Pairwise Separation Axioms And Compact Double Topological Spaces

By

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Abstract

The concept of intuitionistic (double) topological spaces was introduced by Çoker 1996. The aim of this paper is to give a nation of pairwise compactness for double topological spaces and some separation axioms .

الملخص:

فكرة فضاء التبولوجي المضاعف قدم من قبل Coker في العام ١٩٩٦ . الهدف من هذا البحث هو تقديم تعريف التراص الزوجي للفضاءات التبولوجية المضاعفة وبعض بديهيات الفصل.

1-Introduction

The concept of a fuzzy topology was introduced by Change in 1968 [2] after the introduction of fuzzy sets by Zadeh in 1965. Later this concept was extended to intuitionistic fuzzy topological spaces by Çoker in [4] . In [5] Coker studied continuity, connectedness, compactness and separation axioms in intuitionistic fuzzy topological spaces. In this paper we follow the suggestion of J.G. Garcia and S.E. Rodabaugh [7] that (double fuzzy set)is a more appropriate name than (intuitionistic fuzzy set) ,and therefore adopt the term (double-set) for the intuitionistic set , and (double-topology) for the intuitionistic topology of Dogan Çoker , (this issue) we denote by **Dbl-Top** the construct (concrete texture over Set) whose objects are pairs (X,τ) where τ is a double-topology on X .In Section three we discuss making use of this relation between bitopological spaces and double- topological spaces , we generalize a nation of compactness for double- topological space in section four with some theorems about T_1 , T_2 , T_3 .

2-Preliminaries

Throughout the paper by X we denote a non-empty set . In this section we shall present various fundamental definitions and propositions. The following definition is obviously inspired by Atanassov [1].

2.1.Definition. [8] A double-set (Ds in brief) A is an object having the form A=<x,A_1,A_2> .

Where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \phi$. The set A_1 is called the set of members of A , while A_2 is called the set of non- members of A .

Throughout the remainder of this paper we use the simpler A= (A_1,A_2) for a double-set.

2.2.Remark . Every subset A of X may obviously be regarded as a double-set having the form A'= (A,A^c) , where A^c =X\A is the complement of A in X .

We recall several relations and operations between DS' s as follows:

2.3.Definition. [8] Let the DS's A and B on X be the form A=(A_1, A_2) , B=(B_1 , B_2) , respectively . Furthermore, let $\{A_j : j \in J\}$ be an arbitrary family of DS's in X , where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then

(a) $A \subseteq B$ *if and only if* $A_1 \subseteq B_1$ *and* $A_2 \supseteq B_2$;

(b) $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$;

(c) $\overline{A} = (A_2, A_1)$ denotes the complement of A ;

(d) $\bigcap A_j = (\bigcap A_j^{(1)}, \bigcup A_j^{(2)})$;

(e) $\bigcup A_j = (\bigcup A_j^{(1)}, \bigcap A_j^{(2)})$;

(f) $[]A = (A_1, A_1^c)$;

(g) $\langle\rangle A = (A_2^c, A_2)$;

(h) $\underset{\sim}{\phi} = (\phi, X)$ *and* $\underset{\sim}{X} = (X, \phi)$.

In this paper we require the following :

(i) $[\,]A = (A_1, \phi)$, *and* (ii) $\langle\,\rangle A = (\phi, A_2)$.

Is call the image and preimage of DS' s under a function .

2.4.Definition. [3,8] Let $x \in X$ be a fixed element in X. Then:

(a)The DS given by $\underset{\sim}{x} = (\{x\}, \{x\}^c)$ is called a double–point (DP in brief X) .

(b)The DS $\underset{\approx}{x} = (\phi, \{x\}^c)$ is called a vanishing double-point (VDP in brief X) .

2.5.Definition. [3,8]

(a) Let $\underset{\sim}{x}$ be a DP in X and A=(A_1,A_2) be a DS in X . Then $\underset{\sim}{x} \in A$ iff $x \in A_1$.

(b) Let $\underset{\approx}{x}$ be a VDP in X and A=(A_1,A_2) a DS in X . Then $\underset{\approx}{x} \in A$ iff $x \notin A_2$.

It is clear that $\underset{\sim}{x} \in A \Leftrightarrow \underset{\sim}{x} \subseteq A$ and that $\underset{\approx}{x} \in A \Leftrightarrow \underset{\approx}{x} \subseteq A$.

2.6.Definition. [10] A double-topology (DT in brief) on a set X is a family τ of DS's in X satisfying the following axioms :

T1: $\underset{\sim}{\phi}, \underset{\sim}{X} \in \tau$,

T2: $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

T3: $\cup G_j \in \tau$ for any arbitrary family $\{G_j : j \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called a double-topological space (DTS in brief), and any DS in τ is known as a double open set (DOS in brief). The complement $\overline{A}$ of a DOS A in a DTS is called a double closed set (DCS in brief) in X .

2.7.Definition. [10] Let (X, τ) be an DTS and A = (A_1,A_2) be a DS in X.

Then the interior and closure of A are defined by :

$$\text{int}(A) = \cup\{G : G \text{ is a DOS in } X \text{ and } G \subseteq A\},$$

$$cl(A) = \cap\{(H : H \text{ is a DCS in } X \text{ and } A \subseteq H\},$$

respectively .

It is clear that cl(A) is a DCS in X and int(A) a DOS in X . Moreover A is a DCS in X iff cl(A) = A , and A is a DOS in X iff int(A) = A .

2.8. Example. [5] Any topological space (X,τ_0) gives rise to a DT of the form $\tau = \{A' : A \in \tau_0\}$ by identifying a subset A in X with its counterpart A'= (A , A^c), as in Remark 2.2.

3- The Construction of Dbl-Top and Bitop :

We begin by recalling the following results which associates a bitopology with a double topology.

3.1.Prposition. [5] Let (X,τ) be a DTS.

(a) $\tau_1 = \{A_1 : \exists A_2 \subseteq X \; with \; A = (A_1, A_2) \in \tau\}$ is a topology on X.

(b) $\tau_2^* = \{A_2 : \exists A_1 \subseteq X \; with \; A = (A_1, A_2) \in \tau\}$ is the family of closed sets of the topology $\tau_2 = \{A_2^c : \exists A_1 \subseteq X \; with \; A = (A_1, A_2) \in \tau\}$ is a topology on X.

(c) Using (a) and (b) we may conclude that (X, τ_1, τ_2) is a bitopological space.

3.2.Proposition. Let *(X,u,v)* be a bitopological space. Then the family

$$\{(U,V^c) : U \in u,\; V \in v, U \subseteq V\}$$

is a double topology on X .

Proof : The condition $U \subseteq V$ ensures that $U \cap V^c = \phi$,while the given family contains $\underset{\sim}{\phi}$ because $\phi \in u, v$, and it contains $\underset{\sim}{X}$ because $X \in u, v$. Finally this family is closed under finite intersections and arbitrary unions by Definition 2.3 (d,e) and the corresponding properties of the topologies *u* and *v* .

3.3. Definition. Let *(X,u,v)* be a bitopological space. Then we set

$$\tau_{uv} = \{(U,V^c) : U \in u,\; V \in v, U \subseteq V\}$$

and call this the double topology on X associated with *(X,u,v)* .

3.4.Proposition. If *(X,u,v)* is a bitopological space and τ_{uv} the corresponding DT on X , then

$$(\tau_{uv})_1 = u \; and \; (\tau_{uv})_2 = v .$$

Proof. $U \in u$ implies $(U,\phi) \in \tau_{uv}$ since $U \subseteq X \in v, so\, u \subseteq (\tau_{uv})_1$. Conversely, take $U \in (\tau_{uv})_1$. Then $(U,B) \in \tau_{uv}$ for some $B \subseteq X$, and now $U \in u$. Hence $(\tau_{uv})_1 \subseteq u$, and the first equality is proved .

■

The proof of the second equality may be obtained in a similar manner, and we omit the details. Now we define double compact set and we use the link between bitopological space and double topological space to established some theorems .

4.Pairwise Double Compact Set :

4.1. Definition. By an double open cover of a subset A of a double topological space (X,τ), we mean a collection $C = \{G_j : j \in J\}$ of double open subsets of X such that $A \subset \bigcup\{G_j : j \in J\}$ then we say that C covers A . In particular , A collection C is said to be an open cover of the space X iff $X = \bigcup\{(G_j^1, G_j^2) : j \in J\}$ of double open subsets of X .

4.2.Definition. A double-set A of DTS in (X,τ) is said to be double compact set iff every double sub cover , that is iff for every collection $\{G_j : j \in J\}$ of DOS's for which $A \subset \{G_j : j \in J\}$ *for* $A = (A_1, A_2)$ such that $(A_1, A_2) \subset (G_{j1}^1, G_{j1}^2) \bigcup ... \bigcup (G_{jn}^1, G_{jn}^2)$.

4.3.Definition. Let (X,τ) be DTS and let $N \in X$. A double set N of X is said to be τ-nhd of $\underset{\sim}{x}$ iff there exists τ-DOS , G such that $\underset{\sim}{x} \in G \subset N$, similarly N is called a τ-double nhd of $A \subset X$ iff there exists an DOS , G such that $A \subset G \subset N$.

4.4.Definition.[3] The DTS (X,τ) is called pairwise T_2 if given $x \neq y$ *in* X *there exists* $G, H \in \tau$ Satisfying $\underset{\sim}{x} \in G, \underset{\approx}{y} \in H$ *and* $G \subseteq ()\overline{H}$.

4.5. Proposition. If (X,τ) is pairwise T_2 then every double compact set is double closed set.

Proof: We shall show that $\overline{G} \in \tau$ *isdouble open set* . Let $\underset{\sim}{p} \in \bar{G} = (G_1, G_2)$. Since X is T_2 then for.

Then $p \in G_2, y \notin H_2 \Rightarrow y \in H_2^c, \quad G_2 \cap H_2^c = \underset{\sim}{\phi}$

$\exists$ *double open nhds of* p, y , $\mu(p)$ & $N(y)$ *such that* $\mu(p) \cap N(y) = \phi$

Now the collection $\{\mu(p) : p \in G_2\}$ double open cover of G_2

$\because G$ *is compact then* $\{G_2 \subset \cup\mu(p_i)\}$.

let $M = \cup\,\mu(p_i)$, $N = \cap\, N(y_i)$ *then* N *is double open nhd of* y_i

We claim that $M \cap N = \underset{\sim}{\phi}$,

$z \in \mu \Rightarrow z \in \mu(p_i) \Rightarrow z \notin N(y_i) \Rightarrow z \notin N$, *thus* $M \cap N = \underset{\sim}{\phi}$

Since $G_2 \subset M$, then $G_2 \cap N = \phi \Rightarrow N \subset G_2 \Rightarrow N \subset \overline{G}$ this shows that $\overline{G}$ contains a nhd of each of its point and so $\overline{G}$ is DOS otherwise G is DCS .

4.6. Proposition. Let A and B be disjoint double compact subsets of a DTS (X, τ)

Then there exists disjoint DOS's G and H such that $A \subset G$ *and* $B \subset H$. ∎

Proof : First, let $\underset{\sim}{x} \in A$ be fixed . Since X is pairwise T_2 and $\underset{\sim}{x} \notin B$, for each $\underset{\approx}{y} \in B$ $A \subseteq (\,)\overline{B}$.(*clearly* $x \in A_1$, $y \notin B_2 \Rightarrow y \in B_2^c$ *for* $A = (A_1, A_2)$, $B = (B_1, B_2)$)

There exists DOS's $G_{\underset{\approx}{y}}$ *and* $H_{\underset{\approx}{y}}$ *such that* $\underset{\sim}{x} \in G_{\underset{\approx}{y}}$ *and* $\underset{\approx}{y} \in H_{\underset{\approx}{y}}$. The collection $\{H_{\underset{\approx}{y}} : \underset{\approx}{y} \in B\}$ is a double open cover of B . Since B is double compact subspace of X , there exist finitely many points $\underset{\approx}{y_1}, \underset{\approx}{y_2}, \dots, \underset{\approx}{y_n}$ of B such that

$$B \subset \{H_{\underset{\approx}{y_i}} : i = 1,2,\dots,n\}, \quad (B_1, B_2) \subset \{(H^1_{\underset{\approx}{y_i}}, H^2_{\underset{\approx}{y_i}})\, i = 1,2,\dots,n\}$$

let $G_{\underset{\sim}{x}} = \bigcap\{G_{\underset{\approx}{y_i}} : i = 1,2,\dots, n\} = \bigcap\{(G^1_{\underset{\approx}{y_i}}, G^2_{\underset{\approx}{y_i}}) : i = 1,2,\dots, n\}$

$H_{\underset{\sim}{x}} = \bigcup\{(H^1_{\underset{\approx}{y_i}}, H^2_{\underset{\approx}{y_i}}) : i = 1,2,\dots,n\}$ *then* $G_{\underset{\approx}{x}}$, $H_{\underset{\approx}{x}}$ are disjoint open sets such that $\underset{\sim}{x} \in G_{\underset{\sim}{x}}$ *and* $B \subset H_{\underset{\sim}{x}}$.

now let $\underset{\sim}{x} \in A$ be arbitrary and let $G_{\underset{\sim}{x}}$ *and* $H_{\underset{\sim}{x}}$ be as constructed above , then evidently the collection $\{G_{\underset{\sim}{x}} : \underset{\sim}{x} \in A\}$ is a double open cover of A . Since A is a double compact subspace of X. There exist finitely many points, $\underset{\sim}{x_1}, \underset{\sim}{x_2}, \dots, \underset{\sim}{x_m}$ *such that* $A \subset \bigcup\{G_{\underset{\sim}{x_i}} : i = 1,2,\dots,m\}$, *let* $G = \bigcup\{G_{\underset{\sim}{x_i}} : i = 1,2,\dots,m\}$ and $H = \bigcap\{H_{\underset{\sim}{x_i}} : i = 1,2,\dots,m\}$ then G and H are disjoint double open sets such that

$A \subset G$ *and* $B \subset H$. ∎

4.7. Definition. [3] The DTS (X,τ) is called pairwise T_1 if given $x \neq y$ *in* X *there exists*

$G \in \tau$ *with* $\underset{\sim}{x} \in G, \underset{\sim}{y} \notin G$, *and there exists* $H \in \tau$ *with* $\underset{\approx}{y} \in H, \underset{\approx}{x} \notin H$.

4.8. Definition . [6] The DTS (X,τ) is called pairwise T_3 if $\forall\, DCS\ A \in \tau, a \in \operatorname{int} A$ *in* X there exists $G,H \in \tau$ satisfying $\underset{\sim}{a} \in G, \underset{\approx}{a} \notin H$, $A \subseteq H$ *and* $G \subseteq ()\overline{H}$.

4.9. Proposition. The DTS (X,τ) is called pairwise T_1 iff every singleton double set $\{\underset{\sim}{x}\}$ *of* X *is* DCS .

Proof : $\Leftarrow$ Let every singleton double set $\{\underset{\sim}{x}\}$ *of* X *be* DCS *to show that the space* is T_1 .Let x , y be any two disjoint double point of X , then $\overline{\{\underset{\approx}{x}\}}$ is a DOS which contain $\underset{\sim}{y}$

Similarly $\overline{\{\underset{\approx}{y}\}}$ is a DOS which contain $\underset{\sim}{x}$ but does not contain $\underset{\sim}{y}$ .Hence (X,τ) is pairwiseT_1 .

$\Longrightarrow$ Let the space be pairwise $T_1$and let x be any point of X , we want to show that $\{\underset{\sim}{x}\}$ is DCS , that to show X-$\{\underset{\sim}{x}\}$ is DOS . Let $\underset{\sim}{y} \in$ X - $\{\underset{\sim}{x}\}$ then $x \neq y$ since X is pairwise T_1 . There exist an open $G_{\underset{\approx}{y}}$ *such that* $\underset{\sim}{y} \in G_{\underset{\approx}{y}}$ *but* $\underset{\sim}{x} \notin G_{\underset{\approx}{y}}$. *It follows that*

$\underset{\sim}{y} \in G_{\underset{\approx}{y}} \subset$ X-$\{\underset{\sim}{x}\}$ *.Hence* X $-\{\underset{\sim}{x}\}$ is DOS , and to show that $X - \{\underset{\approx}{y}\}$ *is DOS .Let*

$\underset{\approx}{x} \in X - \{\underset{\approx}{y}\}$ *this means* $\underset{\approx}{x} \in X - \{(\phi, \{y\}^c)\} \Longrightarrow x \notin X\{y\}^c, x \in X - \{y\}$ *and*

$\underset{\approx}{y} \notin X - \{\underset{\approx}{y}\}$ then there exists a DOS $H_{\underset{\approx}{x}}$ *such that* $\underset{\approx}{x} \in H_{\underset{\approx}{x}}$ *but* $\underset{\approx}{y} \notin H_{\underset{\approx}{x}}$, it follows that

$\underset{\approx}{x} \in H_{\underset{\approx}{x}} \subset X - \{\underset{\approx}{y}\}$ *.Hence* $X - \{\underset{\approx}{y}\}$ is DOS . Accordingly {x} is DCS .

■

4.10. Proposition. For a DTS (X,τ) pairwise T_3 is pairwise T_1 .

Proof : Let DTS (X,τ) be pairwise T_3 , we have G=(A,B) , H=(C,D)$\in \tau$ with $\underset{\sim}{x} \in G, \underset{\approx}{x} \notin H$ *and* $G \subseteq ()\overline{H}$ *i.e* $A \subseteq D$ *,take* $x \neq y$ *in* X *and* $\underset{\approx}{y} \in H$

$\Rightarrow y \notin D \Rightarrow y \notin A \Rightarrow \underset{\sim}{y} \notin G$ Then (X,τ) is T_1 . ■

4.11. Proposition. For a DTS (X,τ) pairwise T_3 is pairwise T_2 .

Proof : Let DTS (X,τ) be pairwise T_3 . Take $x \neq y$ *in* X
Since X *is pairwise* T_1 *, then there exist* $G \in \tau$ *with* $\underset{\sim}{x}$ *and* $\underset{\sim}{y} \notin G$ *and* $H \in \tau$ *with* $\underset{\approx}{y} \in H$ *and* $\underset{\approx}{x} \notin H$*, and since* X *is pairwise regular there exist* $G, H \in \tau$ *such that* $\underset{\sim}{x} \in G$ *and* $\underset{\approx}{x} \notin$ H , $G \subseteq (\,)\overline{H}$ so that $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ *and* $G \subseteq (\,)\overline{H}$
. Accordingly (X,τ) is T_2 . ■

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