \|f\|_{q,\alpha}.$$

Therefore; $L_{q,\alpha}$ is a normed space.

Now, the following lemma shows some properties of the operator $\beta_{n_1,n_2,\dots,n_m}$.

Lemma (4.2):-

For any $x\in[0,\infty)^m$ and $n_1,n_2,\dots n_m\in N$, the following statements hold:

(1) $\beta_{n_1,n_2,\dots,n_m}(f;x_1,x_2,\dots,x_m)=1$, where $f(x_1,x_2,\dots x_m)=1$.

(2) $\beta_{n_1,n_2,\dots n_m}(f;x_1,x_2,\dots x_m)=\dfrac{n_jx_j+x_j}{n_j}$, where $f(x_1,x_2,\dots x_m)=x_j$ for Some $j\in\{1,2,\dots,m\}$.

(3) $\beta_{n_1,n_2,\dots n_m}(f;x_1,x_2,\dots x_m)=\sum_{i=1}^{m}\dfrac{(n_i+1)(n_i+2)x_i^2+(n_i+1)x_i}{n_i^2}$, where $f(x_1,x_2,\dots x_m)=\sum_{i=1}^{m}x_i^2$.

Proof:-

(1) Let $f(x_1,x_2,\dots x_m)=1$, then

$\int_0^\infty\int_0^\infty\dots\int_0^\infty\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m=(\frac{1}{\alpha q})^m<\infty$. Therefore; $f\in L_{q,\alpha}$. In this case

$$\beta_{n_1,n_2,\dots,n_m}(f;x_1,x_2,\dots,x_m)=\frac{1}{\prod_{i=1}^{m} n_i}\sum_{k_m=0}^{\infty}\ \sum_{k_{m-1}=0}^{\infty}\dots\sum_{k_1=0}^{\infty}\prod_{i=1}^{m}\frac{(n_i+k_i)}{k_i!(n_i-1)!}x_i^{k_i}(1+x_i)^{n_i-k_i-1}$$

$$=\frac{1}{\prod_{i=1}^{m} n_i}\prod_{i=1}^{m}\ \sum_{k_1=0}^{\infty}\frac{(n_i+k_i)}{k_i!(n_i-1)}x_i^{k_i}(1+x_i)^{n_i-k_i-1}$$

$$=\frac{1}{\prod_{i=1}^{m} n_i}\prod_{i=1}^{m} n_i=1.$$

(2) Let $f(x_1,x_2,\dots x_m)=x_j$. for some $j\in\{1,2,\dots,m\}$.

Then it is easy to check that $\int_0^{\infty}\int_0^{\infty}\dots\int_0^{\infty}\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m}\right|^q dx_1dx_2\dots dx_m<\infty$. therefore; $f\in L_{q,\alpha}$ for some $j\in\{1,2,\dots,m\}$.

Consider

$$\beta_{n_1,n_2,\dots,n_m}(f;x_1,x_2,\dots,x_m)=\frac{1}{\prod_{i=1}^{m} n_i}\sum_{k_m=0}^{\infty}\ \sum_{k_{m-1}=0}^{\infty}\dots\sum_{k_1=0}^{\infty}\prod_{i=1}^{m}\frac{(n_i+k_i)}{k_i!(n_i-1)!}x_i^{k_i}(1+x_i)^{n_i-k_i-1}\left(\frac{k_i}{n_i}\right)$$

$$\beta_{n_1,n_2,\dots,n_m}(f;x_1,x_2,\dots,x_m)=\left[\frac{1}{\prod_{\substack{i=1\\i\neq j}}^{m} n_i}\prod_{\substack{i=1\\i\neq j}}^{m}\sum_{k_1=0}^{\infty}\frac{(n_i+k_i)}{k_i!(n_i-1)!}x_i^{k_i}(1+x_i)^{n_i-k_i-1}\right]\frac{1}{n_j}\sum_{k_j=0}^{\infty}\frac{(n_j+x_j)^{k_j}}{k_j(n_j-1)!}x_j^{k_j}(1+x_j)^{n_j-k_j-1}\left(\frac{k_j}{n_j}\right)$$

$$=\frac{1}{\prod_{\substack{i=1\\i\neq j}}^{m} n_i}\left[\prod_{\substack{i=1\\i\neq j}}^{m} n_i\right]\left[\frac{(n_j+1)}{n_j}\right]=\frac{(n_j+1)x_j}{n_j}\quad\text{for some } j\in\{1,2,\dots,m\}$$

(3)Let $f(x_1,x_2,\dots x_m)=\sum_{i=1}^{m} x_i^2$, then it is easy to check that

$\int_0^{\infty}\int_0^{\infty}\dots\int_0^{\infty}\left|\frac{f(x_1,x_2,\dots,x_m)}{\omega_\alpha(x_1,x_2,\dots,x_m)}\right|^q dx_1dx_2\dots dx_m<\infty$. Therefore; $f\in L_{q,\alpha}$

Consider

$$\beta_{n_1,n_2,\dots,n_m}(f;x_1,x_2,\dots,x_m)=\frac{1}{\prod_{i=1}^{m} n_i}\sum_{k_m=0}^{\infty}\ \sum_{k_{m-1}=0}^{\infty}\dots\sum_{k_1=0}^{\infty}\prod_{i=1}^{m}\frac{(n_i+k_i)}{k_i!(n_i-1)!}x_i^{k_i}(1+x_i)^{n_i-k_i-1}.\sum_{i=1}^{m}\left(\frac{k_i}{n_i}\right)^2$$

$$=\frac{1}{\prod_{i=1}^{m} n_i}\sum_{j=1}^{m}\left[\prod_{\substack{i=1\\i\neq 1}}^{m}\ \sum_{k_1=0}^{\infty}\frac{(n_i+k_i)}{k_i!(n_i-1)!}x_i^{k_i}(1+x_i)^{n_i-k_i-1}\right].\qquad\left[\sum_{k_i=0}^{\infty}\frac{(n+k_i)}{k_i!(n_i-1)!}x_i^{k_i}(1+x_i)^{n_i-k_i-1}\left(\frac{k_i}{n_i}\right)^2\right]$$

$$=\sum_{i=1}^{\infty}\frac{(n_i+1)(n_i+2)x_i^{\,2}+(n_i+1)x_i}{n_i^2}.$$

Next, we prove that $\beta_{n_1,n_2,\dots,n_m} f$ is convergent to f as $n_1,n_2,\dots n_m \to \infty$. But before that we need the following lemma. This lemma is a modification of lemma (2.2) and the proof of it is similar to the proof of lemma (2.2), thus we omitted it.

Lemma (4.3):-

Let $L_{n_1,n_2,\dots,n_m}$ be a uniformly bounded sequence of positive linear operators from such that $L_{q,\alpha}(R^m)$ into itself satisfying the condition $\lim\limits_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| L_{n_1,n_2,\dots,n_m} f - f \right\|_{q,\alpha} = 0.$ where $f(x_1,x_2,\dots,x_m) = 1, x_j, \sum_{i=1}^{m} x_i^2$ for some $j \in \{1,2,\dots,m\}$ thus for every $f \in L_{q,\alpha}(R^m)$, $\lim\limits_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| L_{n_1,n_2,\dots,n_m}(f) - f \right\|_{q,\alpha} = 0.$

Theorem (4.4):-

Let $f \in L_{q,\alpha}$, then $\beta_{n_1,n_2,\dots,n_m} f \longrightarrow f$ as $n_1,n_2,\dots n_m \to \infty$.

Proof :-

Let $f(x_1,x_2,\dots x_m) = 1,\ \forall\ x \in [0,\infty)^m$ then from lemma (4.2) one can have:

$$\lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| \beta_{n_1,n_2,\dots,n_m} f - f \right\|_{q,\alpha} = 0.$$

Also, for $f(x_1,x_2,\dots x_m) = x_j$, for some $j \in \{1,2,\dots,m\}$ one can have:

$$\lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| \beta_{n_1,n_2,\dots,n_m} f - f \right\|_{q,\alpha} = \lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left(\int_0^\infty \int_0^\infty \dots \int_0^\infty \left| \frac{\frac{(n_j+1)x_j}{n_j} - x_j}{\omega_\alpha(x_1,x_2,\dots,x_m)} \right|^q dx_1 dx_2 \dots dx_m \right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty \int_0^\infty \dots \int_0^\infty \left| \frac{x_j}{\omega_\alpha(x_1,x_2,\dots,x_m)} \right|^q dx_1 dx_2 \dots dx_m \right)^{\frac{1}{q}} \lim_{n_j\to\infty} \frac{1}{n_j} = 0.$$

Moreover; for $f(x) = \sum_{i=1}^{m} x_i^2$ then

$$\lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \left\| \beta_{n_1,n_2,\dots,n_m} f - f \right\|_{q,\alpha} = \lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \frac{1}{n_i^2} \left(\int_0^\infty \int_0^\infty \dots \int_0^\infty \left| \sum_{i=1}^{m} \frac{(n_i+1)(n_i+2)x_i^2 + (n_i+1)x_i - n_i^2 x_i^2}{\omega_\alpha(x_1,x_2,\dots,x_m)} \right|^q dx_1 dx_2 \dots dx_m \right)^{\frac{1}{q}}$$

$$= \lim_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}} \frac{1}{n_i^2} \left(\int_0^\infty \int_0^\infty \dots \int_0^\infty \left| \sum_{i=1}^{m} \frac{(3n_i+2)x_i^2 + (n_i+1)x_i}{\omega_\alpha(x_1,x_2,\dots,x_m)} \right|^q dx_1 dx_2 \dots dx_m \right)^{\frac{1}{q}}$$

$$\leq \sum_{i=1}^{m}\left(\int_0^\infty\int_0^\infty..\int_0^\infty\left|\frac{x_i^2}{\omega_\alpha(x_1,x_2,\ldots x_m)}\right|^q dx_1dx_2..dx_m\right)^{\frac{1}{q}}\lim_{n_i\to\infty}\frac{3n_i+2}{n_i^2}+\sum_{i=1}^{m}\left(\int_0^\infty\int_0^\infty..\int_0^\infty\left|\frac{x_i^2}{\omega_\alpha(x_1,x_2,\ldots x_m)}\right|^q dx_1dx_2..dx_m\right)^{\frac{1}{q}}\lim_{n_i\to\infty}\frac{n_i+1}{n_i^2}\Bigg]$$

$$=0.$$

Thus, $\lim\limits_{\substack{n_1\to\infty\\ n_2\to\infty\\ \vdots\\ n_m\to\infty}}\left\|\beta_{n_1,n_2,\ldots,n_m}f-f\right\|_{q,\alpha}=0.$ By using lemma (4.3), one can get $\beta_{n_1,n_2,\ldots,n_m}f\longrightarrow f$ as $n_1,n_2,\ldots n_m\to\infty.$

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