Asymptotic behavior of second order non-linear neutral differential equations

Hussain Ali Mohamad*, Intidhar Zamil Mushtt**

University of Baghdad, College of science for women   Al Mustansiriyah University, college of education

* hussainmohamad22@gmail.com   ** intidhar Albayaty@yahoo.com

Abstract

In this paper, we have given some necessary and sufficient conditions for all non-oscillatory solutions to the nonlinear neutral differential equation

$$[y(t) - p(t)y(\tau(t))\ ]'' + q(t)f(y(\sigma(t))) = 0$$

So that converge to zero as $t \to \infty$. Some examples are given to illustrate the obtained results.

1. Introduction

Consider the second order non-linear neutral differential equations

$$[y(t) - p(t)y(\tau(t))]'' + q(t)f(y(\sigma(t))) = 0 \qquad (1.1)$$

Under the standing hypotheses:

$(\mathbf{A_1})$ $p(t) \in C([t_0, \infty); (0, \infty)), q(t) \in C([t_0, \infty); R)$.

$(\mathbf{A_2})$ $\tau(t), \sigma(t) \in C([t_0, \infty); R^+), \lim_{t\to\infty} \tau(t) = \infty, \lim_{t\to\infty} \sigma(t) = \infty,$ and $\tau(t), \sigma(t)$ are increasing functions.

$(\mathbf{A_3})$ $f(u) \in C(R; R);\ uf(u) > 0$ for $u \neq 0;\ |f(u)| \geq \beta|u|,\ \beta > 0$.

By a solution of eq.(1.1),we mean a function $y(t) \in C([\rho(t), \infty); R)$ such that$y(t) - p(t)y(\tau(t))$ is two times continuously differentiable and$y(t)$ satisfies (1.1), where

$\rho(t) = \min \{\tau(t), \sigma(t), t_0\}$. A solution $y(t)$ is said to be oscillatory if it has arbitrarily large zeros otherwise $y(t)$ is said to be non-oscillatory. Hence there has been much research activity concerning oscillatory and nonoscillatory behavior of solutions to different classes of neutral differential equations, we refer the reader to [1-13].

2. Main Results

Before we present the results we begin with the following lemma which is helpful to establish our main results

Lemma 2.1 ([14], Lemma 2.1 and Lemma 2.2, pp.477-478)

Assume that $p \in C([t_0, \infty); R^+), \tau \in C([t_0, \infty); R)$, for $t \geq t_0$,

i. Suppose that $0 < p(t) \leq 1$, for $t \geq t_0$. Let $y(t)$ be a non-oscillatory solution of a functional inequality $y(t)[\, y(t) - p(t)y(\tau(t))] < 0$, in a neighborhood of infinity. Suppose that $\tau(t) < t$ for $t \geq t_0$, then $y(t)$ is bounded. If moreover $0 < p(t) \leq \delta < 1, t \geq t_0$, for some positive constant δ, then $\lim_{t\to\infty} y(t) = 0$.

ii. Suppose that $1 \leq p(t)$ for $t \geq t_0$. Let $y(t)$ be a non-oscillatory solution of a functional inequality $y(t)[y(t) - p(t)y(\tau(t))] > 0$ in a neighborhood of infinity. Suppose that $\tau(t) > t$ for $t \geq t_0$, then $y(t)$ is bounded. If moreover $1 < \delta \leq p(t),\ t \geq t_0$, for some positive constant δ, then $\lim_{t\to\infty} y(t) = 0$.

Let

$$z(t) = y(t) - p(t)y(\tau(t)) \qquad (1.2)$$

Theorem 2.2 Assume that $(A_1)-(A_3)$ hold, $p(t) \geq p > 1, q(t) < 0, \sigma^{-1}(\tau(t)) > t$

$\tau(t) > t,$ and

$$\limsup_{t\to\infty} t\int_{\sigma^{-1}(\tau(t))}^{\infty} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds > \frac{1}{\beta} \qquad (1.3)$$

Whereβ as in(A_3). Then every nonoscillatory solution of equation (1.1) tends to zero as $t \to \infty$.

Proof. Suppose that $y(t)$ be anonoscillatory solution of (1.1). Without loss of generality assume that $y(t) > 0, y(\sigma(t)) > 0, y(\tau(t)) > 0$ for $t \geq t_0$.

Then from (1.1) and (1.2) it follows that

$$z''(t) = -q(t)f\left(y(\sigma(t))\right) \geq 0 \qquad (1.4)$$

Hence $z(t), z'(t)$ are monotone functions, we have two cases for $z'(t)$

1. $z'(t) > 0$ for $t \geq t_1 \geq t_0$; 2. $z'(t) < 0$ for $t \geq t_1 \geq t_0$.

Case 1: In this case $z''(t) \geq 0,\ z'(t) > 0,\ z(t) > 0$, leads to $\lim_{t\to\infty} z(t) = \infty$. Then from (1.2) it follows that $z(t) \leq y(t)$ which implies that $\lim_{t\to\infty} y(t) = \infty$.

On the other side by lemma[2.1-ii],it follows that $y(t)$ is bounded , this is a contradiction.

Case 2: $z''(t) \geq 0,\ z'(t) < 0,$ we have two sub-cases for $z(t)$

Case (a) $z(t) > 0$, for $t \geq t_2 \geq t_1$; Case (b) $z(t) < 0$ for $t \geq t_2 \geq t_1$.

Case (a): In this case we have $z''(t) \geq 0,\ z'(t) < 0,\ z(t) > 0.$

By lemma [2.1-ii], it follows that $\lim_{t\to\infty} y(t) = 0.$

Case (b) $z''(t) \geq 0,\ z'(t) < 0,\ z(t) < 0$

From (1.2) we get $z(t) > -p(t)y(\tau(t))$ that is $y\big(\tau(t)\big) > \frac{-1}{p(t)} z(t)$

then

$$y(\sigma(t)) > \frac{-1}{p\left(\tau^{-1}\big(\sigma(t)\big)\right)} z(\tau^{-1}(\sigma(t))) \tag{1.5}$$

Integrating (1.4) from t to ∞ we get

$$-z'(t) \geq -\int_t^\infty q(s) f\left(y\big(\sigma(s)\big)\right) ds \tag{1.6}$$

Using (A_3) in (1.6) it follows that

$$-z'(t) \geq -\beta \int_t^\infty q(s) y\big(\sigma(s)\big) ds \tag{1.7}$$

Substituting (1.5) in (1.7)we obtain

$$-z'(t) \geq \beta \int_t^\infty \frac{q(s)}{p(\tau^{-1}(\sigma(s)))} z(\tau^{-1}(\sigma(s))) ds \tag{1.8}$$

Now from condition (1.3) we have

$$1 < \beta t \int_t^\infty \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds \leq \beta \int_{\sigma^{-1}(\tau(t))}^\infty \frac{s|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds$$

We claim that the condition (1.3) implies that

$$\int_{t_2}^{\infty} \frac{s|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds = \infty, \text{ for } t \geq t_2 \tag{1.9}$$

Otherwise

$$\int_{t_2}^{\infty} \frac{s\,|q(s)|}{p(\tau^{-1}(\sigma(s))} ds < \infty$$

We can choose $t_3 \geq t_2$ large enough such that

$\int_{t_3}^{\infty} \frac{s|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds < 1$ which is a contradiction.

Multiplying (1.4) by t and integrating from t_2 to t, we have for all $t \geq t_2$.

$$\int_{t_2}^{t} s\, z''(s)\, ds = -\int_{t_2}^{t} s\, q(s) f(y(\sigma(s))) ds$$

$$tz'(t) - t_2 z'(t_2) - z(t) + z(t_2) \geq -\beta \int_{t_2}^{t} s\, q(s) y(\sigma(s))\, ds$$

$$\geq \beta z(\tau^{-1}(\sigma(t_2))) \int_{t_2}^{t} \frac{s\, q(s)}{p(\tau^{-1}(\sigma(s)))} ds$$

As $t \to \infty$ the last inequality yields to

$$\lim_{t\to\infty} [\, tz'(t) - z(t) - t_2 z'(t_2) + z(t_2)] \geq \beta z(\tau^{-1}(\sigma(t_2))) \int_{t_2}^{\infty} \frac{s\, q(s)}{p(\tau^{-1}(\sigma(s)))} ds$$

Then $\lim_{t\to\infty} [\, tz'(t) - z(t)] = \infty$

Hence

$$tz'(t) \geq z(t) \quad \text{for} \quad t \geq t_3 \geq t_2 \tag{1.10}$$

From (1.8) we get

$$-t\, z'(t) \geq \beta t \int_t^{\infty} \frac{q(s)}{p\left(\tau^{-1}(\sigma(s))\right)} z\left(\tau^{-1}(\sigma(s))\right) ds, \qquad (1.11)$$

Substation (1.10) in (1.11) we get

$$-z(t) \geq -\beta t \int_t^{\infty} \frac{|q(s)|}{p\left(\tau^{-1}(\sigma(s))\right)} z(\tau^{-1}(\sigma(s)))\, ds$$

$$\geq -\beta t \int_{\sigma^{-1}(\tau(t)))}^{\infty} \frac{|q(s)|}{p\left(\tau^{-1}(\sigma(s))\right)} z(\tau^{-1}(\sigma(s)))\, ds$$

$$\geq -\beta\; t\; z(t) \int_{\sigma^{-1}(\tau(t))}^{\infty} \frac{|q(s)|}{p\left(\tau^{-1}(\sigma(s))\right)} ds$$

$$1 \geq \beta\; t \int_{\sigma^{-1}(\tau(t))}^{\infty} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds$$

Which is a contradiction. The proof is complete. □

Example 1: Consider the following neutral delay equation

$$\left[y(t) - \left(\frac{3}{2} - \frac{1}{8t}\right) y(2t)\right]'' - \frac{1}{6t^2}\left[1 + \frac{1}{4} y\left(\frac{t}{3}\right)\right] y\left(\frac{t}{3}\right) = 0, \qquad t > \frac{1}{2} \qquad (E1)$$

$p(t) = \frac{3}{2} - \frac{1}{8t}$, $\tau(t) = 2t,$ $\sigma(t) = \frac{t}{3}$, $q(t) = \frac{-1}{6t^2}$, $f(y(\sigma(t))) = [1 + \frac{1}{4} y(\frac{t}{3})] y(\frac{t}{3})$,

$\sigma^{-1}(\tau(t)) = 6t$, $\tau^{-1}(\sigma(s)) = \frac{t}{6}$, and $\beta = 1$.

$$\int_{\sigma^{-1}(\tau(t))}^{\infty} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds = \int_{\frac{9}{2}t}^{\infty} \frac{\frac{1}{6s^2}}{\frac{3}{2}\left[1 - \frac{1}{2s}\right]} ds$$

$$\limsup_{t\to\infty} \beta t \int_{\sigma^{-1}(\tau(t))}^{\infty} \frac{|q(s)|}{p(\tau^{-1}(\sigma(s)))} ds = \frac{2}{9} \lim_{t\to\infty} t \int_{6t}^{\infty} \left[\frac{2}{2s-1} - \frac{1}{s}\right] ds = \infty$$

All condition of theorem 2.2 hold. Then every solution of $(E1)$ tends to zero as $t \to \infty$. For instance $y(t) = \frac{1}{t}$ is such a solution.

Theorem 2.3 Assume that $0 < p(t) \leq p, q(t) \geq 0,\ \tau(t) < t, \sigma^{-1}(\tau^{-n}(t)) > t$ and

$$\limsup_{t\to\infty} \beta t \int_{\sigma^{-1}(\tau^{-n}(t))}^{\infty} f(1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^{n}(\sigma(s))))q(s)\, ds > 1 \qquad (1.12)$$

Whereβ as in(A_3), Then every nonoscillatory solution of (1.1) tends to zero as $t \to \infty$.

Proof. Suppose that $y(t)$ be nonoscillatory solution of (1.1). Without loss of generality assume that $y(t)$ is an eventually positive so there exists t_0 such that $y(t) > 0, y(\tau(t)) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_0$.

From (1.1) it follows that

$$z''(t) = -q(t)f\left(y(\sigma(t))\right) \leq 0 \qquad (1.13)$$

Then we have two cases for $z'(t)$

Case 1. $z'(t) < 0$ for $t \geq t_1 \geq t_0$; Case 2. $z'(t) > 0$ for $t \geq t_1 \geq t_0$.

Case 1: We have $z''(t) \leq 0,\ z'(t) < 0,\ z(t) < 0$, it follows that $\lim_{t\to\infty} z(t) = -\infty$,since $z(t) > -p(t)y(\tau(t))$ then $\lim_{t\to\infty} y(t) = \infty$.

On the other hand by lemma [2.1-i] it follows $y(t)$ is bounded, which is a contradiction.

Case 2: In this case $z''(t) \leq 0,\ z'(t) > 0,$ we have two sub-cases for $z(t)$

Case (a) $z(t) < 0$ for $t \geq t_2 \geq t_1$; Case (b) $z(t) > 0$ for $t \geq t_2 \geq t_1$.

Case (a): $z''(t) \leq 0,\ z'(t) > 0,\ z(t) < 0$

By lemma[2.1-i] it follows that $\lim_{t\to\infty} y(t) = 0.$

Case (b): $z''(t) \leq 0,\ z'(t) > 0,\ z(t) > 0$

$$y(t) = z(t) + p(t)y(\tau(t))$$

$$= z(t) + p(t)[z(\tau(t)) + p(\tau(t))y(\tau^2(t))]$$

$$= z(t) + p(t)z(\tau(t)) + p(t)p(\tau(t))[z(\tau^2(t)) + p(\tau^2(t))y(\tau^3(t))]$$

As in thermo [2.4] From [15] we can written the following inequality

$$y(\sigma(t)) \geq f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p\left(\tau^k(\sigma(t))\right)]\, z\left(\tau^n(\sigma(t))\right) \tag{1.14}$$

Integrating (1.13) from t to ∞ we get

$$-z'(t) \leq -\int_t^\infty q(s) f(y(\sigma(s)))ds$$

Using (A_2) the last inequality implies

$$-z'(t) \leq -\beta \int_t^\infty q(s) y(\sigma(s))ds \tag{1.15}$$

Using (1.14) we get

$$-z'(t) \leq -\beta \int_t^\infty q(s) f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(s))] z(\tau^n(\sigma(s)) ds \qquad (1.16)$$

We claim that

$$\int_t^\infty s f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(s))] q(s) ds = \infty \qquad (1.17)$$

Otherwise

$$\int_{t_2}^\infty s f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(s))] q(s) ds < \infty \quad \text{for} \quad t > t_2$$

We can find $t_* > t_2$ large enough such that

$$\int_{t_*}^\infty s f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(s))] q(s) ds < 1$$

Which is a contradiction. Then (1.17) holds.

From (1.13) we obtain

$$\int_{t_2}^t s\, z''(s)\, ds = -\int_{t_2}^t s\, q(s) f(y(\sigma(s))) ds$$

$$t z'(t) - t_2 z'(t_2) - z(t) + z(t_2) \leq -\beta \int_{t_2}^t s\, q(s) y(\sigma(s)) ds$$

$$\leq -\beta \int_{t_2}^t s q(s) f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(s))] z(\tau^n(\sigma(s))\, ds$$

$$\leq -\beta z(\tau^n(\sigma(t_2))) \int_{t_2}^t s f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(s))] q(s)\, ds$$

hence as $t \to \infty$ the last inequality implies that

$$\lim_{t\to\infty}[z(t) - tz'(t)] = \infty$$

Then for $t \geq t_3 \geq t_2$

$$z(t) > tz'(t) \tag{1.18}$$

From (1.16)

$$t\,z'(t) \geq \beta t \int_t^\infty q(s)\,f[1 + \sum_{i=1}^{n}\prod_{k=0}^{i-1} p(\tau^k(\sigma(s))]\,z(\tau^n(\sigma(s))ds \tag{1.19}$$

Substituting (1.18) in (1.19) we get

$$z(t) \geq \beta t \int_t^\infty q(s) f[1 + \sum_{i=1}^{n}\prod_{k=0}^{i-1} p(\tau^k(\sigma(s))] z(\tau^n(\sigma(s))ds$$

$$\geq \beta t \int_{\sigma^{-1}(\tau^{-n}(t))}^\infty q(s) f[1 + \sum_{i=1}^{n}\prod_{k=0}^{i-1} p(\tau^k(\sigma(s))] z(\tau^n(\sigma(s))ds$$

$$\geq \beta t z(t) \int_{\sigma^{-1}(\tau^{-n}(t))}^\infty q(s) f[1 + \sum_{i=1}^{n}\prod_{k=0}^{i-1} p(\tau^k(\sigma(s))]\ ds$$

$1 \geq \beta \int_{\sigma^{-1}(\tau^{-n}(t))}^\infty q(s) f[1 + \sum_{i=1}^{n}\prod_{k=0}^{i-1} p(\tau^k(\sigma(s))]\ ds$ which is a contradiction.

The proof is complete. □

Example 2: Consider the following neutral delay equation

$$[y(t) - 8y(\sqrt{t})]'' + \left[\frac{8}{t^2} - \frac{3}{t^3}\right]\sqrt[4]{y\left(\frac{t^2}{4}\right)} = 0, \quad t \geq \frac{1}{2} \tag{E2}$$

$\beta = 1, n = 1, \sigma(t) = \frac{t^2}{4}, \tau(t) = \sqrt{t}, \sigma^{-1}(\tau^{-1}(t)) = 2t, p(t) = 8,\ q(t) = \frac{8}{t^2} - \frac{3}{t^3},$

$f(y(t)) = \sqrt[4]{y(t)},\ f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(t))] = \sqrt{3}$

$$\int_{\sigma^{-1}(\tau^{-n}(t))}^{\infty} f[1 + \sum_{i=1}^{n} \prod_{k=0}^{i-1} p(\tau^k(\sigma(t))]\ q(s)ds = \sqrt{3} \int_{2t}^{\infty} [\frac{8}{s^2} - \frac{3}{s^3}]\ ds$$

$$\limsup_{t\to\infty} \beta t \int_{\sigma^{-1}(\tau^{-n}(t))}^{\infty} f(1 + \sum_{i=0}^{n} \prod_{k=1}^{i-1} p(\tau^n(\sigma(s))\, q(s)\, ds = \sqrt{3} \lim_{t\to\infty} [4 - \frac{3}{8t}] = 4\sqrt{3} > 1$$

All condition of theorem [2.3] hold. Then every solution of $(E2)$ tends to zero as $t \to \infty$. For instance $y(t) = \frac{1}{t^2}$ is such a solution.

References

[1] R.P. Agarwal, S.R. Grace and D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic Publisheres, Drdrechet 2000.

[2] R.P. Agarwal, S.R. Grace and D. O'Regan, Oscillation and Asymptotic Behavior for Second-Order Nonlinear Perturbed Differential Equations, Mathematical and Computer Science Modeling, 39 (2004) 1477-1490.

[3] D.D. Bainov and D.P. Mishev, Oscillation Theory for Neutral Differential Equations with Delay, Adam Hilger, New York 1991.

[4]J.Dzurina, Oscillation theorems for second order advanced neutral differential equations ,mathematical publication ,48(2011)pp 61-71.

[5] J. Dzurina and B. Mihalikova, Oscillation criteria for second order neutral differential equations, Math. Boh. 125 (2000), 145-153.

[6] J.R. Graef, M.K. Grammatikopoulos and P.W. Spikes, Asymptotic properties

of solutions of nonlinear neutral delay differential equations of the second order, Radovi Mat. 4 (1988), 133-149.

[7]J.Jaroš and T.Kusano, On a class of first order nonlinear functional differential equations of neutral type,Czechoslovak Mathematical Journal, Vol. 40 (1990), No. 3, 475-490.

[8]Y. Jiangand T. Li,Asymptotic behavior of a third-order nonlinear neutral delay differential equation, Journal of Inequalities and Applications (2014),7 page.

[9]T. Li, B.Baculkov and J.Džurina,Oscillatory behavior of second-order nonlinear neutral differential equations with distributed deviating arguments, Boundary Value Problems(2014),15 page.

[10] T. Liand Y. V. Rogovchenko, Oscillation Theorems for Second-Order Nonlinear Neutral Delay Differential Equations, Abstract and Applied Analysis Volume (2014), 5 pages.

[11]T. Li, Z.Han, P.Zhao and S. Sun1,Oscillation of Even-Order Neutral Delay Differential Equations, Advances in Difference equation Volume (2010), 9 pages.

[12] E. Thandapani, V. Ganesan,oscillatory and asymptotic behavior of solutions of second order neutral delay differential equations with "maxima", International Journal of Pure and Applied Mathematics Volume 78 No. 7(2012), pp1029-1039.

[13] M.S. Zaki,On asymptotic behavior of a second order delay differential equation, International Journal of Physical Sciences Vol. 2 (7),(2007) pp. 185-187.

[14] M.K. Yildiz, E. Karaman, H. Durur,Oscillation and Asymptotic Behavior of a Higher-Order Nonlinear Neutral-Type Functional Differential Equation with Oscillating Coefficients. Journal of Applied Mathematics, Vol. (2011),pp 1-9.

[15] H. A. Mohamad, I. Z. Mushtt, Oscillation of second order non-linear neutral differential equations, to appear.

السلوك المحاذي للمعادلات التفاضليه التباطؤيه المحايده غير الخطيه من الرتبه الثانيه

حسين علي محمد　　　　انتظار زامل مشتت

كليه العلوم للبنات –جامعه بغداد　　　　كليه التربيه –الجامعه المستنصريه

الخلاصه :

قدمنا في هذا البحث بعض الشروط الضرويه والكافيه للحلول غير المتذبذبه للمعادله التباطؤيه غير الخطيه من الرتبه الثانيه

$$[y(t) - p(t)y(\tau(t))\,]'' + q(t)f(y(\sigma(t))) = 0$$

التي تضمن تقارب هذه الحلول الى الصفر عندما $t \to \infty$.وكما قدمنا بعض الامثله التوضيحيه اللتي تحقق النتائج التي حصلنا عليها .