On Fixed points of absorbing maps in fuzzy metric spaces

By:

Prof. Younus J. Yaseen* and Saja S. Mohsun**

Abstract : *In this paper , we prove a common fixed point theorem for eight self mappings using absorbing maps and reciprocal continuous maps in fuzzy metric space . Our paper extends the results of Anju Rani [1] .*

---

*This thesis is a part of M.Sc. thesis written by ** supervised by **

Introduction:

In 1965 Zadeh [7] introduced the notion of fuzzy set . Later many authors have extensively developed the theory of fuzzy sets and applications .

The purpose of this work is to give a comprehensive study of fixed point theorem and we supply the details of the proofs for most of the results that were given by Anju Rani [1] that's depended on the papers of Cho , "Common fixed points of compatible maps of type (α) on fuzzy metric space , 1998" . And Chung , "On common fixed point theorem in fuzzy metric spaces , 2002" . And Kutukchu , "On common fixed points in Menger probabilistic and fuzzy metric spaces , 2007" . The Anju Rani work was proof the six self maps have a unique common fixed point . We add also some results that seem to be new to the best of our knowledge that's we are tried to generalize this paper for eight self maps have a unique common fixed point and given some corollaries .

$1:Definitions on fuzzy normed linear spaces

In this section we shall introduce some basic concepts and definitions with some illustrate examples which are necessary to our work .

Definition 1-1:[7]

Let X be any non empty set . A fuzzy set A in X is a function with domain X and values in $[0,1]$.

Definition 1-2:[12]

Let X be a linear space over R . A fuzzy subset $\mu : X \times (0,\infty) \to [0,1]$ is said to be fuzzy norm on X if for all $x , y \in X$ and all $s , t \in (0,\infty)$:

1) $\mu(x,t) > 0$

2) $\mu(x,t) = 1 \quad if\ and\ only\ if\ x = 0$

3) $\mu(cx,t) = \mu\left(x, \frac{t}{|c|}\right) \quad ; \quad c \neq 0$

4) $\mu(x+y, s+t) \geq min\{\mu(x,s), \mu(y,t)\}$

5) $\mu(x,.) : (0,\infty) \to [0,1]$ is a non-decreasing function on R and $lim_{t\to\infty} \mu(x,t) = 1$

Then (X,μ) is called a fuzzy normed linear space .

Example 1-3:[5]

Let $(X, \|\ .\ \|)$ is a normed linear space . define a fuzzy subset

$\mu(x,t) = \frac{t}{t+\|x\|}$. *Then (X,μ) is a fuzzy normed linear space .*

Definition 1-4:[9]

Let X be any non-empty set and let $F(\tilde{X})$ *be the set of all fuzzy set on* X, *for* $f, g \in F(\tilde{X})$ *such that* $f: X \times (0,1] \to [0,1]$, $g: X \times (0,1] \to [0,1]$ *and* $k \in R$ *define*

$f + g = \{(x + y, \mu \wedge \lambda) : (x, \mu) \in f \text{ and } (y, \lambda) \in g\}$ *and* $kf = \{(kx, \mu) : (x, \mu) \in f\}$

Definition 1-5:[9]

A fuzzy linear space $\tilde{X} = X \times (0,1]$ *over the field* F (R *or* C) *where the addition and scalar multiplication operation on* X *are define by :*

$$(x, \lambda) + (y, \mu) = (x + y, \lambda \wedge \mu) \quad \text{and} \quad k(x, \lambda) = (kx, \lambda)$$

is a fuzzy normed space if to every $(x, \lambda) \in \tilde{X}$ *there is associated a non-negative real number* $\|(x, \lambda)\|$ *called the fuzzy norm of* (x, λ), *in such a way that :*

1) $\|(x, \lambda)\| = 0$ *if and only if* $x = 0$, $\lambda \in (0,1]$

2) $\|k(x, \lambda)\| = |k|\|(x, \lambda)\|$ *for all* $(x, \lambda) \in \tilde{X}$, $k \in F$

3) $\|(x, \lambda) + (y, \mu)\| \leq \|(x, \lambda \wedge \mu)\| + \|(y, \lambda \wedge \mu)\|$ *for all* $(x, \lambda), (y, \mu) \in \tilde{X}$

4) $\|x, \vee_t \lambda_t\| = \wedge_t \|(x, \lambda)\|$ *for all* $\lambda_t \in (0,1]$.

Definition 1-6:[9]

The linear space $F(\tilde{X})$ *is said to be normed space if for every* $f \in F(\tilde{X})$, *there is associated a non-negative real number* $\|f\|$ *called the norm of* f *in such a way that :*

1) $\|f\| = 0$ *if and only if* $f = 0$ *.For*

$\|f\| = 0$ *if and only if* $\sup\{\|(x, \mu)\| : (x, \mu) \in f\} = 0$

if and only if $x = 0$, $\mu \in (0,1]$

if and only if $f = 0$

2) $\|kf\| = \sup\{\|k(x, \mu)\| : (x, \mu) \in f, \ k \in R\}$

$= \sup\{|k|\|(x, \mu)\| : (x, \mu) \in f\} = |k|\|f\|$

3) $\|f + g\| \leq \|f\| + \|g\|$. *For*

$$\|f + g\| = \sup\{\|(x, \mu) + (y, \lambda)\| : (x, \mu) \in f \text{ and } (y, \lambda) \in g\}$$
$$= \sup\{\|(x + y, \mu \wedge \lambda)\| : x, y \in X \text{ and } \mu, \lambda \in (0,1]\}$$

$$\leq sup\{\|(x,\mu\wedge\lambda)\| + \|(y,\mu\wedge\lambda)\| : (x,\mu) \in f and\ (y,\lambda) \in g\}$$
$$\leq sup\{\|(x,\mu\wedge\lambda)\| : (x,\mu) \in f\} + \sup\{\|(y,\mu\wedge\lambda)\| : (y,\lambda) \in g\} = \|f\| + \|g\|$$

Then $(F(\tilde{X}), \|.\|)$ *is a normed linear space .*

Definition 1-7:[9]

Let $F(\tilde{X})$ *be a linear space over* R *. A fuzzy subset* $\mu : F(\tilde{X}) \times (0,\infty) \to [0,1]$ *is called fuzzy norm on X if the following are satisfies :*

1) $\mu(f,t) > 0 \qquad \forall f \in F(\tilde{X})$
2) $\mu(f,t) = 1 \quad if\ and\ only\ if\ f = 0$
3) $\mu(cf,t) = \mu\left(f,\frac{t}{|c|}\right) \quad ; c \neq 0$
4) $\mu(f+g, s+t) \geq min\{\mu(f,s), \mu(g,t)\}$
5) $\mu(f,.) : (0,\infty) \to [0,1]$ *is a non-decreasing function on* R *and* $lim_{t\to\infty}\, \mu(f,t) = 1$

Then $(F(\tilde{X}), \mu)$ *is called a fuzzy normed linear space .*

Example 1-8:

Let X be a non-empty set and $(F(\tilde{X}), \|.\|)$ *be a normed linear space . define* $\mu: F(\tilde{X}) \times (0,\infty) \to [0,1]$ *by :* $\mu(f,t) = \frac{t}{t+\|f\|}$ $where \quad t \in (0,\infty)$ *such that* $\|f\| = \{\|(x,\lambda)\| : (x,\lambda) \in f\}$ *, Then* $(F(\tilde{X}), \mu)$ *is a fuzzy normed linear space .*

Definition 1-9:[4]

Let X *be a non-empty set , then* (X,M) *is a fuzzy metric space if* M *is a fuzzy set in* $X^2 \times (0,\infty)$ *satisfying the following condition , for all* $x,y,z \in X\ and\ \ s,t \in (0,\infty)$ *:*

1) $M(x,y,t) > 0$
2) $M(x,y,t) = M(y,x,t)$
3) $M(x,y,t) = 1 \ \ if\ and\ only\ if\ \ x = y$
4) $M(x,y,t+s) \geq min\{\ M(x,z,t),\ M(z,y,s)\}$
5) $M(x,y,t) : (0,\infty) \to [0,1]$ *is a non-decreasing function and* $lim_{t\to\infty} M(x,y,t) = 1$

Example 1-10:[4]

Let (X, d) *be a metric space . define* $M: X^2 \times (0,\infty) \to [0,1]$ *by :*

$M(x,y,t) = \frac{t}{t+d(x,y)}$. Then (X, M) is a fuzzy metric space .

$2: Fixed points of absorbing maps in fuzzy metric space

In this section we recall some definitions and known results in fuzzy metric space .

Definition 2-1:[12]

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called continuous t-norm if $*$ is satisfying the following conditions :

1) $*$ is commutative and associative ,

2) $*$ is continuous ,

3) $a * 1 = a$ for all $a \in [0,1]$,

4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$

and $a, b, c, d \in [0,1]$.

Definition 2-2:[4]

The 3-tuple $(X, M, *)$ is said to be fuzzy metric space if X is an arbitrary set , $*$ is continuous t-norm and M is a fuzzy set in $X^2 \times (0,\infty)$ satisfying the following ; for all $x, y, z \in X$ and $t, s \in (0,\infty)$:

1) $M(x,y,t) > 0$

2) $M(x,y,t) = M(y,x,t)$

3) $M(x,y,t) = 1$ if and only if $x = y$

4) $M(x,y,s+t) \geq M(x,z,s) * M(z,y,t)$

5) $M(x,y,.) : (0,\infty) \to [0,1]$ is a left continuous function and $\lim_{t\to\infty} M(x,y,t) = 1$.

Definition 2-3:[1]

Let $(X, M, *)$ be a fuzzy metric space :

i) A sequence $\langle x_n \rangle$ in X is said to be convergent to a point $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$.

ii) A sequence $\langle x_n \rangle$ in X is said to be Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $p > 0$.

iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete .

Definition 2-4:[1]

A pair (A, S) *of self maps of a fuzzy metric space* $(X, M, *)$ *is said to be reciprocal continuous if* $lim_{n\to\infty} ASx_n = Ax$ *and* $lim_{n\to\infty} SAx_n = Sx$, *whenever there exists a sequence* $\langle x_n \rangle \in X$ *such that* : $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x$ *for some* $x \in X$

If A *and* S *are both continuous then they are reciprocally continuous but the converse need not be true .*

Example 2-5

Let $X = [\ 5,\ 25\ ]$ *and* d *be the usual metric space* X . *Define mapping* $A , S : X \to X$ *by :*

$$A(x) = \begin{cases} 5 & if \quad x = 5 \\ 7 & if \quad x > 5 \end{cases} \qquad and \quad S(x) = \begin{cases} 5 & if \quad x = 5 \\ 9 & if \quad x > 5 \end{cases}$$

It may be noted that A *and* S *are reciprocally continuous mapping , But neither* A *nor* S *is continuous mapping .*

Definition 2-6:[2]

The two maps A and B from a fuzzy metric space $(X, M, *)$ *into itself are said to be compatible if :* $lim_{n\to\infty} M(ABx_n, BAx_n) = 1$

Definition 2-7:[1]

Let f , g *are two self maps on a fuzzy metric space* $(X, M, *)$ *then* f *is called* g-*absorbing if there exists a positive integer* $K > 0$ *such that :* $M(gx, gfx, t) \geq M\left(gx, fx, \frac{t}{K}\right)$, *for all* $x \in X$.

Similarly , g *is called* f-*absorbing if there exists a positive integer* $K > 0$ *such that :* $M(fx, fgx, t) \geq M\left(fx, gx, \frac{t}{K}\right)$ *for all* $x \in X$.

The map f *is called point wise* g-*absorbing if for given* $x \in X$, *there exists a positive integer* $K > 0$ *such that :* $M(gx, gfx, t) \geq M\left(gx, fx, \frac{t}{K}\right)$

Similarly , g *is called point wise* f-*absorbing if for given* $x \in X$, *there exists a positive integer* $K > 0$ *such that :* $M(fx, fgx, t) \geq M\left(fx, gx, \frac{t}{K}\right)$

Lemma 2-8:[6],[11]

If for all $x , y \in X$, $t \in (0, \infty)$ *and* $0 < k < 1$ $M(x, y, kt) \geq M(x, y, t)$, *then* $x = y$.

Lemma 2-9:[8],[13]

For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing .

Theorem 2-10:[1]

Let P be a point wise AB-absorbing and Q be a point wise ST-absorbing self maps on complete fuzzy metric space $(X, M, *)$ with continuous t-norm defined by $a * b = min\{a, b\}$ where $a, b \in [0,1]$, satisfying the conditions :

1) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$

2) There exists $k \in (0,1)$ such that for every $x, y \in X$ and $t \in (0, \infty)$

$$M(Px, Qy, kt) \geq min\{M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, t), M(Px, STy, t)\}$$

3) For all $x, y \in X$, $lim_{t \to \infty} M(x, y, t) = 1$

4) $AB = BA$, $ST = TS$, $PB = BP$, $SQ = QS$, $QT = TQ$.

If the pair of maps (P, AB) is reciprocal continuous compatible maps then P , Q , S , T , A , and B have a unique common fixed point in X .

Proof : we can see the proof in [1] . #

Theorem 2-12:[1]

Let P be a point wise AB-absorbing and Q be a point wise ST-absorbing pairs of self mappings of a fuzzy metric space $(X, M, *)$ satisfying conditions :

1) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$

2) There exists $k \in (0,1)$ such that for every $x, y \in X$ and $t \in (0, \infty)$.

$$M(Px, Qy, kt) \geq min\{ M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, t), M(Px, STy, t)\}$$

3) For all $x, y \in X$, $lim_{t \to \infty} M(x, y, t) = 1$

4) $AB = BA$, $ST = TS$, $PB = BP$, $SQ = QS$, $QT = TQ$

If the range of one of the mappings $P(X)$, $Q(X)$, $AB(X)$ or $ST(X)$ be a complete subspace of X then P, Q, S, T, A and B have a unique common fixed point in X.

Proof : we can see the proof in [1]. #

Example 2-13:[1]

Let (X, d) be usual metric space where $X = [0,1]$ and for each $t \in [0,1]$ and M be the usual fuzzy metric on $(X, M, *)$ where $*$ is defined by

$$a * b = ab \quad with \quad M(x,y,t) = \frac{t}{t + d(x,y)} \quad for\ x, y \in X,$$

Let A, B, S, T, P *and* Q *be self maps defined as ,*

$Ax = \frac{x}{5}$ $\qquad$ $Bx = \frac{x}{3}$

$Sx = x$ $\qquad$ $Tx = \frac{x}{2}$

$Px = \frac{x}{6}$ $\qquad$ $Qx = 0 \ , \ \forall x, y \in X .$

Then $P(X) = \left[0, \frac{1}{6}\right] \subset \left[0, \frac{1}{2}\right] = ST(X)$ *, and* $Q(X) = 0 \subset \left[0, \frac{1}{15}\right] = AB(X)$ *. Hence* P *is* AB*- absorbing and* Q *is* ST*-absorbing with* $K > 0$ *.*
If we take $k = \frac{1}{2}$ *and* $t = 1$*, then the contractive condition* (2) *in theorem (2-12) is satisfied and zero is the unique common fixed point.* #

Now we going to generalize the result given in the theorem (2-10) for eight self maps:

Theorem 2-14

Let H *and* P *be a point wise* AB*-absorbing and* L *and* Q *be a point wise* ST*-absorbing self maps on complete fuzzy metric space* $(X, M, *)$ *with continuous t-norm defined by* $a * b = min\{a, b\}$ $where$ $a, b \in [0,1]$ *, satisfying the conditions :*

1) $H(X), P(X) \subseteq ST(X)$ and $L(X), Q(X) \subseteq AB(X)$

2) *There exists* $k \in (0,1)$ $such\ that\ for\ every\ x, y \in X\ and\ t \in (0, \infty)$

a) $M(Px, Qy, kt) \geq min\ \{M($ …
$M(Qy, STy, t), M(Px, STy, t)\}$

b) $M(Px, Ly, kt) \geq min\ \{M($ …
$M(Ly, STy, t), M(Px, STy, t)\}$

c)
$$M(Hx, Qy, kt) \geq min\ \{M(ABx, STy, t), M(Hx, ABx, t), M(Qy, STy, t), M(Hx, STy, t)\}$$

d)
$$M(Hx, Ly, kt) \geq min\ \{M(ABx, STy, t), M(Hx, ABx, t), M(Ly, STy, t), M(Hx, STy, t)\}$$

3) $For\ all\ x, y \in X$, $lim_{t \to \infty} M(x, y, t) = 1$

4)
$AB = BA$, $ST = TS$, $PB = BP$, $HB = BH$, $SQ = QS$, $SL = LS$, $QT = TQ$, $LT = TL$

.

If one of the pair of maps $(P,AB),(H,AB),(Q,ST)$ *or* (L,ST) *is reciprocal continuous compatible maps then* $P, Q, H, L, S, T, A,$ *and* B *have a unique common fixed point in* X .

Proof :

Let x_0 *be any arbitrary point in* X *, construct a sequence* $\langle y_n \rangle$ *in* X *such that*

$$y_{2n-1} = STx_{2n-1} = Px_{2n-2} = Hx_{2n+2} \quad and$$
$$y_{2n} = ABx_{2n} = Qx_{2n+1} = Lx_{2n+3} \;, n = 1,2,3,\cdots.$$

This can be done by the virtue of (1) . By using the same techniques of theorem (2-10) we can show that $\langle y_n \rangle$ *is Cauchy sequence in* X *. Since* $(X,M,*)$ *is complete so there exists a point (say)* z *in* X *such that* $\langle y_n \rangle \to z$ *Also we have* $\langle Px_{2n-2}\rangle, \langle Hx_{2n+2}\rangle, \langle STx_{2n-1}\rangle, \langle ABx_{2n}\rangle, \langle Qx_{2n+1}\rangle, \langle Lx_{2n+3}\rangle \to z$.

Let the pair (P,AB) *is reciprocally continuous mappings , then from theorem (2-10) we have ,* Pz *is a common fixed point of* P, Q, S, T, A *and* B.

Now , we have to show that H *and* L *have the same common fixed point :*

Now putting $x = Pz$, $y = u$ *in the contactive condition (c) we get ,*

$$M(HPz,Qu,kt) \geq \min\{M(ABPz,STu,t), M(HPz,ABPz,t),$$
$$M(Qu,STu,t), M(HPz,STu,t)\}$$
$$M(HPz,Pz,kt) \geq \min\{M(Pz,Pz,t), M(HPz,Pz,t), M(Qu,Qu,t), M(HPz,Pz,t)\}$$
$$\geq M(HPz,Pz,t)$$

so $HPz = Pz$

Now putting $x = z$, $y = Pz$ *in the contactive condition (b) we get ,*

$$M(Pz,LPz,kt) \geq \min\{M(ABz,STPz,t), M(Pz,ABz,t),$$
$$M(LPz,STPz,t), M(Pz,STPz,t)\}$$
$$\geq \min\{M(Pz,Pz,t), M(Pz,Pz,t), M(LPz,Pz,t), M(Pz,Pz,t)\}$$
$$\geq M(LPz,Pz,t) = M(Pz,LPz,t)$$

so $LPz = Pz$ *. Hence* $Pz = PPz = QPz = BPz = APz = TPz = SPz = HPz = LPz$ *. Hence* Pz *is a common fixed point of* P, Q, S, T, A, B, H *and* L.

Uniqueness is the same of theorem (2-10) .

The proof when the pair of maps (H,AB) *is reciprocal continuous compatible maps .*

Since $(X,M,*)$ *is complete so there exists a point (say)* z *in* X *such that* $\langle y_n \rangle \to z$ *Also we have* $\langle Px_{2n-2}\rangle, \langle Hx_{2n+2}\rangle, \langle STx_{2n-1}\rangle, \langle ABx_{2n}\rangle, \langle Qx_{2n+1}\rangle, \langle Lx_{2n+3}\rangle \to z$.

since the pair (H, AB) *is reciprocally continuous mappings , then we have ,* $\lim_{n\to\infty} HABx_{2n} = Hz$ *and* $\lim_{n\to\infty} ABHx_{2n} = ABz$ *, and compatibility of* H *and* AB *yields ,* $\lim_{n\to\infty} M(HABx_{2n}, ABHx_{2n}, t) = 1$ $(i.e)$ $M(Hz, ABz, t) = 1$ *. Hence* $Hz = ABz$ *. Since* $H(X) \subseteq ST(X)$ *then there exists a point* u *in* X *such that* $Hz = STu$ *.*

Now by contractive condition (d) , we get ,

$$M(Hz, Lu, kt) \geq \min\{M(ABz, STu, t), M(Hz, ABz, t), M(Lu, STu, t), M(Hz, STu, t)\}$$
$$\geq \min\{M(Hz, Hz, t), M(Hz, Hz, t), M(Lu, Hz, t), M(Hz, Hz, t)\}$$
$$\geq M(Hz, Lu, t)$$

(i.e) $Hz = Lu$ *thus* $Hz = ABz = Lu = STu$ *.*

Since H *is* AB*-absorbing then for* $k > 0$ *, we have* $M(ABz, ABHz, t) \geq M\left(ABz, Hz, \frac{t}{k}\right) \geq M\left(Hz, Hz, \frac{t}{k}\right) = 1$ *(i.e)* $Hz = ABz = ABHz$

Now by contractive condition (d) we have ,

$$M(HHz, Hz, kt) = M(HHz, Lu, kt) \geq \min\{M(ABHz, STu, t), M(HHz, ABHz, t), M(Lu, STu, t), M(HHz, STu, t)\}$$
$$\geq \min\{M(Hz, Hz, t), M(HHz, Hz, t), M(Lu, Lu, t), M(HHz, Hz, t)\}$$
$$\geq M(HHz, Hz, t)$$

(i.e) $HHz = Hz = ABHz$ *. Therefore* Hz *is a common fixed point of* H *and* AB *. Similarly ,* L *is* ST*-absorbing therefore we have ,* $M(STu, STLu, t) \geq M\left(STu, Lu, \frac{t}{k}\right) = M\left(Lu, Lu, \frac{t}{k}\right) = 1$ *(i.e)* $STu = STLu = Lu$ *.*

Now by contractive condition (d) we have ,

$$M(Lu, LLu, kt) = M(Hz, LLu, kt) \geq \min\{M(ABz, STLu, t), M(Hz, ABz, t), M(LLu, STLu, t), M(Hz, STLu, t)\}$$
$$\geq \min\{M(Hz, Lu, t), M(Hz, Hz, t), M(LLu, Lu, t), M(Hz, Lu, t)\}$$
$$\geq \min\{M(Hz, Hz, t), M(Hz, Hz, t), M(LLu, Lu, t), M(Hz, Hz, t)\}$$
$$\geq M(LLu, Lu, t)$$

(i.e) $LLu = Lu = STLu$

Now putting $Hz = Lu$ *, we have* $LHz = Hz = STHz$

Now putting $x = z$ *,* $y = Hz$ *in the contactive condition* (c) *we get ,*

$M(Hz,QHz,kt) \geq min\{M(ABz,STHz,t), M(Hz,ABz,t),$
$M(QHz,STHz,t), M(Hz,STHz,t)\}$

$$\geq min\{M(Hz,Hz,t), M(Hz,Hz,t), M(QHz,Hz,t), M(Hz,Hz,t)\}$$
$$\geq M(QHz,Hz,t) = M(Hz,QHz,t)$$

so $QHz = Hz$

Now putting $x = Hz$, $y = u$ in the contactive condition (b) we have ,

$$M(PHz,Lu,kt) \geq \min\{M(ABHz,STu,t), M(PHz,ABHz,t),$$
$$M(Lu,STu,t), M(PHz,STu,t)\}$$
$$M(PHz,Hz,kt) \geq min\{M(Hz,Hz,t), M(PHz,Hz,t), M(Lu,Lu,t), M(PHz,Hz,t)\}$$
$$\geq M(PHz,Hz,t)$$

so $PHz = Hz$

Now putting $x = BHz$, $y = Hz$ in the contractive condition (d) , we have ,

$$M(H(BHz),L(Hz),kt) \geq \min\{M(AB(BHz),ST(Hz),t),$$
$$M(H(BHz),AB(BHz),t), M(L(Hz),ST(Hz),t), M(H(BHz),ST(Hz),t)\}$$

As $HBHz = BHHz = BHz$ and $ABBHz = BABHz = BHz$, We have,

$$M(BHz,Hz,kt) \geq \min\{M(BHz,Hz,t), M(BHz,BHz,t)$$
$$M(Hz,Hz,t), M(BHz,Hz,t) \geq M(BHz,Hz,t)$$

By lemma (2-8) we have , $BHz = Hz$. Hence ,$Hz = HHz = LHz = PHz = QHz = ABHz = AHz$ .Hence ,$Hz = HHz = LHz = PHz = QHz = BHz = AHz$.

Now putting $x = Hz$,$y = THz$ in the contractive condition (d) , we have ,

$$M(HHz,LTHz,kt) \geq min\{M(ABHz,STTHz,t), M(HHz,ABHz,t),$$
$$M(LTHz,STTHz,t), M(HHz,STTHz,t)\}$$

As $STTHz = TSTHz = THz$ and $LTHz = TLHz = THz$ We have ,

$$M(Hz,THz,kt) \geq min\{M(Hz,THz,t), M(Hz,Hz,t),$$
$$M(THz,THz,t), M(Hz,THz,t)\} \geq M(Hz,THz,t)$$

By lemma (2-8) we have , $THz = Hz$. Since $Hz = HHz = LHz = PHz = QHz = BHz = AHz = THz = STHz$. Hence $Hz = HHz = LHz = PHz = QHz = BHz = AHz = THz = SHz$. Hence Hz is a common fixed point of H, L, P, Q, S, T, A and B .

Uniqueness , let Hw be another fixed point of H, L, P, Q, S, T, A and B then putting $x = Hz$ and $y = Hw$ in the contractive condition (d) we have ,

$$M(HHz, LHw, kt) \geq \min\{M(ABHz, STHw, t), M(HHz, ABHz, t), \\ M(LHw, STHw, t), M(HHz, STHw, t)\}$$

$$M(Hz, Hw, t) \geq \min\{M(Hz, Hw, t), M(Hz, Hz, t), M(Hw, Hw, t), M(Hz, Hw, t)\} \\ \geq M(Hz, Hw, t)$$

Therefore $M(Hz, Hw, kt) \geq M(Hz, Hw, t)$. *Hence* $Hz = Hw$.

The proof when the pair of maps (Q, ST) *is reciprocal continuous compatible maps . From theorem (2-10) we have ,* Qu *is a common fixed point of* P, Q, S, T, A *and* B *.Now , we have to show that* H *and* L *have the same common fixed point : putting* $x = Qu$, $y = u$ *in the contactive condition* (c) *we get ,*

$$M(HQu, Qu, kt) \geq \min\{M(ABQu, STu, t), M(HQu, ABQu, t), \\ M(Qu, STu, t), M(HQu, STu, t)\}$$

$$M(HQu, Qu, kt) \geq \min\{M(Qu, Qu, t), M(HQu, Qu, t), \\ M(Qu, Qu, t), M(HQu, Qu, t)\} \geq M(HQu, Qu, t)$$

so $HQu = Qu$.

Now putting $x = z$, $y = Qu$ *in the contactive condition* (b) *we have ,*

$$M(Pz, LQu, kt) \geq \min\{M(ABz, STQu, t), M(Pz, ABz, t), \\ M(LQu, STQu, t), M(Pz, STQu, t)\}$$

$$M(Qu, LQu, kt) \geq \min\{M(Qu, Qu, t), M(Qu, Qu, t), M(LQu, Qu, t), M(Qu, Qu, t)\} \\ \geq M(LQu, Qu, t) = M(Qu, LQu, t)$$

so $Qu = LQu$. *Hence* $Qu = PQu = QQu = BQu = AQu = TQu = SQu = HQu = LQu$. *Hence* Qu *is a common fixed point of* P, Q, S, T, A, B, H *and* L.

Uniqueness is the same of theorem (2-10) .

The proof when the pair of maps (L, ST) *is reciprocal continuous compatible maps .*

Since $(X, M, *)$ *is complete so there exists a point (say)* u *in* X *such that* $\langle y_n \rangle \to u$. *Also we have* $\langle Px_{2n-2} \rangle, \langle Hx_{2n+2} \rangle, \langle STx_{2n-1} \rangle, \langle ABx_{2n} \rangle, \langle Qx_{2n+1} \rangle, \langle Lx_{2n+3} \rangle \to u$.

Since the pair (L, ST) *is reciprocally continuous mappings , then we have ,* $\lim_{n\to\infty} LSTx_{2n} = Lu$ *and* $\lim_{n\to\infty} STLx_{2n} = STu$ *, and compatibility of* L *and* ST *yields ,* $\lim_{n\to\infty} M(LSTx_{2n}, STLx_{2n}, t) = 1$ $(i.e)$ $M(Lu, STu, t) = 1$. *Hence* $Lu = STu$.

Since $L(X) \subseteq AB(X)$ *then there exists a point* z *in* X *such that* $Lu = ABz$.

Now by contractive condition (d) , we get ,

$$M(Hz,Lu,kt) \geq \min\{M(ABz,STu,t), M(Hz,ABz,t),$$
$$M(Lu,STu,t), M(Hz,STu,t)\}$$
$$\geq min\{M(Lu,Lu,t), M(Hz,Lu,t), M(Lu,Lu,t), M(Hz,Lu,t)\}$$
$$\geq M(Hz,Lu,t)$$

(i.e) $Hz = Lu$ *thus* $Lu = STu = Hz = ABz$.

Since L *is* ST*-absorbing then for* $k > 0$ *we have* , $M(STu,STLu,t) \geq M\left(STu,Lu,\frac{t}{k}\right) = M\left(Lu,Lu,\frac{t}{k}\right) = 1$

(i.e) $Lu = STLu = STu$.

Now by contractive condition (d) we have ,

$$M(Lu,LLu,kt) = M(Hz,LLu,kt) \geq min\{M(ABz,STLu,t),$$
$$M(Hz,ABz,t), M(LLu,STLu,t), M(Hz,STLu,t)\}$$
$$\geq min\{M(Lu,Lu,t), M(Lu,Lu,t), M(LLu,Lu,t), M(Lu,Lu,t)\}$$
$$\geq M(LLu,Lu,t) = M(Lu,LLu,t)$$

(i.e) $LLu = Lu = STLu$

Therefore Lu *is a common fixed point of* L *and* ST . *Similarly* H *is* AB*-absorbing therefore we have* , $M(ABz,ABHz,t) \geq M\left(ABz,Hz,\frac{t}{k}\right) \geq M\left(Hz,Hz,\frac{t}{k}\right) = 1$
(i.e) $ABz = ABHz = Hz$

Now by contractive condition (d) we have ,

$$M(HHz,Hz,kt) = M(HHz,Lu,kt) \geq min\{M(ABHz,STu,t),$$
$$M(HHz,ABHz,t), M(Lu,STu,t), M(HHz,STu,t)\}$$
$$\geq min\{M(Hz,Hz,t), M(HHz,Hz,t), M(Lu,Lu,t), M(HHz,Hz,t)\}$$
$$\geq M(HHz,Hz,t)$$

(i.e) $HHz = Hz = ABHz$. *Now putting* $Lu = Hz$ *we have* $HLu = Lu = ABLu$.

Now putting $x = z$, $y = Lu$ *in the contactive condition* (c) *we have ,*

$$M(Lu,QLu,kt) = M(Hz,QLu,kt) \geq min\{M(ABz,STLu,t), M(Hz,ABz,t),$$
$$M(QLu,STLu,t), M(Hz,STLu,t)\}$$
$$\geq min\{M(Lu,Lu,t), M(Lu,Lu,t), M(QLu,Lu,t), M(Lu,Lu,t)\}$$
$$\geq M(QLu,Lu,t) = M(Lu,QLu,t)$$

so $QLu = Lu$

Now putting $x = Lu$, $y = u$ in the contactive condition (b) we have ,

$$M(PLu, Lu, kt) \geq \min\{M(ABLu, STu, t),$$
$$M(PLu, ABLu, t), M(Lu, STu, t), M(PLu, STu, t)\}$$
$$M(PLu, Lu, kt) \geq \min\{M(Lu, Lu, t), M(PLu, Lu, t), M(Lu, Lu, t), M(PLu, Lu, t)\}$$
$$\geq M(PLu, Lu, t)$$

so $PLu = Lu$

Now putting $x = BLu$, $y = Lu$ by contractive condition (d) ,we have ,

$$M(H(BLu), L(Lu), kt) \geq \min\{M(AB(BLu), ST(Lu), t), M(H(BLu), AB(BLu), t),$$
$$M(L(Lu), ST(Lu), t), M(H(BLu), ST(BLu), t)\}$$

As $HBLu = BHLu = BLu$ and $ABBLu = BABLu = BLu$, we have,

$$M(BLu, Lu, kt) \geq \min\{M(BLu, Lu, t), M(BLu, BLu, t),$$
$$M(Lu, Lu, t), M(BLu, Lu, t)\} \geq M(BLu, Lu, t)$$

by lemma (2-8) we have , $BLu = Lu$. Since , $Lu = LLu = HLu = QLu = PLu = BLu = ABLu$. Hence , $Lu = LLu = HLu = QLu = PLu = BLu = ALu$.

Now putting $x = Lu$, $y = TLu$ in the contractive condition (d) ,we have ,

$$M(H(Lu), L(TLu), kt) \geq \min\{M(ABLu, STTLu, t), M(HLu, ABLu, t),$$
$$M(LTLu, STTLu, t), M(HLu, STTLu, t)\}$$

As $STTLu = TSTLu = TLu$ and $LTLu = TLLu = TLu$ We have ,

$$M(Lu, TLu, kt) \geq \min\{M(Lu, TLu, t), M(Lu, Lu, t), M(TLu, TLu, t), M(Lu, TLu, t)\}$$
$$\geq M(Lu, TLu, t)$$

By using lemma (2-8) we have $TLu = Lu$. Since $Lu = LLu = HLu = PLu = QLu = ALu = BLu = TLu = STLu.$ Hence $Lu = LLu = HLu = PLu = QLu = ALu = BLu = TLu = SLu$. Hence Lu is a common fixed point of P, Q, L, H, S, T, A and B .

Uniqueness , let Lw be another fixed point of L, H, S, T, P, Q, A and B then putting $x = Lu$ and $y = Lw$ in the contractive condition (d) we have ,

$$M(HLu, LLw, kt) \geq min\ \{M(ABLu, STLw, t), M(HLu, ABLu, t),$$
$$M(LLw, STLw, t), M(HLu, STLw, t)\}$$
$$M(Lu, Lw, t) \geq min\ \{M(Lu, Lw, t), M(Lu, Lu, t), M(Lw, Lw, t), M(Lu, Lw, t)\}$$
$$\geq M(Lu, Lw, t)$$

Therefore $M(Lu, Lw, kt) \geq M(Lu, Lw, t)$. *Hence* $Lu = Lw$. #

Corollary 2-15:

In theorem (2-14) , when $P = H$ *and* $Q = L$ *then we have the same result as in theorem (2-10) .* #

Now we going to the result for eight mappings , using absorbing maps , which are not necessarily continuous .

Corollary 2-16 :

Let H *and* P *be a point wise* AB*-absorbing and* L *and* Q *be a point wise* ST*-absorbing self maps on complete fuzzy metric space* $(X, M, *)$ *with continuous t-norm defined by* $a * b = min\{a, b\}$ *where* $a, b \in [0,1]$ *, satisfying the conditions :*

1) $H(X), P(X) \subseteq ST(X)$ *and* $L(X), Q(X) \subseteq AB(X)$

2) There exists $k \in (0,1)$ *such that for every* $x, y \in X$ *and* $t \in (0, \infty)$

$$M(Px, Qy, kt) \geq min\ \{M(ABx, STy, t), M(Px, ABx, t),$$
$$M(Qy, STy, t), M(Px, STy, t)\}$$
$$M(Px, Ly, kt) \geq min\ \{M(ABx, STy, t), M(Px, ABx, t),$$
$$M(Ly, STy, t), M(Px, STy, t)\}$$
$$M(Hx, Qy, kt) \geq min\ \{M(ABx, STy, t), M(Hx, ABx, t),$$
$$M(Qy, STy, t), M(Hx, STy, t)\}$$
$$M(Hx, Ly, kt) \geq min\ \{M(ABx, STy, t), M(Hx, ABx, t),$$
$$M(Ly, STy, t), M(Hx, STy, t)\}$$

3) $For\ all\ x, y \in X,\ lim_{t \to \infty} M(x, y, t) = 1$

4)
$AB = BA$, $ST = TS$, $PB = BP$, $HB = BH$, $SQ = QS$, $SL = LS$, $QT = TQ$, $LT = TL$

.

If the range of one of the mappings $P(X), H(X), Q(X), L(X), AB(X), ST(X)$ *be a complete subspace of* X *then* P, Q, H, L, S, T, A *, and* B *have a unique common fixed point in* X *.* #

References :

[1] *Anju R. , R. C. and S. M. , " Fixed points of absorbing maps in fuzzy metric space" , International Mathematical Forum, 4 , 2009 , No. 32, 1591 – 1601.*

[2] *Bijendra S. and S. J. , "Semi-compatibility , compatibility and fixed point theorems in fuzzy metric space" , Journal of the Chungcheong Mathematical Society volume 18, No. 1, April 2005.*

[3] *Bivas D. , T.K. S. and I. H. J. , "Fuzzy anti-norm and fuzzy α-anti-convergence" , arxiv:1002.3818v1 [Math.GM] 19 Feb 2010 .*

[4] *Hakan E. , "Round fuzzy metric spaces" , International Mathematical Forum, 2, 2007, No. 35, 1717 – 1721.*

[5] *Iqbal H. J. ,"Level n-fuzzy bounded set and α-completeness in fuzzy n-normed linear space" , Journal of Mathematics research Vol. 2, No. 2, May 2010 .*

[6] *Ishak A. and D. T. , "Some fixed point theorems on fuzzy metric spaces with implicit relations" , Commun. Korean Math. Soc. 23 (2008), No. 1, pp. 111-124.*

[7] *L. A. Zadeh , " Fuzzy sets" , Information and control 8 , 338-353 (1965) .*

[8] *Mariusz G. , "Fixed points in fuzzy metric spaces " , Fuzzy sets and systems 27 (1988) , 385-389 .*

[9] *R. M. Somasundaram and T. B. , "Some aspects of 2-fuzzy 2-normed linear spaces" , Bull. Malays. Math. Sci. Soc. (2) 32(2) (2009), 211–221 .*

[10] *S.K. Elagan , E.M.E. Z. , and T.A. N. , "Some remarks on series in fuzzy n-normed spaces" , International Mathematical Forum, 5, 2010, No. 3 , 117 – 124 .*

[11] *S.N. Mishra , "Common fixed point of maps on fuzzy metric spaces" , Internat. J. Math. & Math. Sci. VOL. 17 NO. 2 (1994) 253-258 .*

[12] *S. Vijayabalaji , and N. T. , "Complete fuzzy n-normed linear space" , Journal of Fundamental sciences 3 (2007) 119-126 .*

[13] *U. Mishra, A. S. Ranadive and D. Gopal , "Fixed Point Theorems via Absorbing Maps" , Thai Journal of Mathematics Volume 6 (2008) Number 1 : 49-60 .*

حول النقاط الثابتة للدوال الماصة في الفضاءات المترية المضببة

أ.يونس جهاد ياسين و سجى سعد محسن

المستلخص :

في هذا البحث ، برهنا مبرهنة النقطة الثابتة لثمانية دوال ذاتية باستخدام الدوال الماصة والدوال المستمرة تبادليا في الفضاء المتري المضبب . بحثنا عبارة عن توسيع لنتائج الباحث أنجو راني .