Journal of AL-Qadisiyah for computer science and mathematics
Vol.6 No.2 Year 2014

Page 149-156

On Essentially Normality of the Composition Operator C_σ

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Recived :28\2\2013 Revised : 16\9\2014 Accepted :28\9\2014

Abstract

In this paper we have studied the composition operator induced by the automorphism **σ** and discussed the adjoint of the composition of the symbol **σ**.We have look also at some known properties on composition operators and tried to see the analogue properties in order to show how the results are changed by changing the function **σ** on U.

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties .

Mathematics Subject Classification: 47B38+47A58

Introduction

In this paper we are going to the composition operator $C_\sigma$induced by the symbol **σ** and properties of C_σ and also discuss the adjoint of Composition Operator C_σ induced by the symbol **σ** and we discuss the normality of C_σ **.** Moreover, we study the essential normality of C_σ .

Definition(1.1) : [4]

Let $U = \{z \in C : |z| < 1\}$ is called unit ball in complex numbers C and $\partial U = \{z \in C : |z| = 1\}$ is called boundary of U

Definition (1.2):

For $\gamma \in U$, define $\sigma(z) = \frac{z}{-1+\gamma z}$ $(z \in U)$. Since the denominator equal zero only at $z = \frac{1}{\gamma}$, the function σ is holomorphic on the ball $\left\{|z| < \frac{1}{|\gamma|}\right\}$. Since $(\gamma \in U)$ then this ball contains U. Hence σ take U into U and holomorphic on U .

Definition (1.3): [4]

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z) = \sum_{n=0}^{\infty} f\hat{}(n)\, z^n$ holomorphic on U such that $\sum_{n=0}^{\infty} |f\hat{}(n)|^2 < \infty$ with $f\hat{}(n)$ denotes then the Taylor coefficient of f.

Remark (1.4) : [1]

We can define an inner product of the Hardy space H^2 as follows:
$f(z) = \sum_{n=0}^{\infty} f\hat{}(n)\, z^n$ and $g(z) = \sum_{n=0}^{\infty} g\hat{}(n)\, z^n$, then inner product of f and g is:
$\langle f, g\rangle = \sum_{n=0}^{\infty} f\hat{}(n)\, \overline{g\hat{}(n)}$

Example (1.5) :[10]

Let $K_\alpha(z) = \frac{1}{1-\overline{\alpha}z}$. Since $\alpha \in U$,then $|\alpha| < 1$, hence the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and thus $K_\alpha \in H^2$ and $K_\alpha(z) = \sum_{n=0}^{\infty} (\overline{\alpha})^n z^n$.

Definition(1.6) : [4]

Let $\psi : U \rightarrow U$ and holomorphic on U, the composition operator C_ψ induced by ψ is defined on H^2 by the equation $C_\psi f = f \circ \psi$ $(f \in H^2)$

Definition(1.7) : [2]

Let T be a bounded operator on a Hilbert space H, then the norm of an operator T is defined by $\|T\| = \sup\{ \|Tf\| : f \in H, \|f\| = 1\}$.

Littlewood′s Subordination principle (1.8) : [11]

Let $\psi : U \to U$ and holomorphic on U with $\psi(0) = 0$, then for each $f \in H^2$, $f \circ \psi \in H^2$ and $\|f \circ \psi\| \leq \| f \|$.

Remark (1.9) : [4]

1) One can easily show that $C_\phi C_\psi = C_{\psi \circ \phi}$ and hence $C_\phi^n = C_\phi C_\phi \ldots \ldots C_\phi = C_{\phi \circ \phi \ldots\ldots \circ \phi} = C_{\phi_n}$
2) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U.
3) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Definition(1.10): [3]

Let T be an operator on a Hilbert space H , The operator T^* is the adjoint of T if $\langle T^* x, y\rangle = \langle x, Ty\rangle$ for each $x, y \in H$.

Theorem (1.11) : [5]

$V_{\alpha \in U}\{K_\alpha\}$ forms a dense subset of H^2.

Theorem (1.12) : [10]

Let $\psi : U \to U$ and holomorphic on U, then for all $\alpha \in U$
$C_\psi^* K_\alpha = K_{\psi(\alpha)}$

Definition(1.13): [11]

Let H^∞ be the set of all bounded holomorphic on U .

Definition(1.14): [6]

Let $g \in H^\infty$, the Toeplits operator T_g is the operator on H^2 is given by :
$(T_g f)(z) = g(z)f(z) \; (f \in H^2, z \in U)$

Remark (1.15) : [7]

For each $f \in H^2$, it is well- know that $T_h^* f = T_{\bar{h}} f$ such that $h \in H^\infty$.

Proposition(1.16) :
Let $\alpha \in U, C_\sigma^* = T_g C_\beta T_h^*$, where $h(z) = 1 - \gamma z$, $g(z) = 1$, $\beta(z) = \bar{\gamma} - z$

Proof :

By (1.15), $T_h^* f = T_{\bar{h}} f$ for each $f \in H^2$. Hence for all $\alpha \in U$,

$$\langle T_h^* f,\ K_\alpha \rangle = \langle T_{\bar{h}} f, K_\alpha \rangle = \langle f, T_{\bar{h}}^* K_\alpha \rangle \ldots. (1-1)$$

On the other hand ,

$$\langle T_h^* f,\ K_\alpha \rangle = \langle f, T_h K_\alpha \rangle = \langle f, h(\alpha)\, K_\alpha \rangle \ldots. (1-2)$$

From (1-1)and (1-2) one can see that $T_{\bar{h}}^* K_\alpha = h(\alpha) K_\alpha$. Hence $T_h^* K_\alpha = \overline{h(\alpha)} K_\alpha$.

Calculation give:

$$C_\sigma^* K_\alpha(z) = K_{\sigma(\alpha)}(z)$$
$$= \frac{1}{1-\overline{\sigma(\alpha)}z} = \frac{1}{1-\frac{\bar{\alpha}\, z}{-1+\bar{\gamma}\bar{\alpha}}} = \frac{1}{\frac{-1+\bar{\gamma}\bar{\alpha}-\bar{\alpha}z}{-1+\bar{\gamma}\bar{\alpha}}}$$
$$= \frac{1}{\frac{-1+\bar{\gamma}\bar{\alpha}-\bar{\alpha}z}{-1+\bar{\gamma}\bar{\alpha}}} = \frac{1-\bar{\gamma}\bar{\alpha}}{1-\bar{\alpha}(\bar{\gamma}-z)} = \frac{\overline{(1-\gamma\alpha)}}{1-\bar{\alpha}(\bar{\gamma}-z)}$$
$$= \overline{(1-\gamma\alpha)}\, .1. \frac{1}{1-\bar{\alpha}(\bar{\gamma}-z)}$$
$$= \overline{h(\alpha)}\, T_g\, K_\alpha\big(\beta(z)\big) = \overline{h(\alpha)}\, T_g\, C_\beta K_\alpha(z)$$
$$= T_g\, \overline{h(\alpha)}\, C_\beta K_\alpha(z) = T_g\, C_\beta\, \overline{h(\alpha)}\, K_\alpha(z)$$
$$= T_g\, C_\beta\, T_h^*\, K_\alpha(z)\text{ , therefore}$$

$$C_\sigma^* K_\alpha(z) = T_g\, C_\beta\, T_h^*\, K_\alpha(z)\ .$$

But $\overline{V_{\alpha\in U}\{K_\alpha\}} = H^2$ then $C_\sigma^* = T_g\, C_\beta\, T_h^*$.

Definition (1.17) : [3]

Let T be an operator on a Hilbert space H , T is called normal operator if $TT^* = T^*T$ and T is called unitary operator if if $TT^* = T^*T = I$, and T is called isometric operator if $T^*T = I$

Theorem (1.18) : [9]

If $\phi : U \to U$ is holomorphic map on U, then C_ϕ is normal if and only if $\phi(z) = \lambda z$ for some λ , $|\lambda| \le 1$.

Theorem (1.19) :

If $\phi : U \to U$ be holomorphic map on U, then C_ϕ is unitary if and only if $\phi(z) = \lambda z$ for some λ , $|\lambda| = 1$

Proof :

Suppose C_ϕ is unitary , hence by (1.17) $C_\phi C_\phi^* = C_\phi^* C_\phi = I$, hence $C_\phi C_\phi^* = C_\phi^* C_\phi$, hence C_ϕ is normal operator, hence by (1.18) $\phi(z) = \lambda z$ for some λ , $|\lambda| \leq 1$. It is enough to show that $|\lambda| = 1$

$$C_\phi^* C_\phi K_\beta(z) = C_\phi^* K_\beta(\phi(z)) = K_{\phi(\beta)}\left(\phi(z)\right) .$$

$$= \frac{1}{1-\overline{\phi(\alpha)}\phi(z)} = \frac{1}{1-\bar{\lambda}\bar{\beta}\lambda z} = \frac{1}{1-|\lambda|^2\bar{\beta}z}$$

On the other hand $C_\phi^* C_\phi K_\beta(z) = K_\beta(z)$, hence $\frac{1}{1-|\lambda|^2\bar{\beta}z} = K_\beta(z) = \frac{1}{1-\bar{\beta}z}$.

Thus $|\lambda|^2\bar{\beta} = \bar{\beta}$, then $|\lambda| = 1$.

Conversely, Suppose $\phi(z) = \lambda z$ for some λ , $|\lambda| = 1$. For $\beta \in U$, for every $z \in U$

$$C_\phi^* C_\phi K_\beta(z) = C_\phi^* K_\beta(\phi(z)) = K_{\phi(\beta)}\left(\phi(z)\right) .$$

$$= \frac{1}{1-\overline{\phi(\beta)}\phi(z)} = \frac{1}{1-\bar{\lambda}\bar{\beta}\lambda z} = \frac{1}{1-|\lambda|^2\bar{\beta}z}$$

$$= = \frac{1}{1-\bar{\beta}z} = K_\beta(z)$$

Moreover for every $z \in U$

$$C_\phi C_\phi^* K_\beta(z) = C_\phi K_{\phi(\beta)}(\phi(z)) = K_{\phi(\beta)}\left(\phi(z)\right) .$$

$$= \frac{1}{1-\overline{\phi(\beta)}\phi(z)} = \frac{1}{1-\bar{\lambda}\bar{\beta}\lambda z} = \frac{1}{1-|\lambda|^2\bar{\beta}z} = \frac{1}{1-\bar{\beta}z} = K_\beta(z)$$

hence $C_\phi C_\phi^* = C_\phi^* C_\phi = I$ on the family $V_{\alpha\in U}\{K_\alpha\}$. But by (1.11) $V_{\alpha\in U}\{K_\alpha\}$ forms a dense subset of H^2, hence $C_\phi C_\phi^* = C_\phi^* C_\phi = I$ on H^2.Therefore C_ϕ is unitary composition operator on H^2.

Proposition(1.20) :

If $\gamma = 0$, then C_σ is an unitary composition operator .

Proof :

Since $\sigma(z) = \frac{z}{-1+\gamma z}$ since $\gamma = 0$, $\sigma(z) = \frac{z}{-1+\gamma z} = -z = \lambda z$. $\lambda = -1$, $|\lambda| = 1$, hence by (1.19) C_σ is unitary composition

Remark(1.21) :

From Definition (1.17), we note every unitary composition operator is a normal composition operator.

Proposition(1.22) :

If $\gamma = 0$, then C_σ is a normal composition operator .

Proof:

Since $\gamma = 0$, then C_σ is an unitary composition operator by **(**1.20) , hence by (1.21) C_σ is a normal composition operator

Definition (1.23) : [7]

Let T be an operator on a Hilbert space H is subnormal if there exists a normal operator S on a Hilbert space K such that H is a subspace of K, the subspace H is invariant under the operator S and the restriction of S to H coincides with T , and every normal operator is subnormal operator.

Proposition(1.24) :

If $\gamma = 0$, then C_σ is a subnormal composition operator .

Proof:

If $\gamma = 0$, then C_σ is a normal composition operator by (1.22), by (1.23) C_σ is a subnormal composition operator.

Definition (1.25) : [13]

Let T be an operator on a Hilbert space H , T is called compact if every sequence x_n in H is weakly converges to x in H , then Tx_n is strongly converges to Tx .Moreover ($x_n \overset{w}{\rightarrow} x$ if $\langle x_n, u\rangle \rightarrow \langle x, u\rangle$ and $x_n \overset{s}{\rightarrow} x$ if $\|x_n - x\| \rightarrow 0$) .

Definition (1.26) : [13]

Let T be an operator on a Hilbert space H , T is called essentially normal if $TT^* = T^* T$ is compact . It is well-known that every normal operator is an essentially normal.

Proposition(1.27) :

If $\gamma = 0$, then C_σ is an essentially normal composition operator .

Proof :

If $\gamma = 0$, then C_σ is a normal composition operator by (1.22) , by (1.26) C_σ is an essentially normal composition operator.

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Journal of AL-Qadisiyah for computer science and mathematics
Vol.6 No.2 Year 2014

حول الطبيعة الجوهرية للمؤثر التركيبي C_σ

من قبل

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قسم الرياضيات

كلية التربية

جامعة القادسية

المستخلص

حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المحتث **σ** درسنا في هذا البحث المؤثر التركيبي المحتث من الدالة . بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من **σ**من الدالة . **σ** ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج التي أعطيت بدون برهان.