Warith Scientific Journal

A Comparison between Bayesian method and maximum likelihood methods to estimate Consul Kumaraswamy Distribution parameters

Noor Aamer Harp Al_bazzony
noor.aamer@s.uokerbala.edu.iq
College of Administration and Economics
University of Karbala

Prof. Awad kadim Shaalan Al_khalidy
alkhalidyawad16@gmail.com
College of Administration and Economics
University of Warith Al_Anbiyaa

Abstract:

In this paper, Monte Carlo simulation technique were used to compare the performance of the maximum likelihood method and the Bayesian method to estimate Consul kumaraswamy distribution paramerers . under two types of loss functions, the general squared loss function and the general entropy loss function. The mean squares error and the integration mean squares errors criterion were used to determined which one is the best.

The main terms of the research :MLE , IMSE, Bayes estimator , Loss function , Consul Kumaraswamy dist.

Introduction:[10]

A random variable X is said to have Consul kumaraswamy distribution iff the p.d.f f(x) is defined as in equation 1

$$f_{CKSD}(x;m,\propto,\beta)=\frac{\beta}{x}\binom{mx}{x-1}\sum_{j=0}^{mx-x+1}\binom{mx-x+1}{j}(-1)^j\,B\left(\beta,\frac{x+j-1}{\propto}+1\right) \qquad 1$$

$$x=1,2,\dots,\propto,\beta>0 \qquad and \qquad m\in N$$

and then, the Cummulative distribution function CD is defined as in equation 2

$$F(t)=\sum_{i=0}^{t}\frac{\beta}{t}\binom{mt}{x-1}\sum_{j=0}^{mt-t+1}\binom{mt-t+1}{j}(-1)^j\;B\left(\beta,\frac{t+j-1}{\alpha}+1\right) \qquad 2$$

where

t: the time required for failure to occur and it is a random variable that represents the time an organism survives to death

2-Maximum Likelihood estimator :[10]

Let $t1,t2, \dots ,tn$ be the set of n random lifetimes from the three parameter consul kumaraswamy dist. Then ,

$$L(t_1,t_2,\dots,t_n)=\prod_{i=1}^{n} f(t;m,\alpha,\beta)$$

$$=\frac{\beta^n}{x_i}\binom{mx_i}{x_i-1}\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\Gamma(\beta)\Gamma(\frac{x_i+j-1}{\alpha})}{\Gamma(\beta+\frac{x_i+j-1}{\alpha})}$$

$$=\frac{\beta^n}{x_i}\frac{(mx_i)!}{(x_i-1)!\,(mx_i-x_i+1)!}\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\Gamma(\beta)\Gamma(\frac{x_i+j-1}{\alpha})}{\Gamma(\beta+\frac{x_i+j-1}{\alpha})} \qquad 3$$

And the maximum likelihood function is defined as in equation 4

$$\log L(t_1,t_2,\dots,t_n)$$
$$=n\log\beta+\sum_{i=1}^{n}\log\Gamma(mx_i+1)-\sum_{i=1}^{n}\log\Gamma(mx_i-x_i+2)-\sum_{i=1}^{n}\log\Gamma(x_i+1)$$
$$+\sum_{i=1}^{n}\log\left[\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}\right)}\right] \qquad 4$$

Derivation equation 4 with respect to m, α, and β we have

$$\frac{\partial\log L(f(t))}{m}=\sum_{i=1}^{n}\frac{\frac{\partial}{\partial m}(\Gamma(mx_i+1))}{\Gamma(mx_i+1)}-\sum_{i=1}^{n}\frac{\frac{\partial}{\partial m}(\Gamma(mx_i-x_i+2))}{\Gamma(mx_i-x_i+2)}$$
$$+\sum_{i=1}^{n}\left[\frac{\frac{\partial}{\partial m}\left[\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}\right)}\right]}{\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}\right)}}\right] \qquad 5$$

$$\frac{\partial\log L(f(t))}{\alpha}=\sum_{i=1}^{n}\left[\frac{\left[\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\partial}{\partial\alpha}\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}\right)}\right]}{\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}\right)}}\right] \qquad 6$$

$$\frac{\partial \log L(f(t))}{\beta}=\frac{n}{\beta}\sum_{i=1}^{n}\left[\frac{\left[\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\partial}{\partial\beta}\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}\right)}\right]}{\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^j\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}\right)}}\right] \quad 7$$

Equations 5, 6, and 7 are difficult to solve analytically, therefore $m^{\wedge}, \alpha^{\wedge}$ *and* $\beta^{\wedge}$ were obtained by maximizing the likelihood function with respect to Newton-Raphson method which is a very powerful technique for solving equations iteratively and numerically.

3-Standard Bayesian Estimator Under Squared Error Loss function [6][1]

The standard Bayes estimator SB for the parameter Ө can be defined as the Posterior mean of the random parameter Ө. The SB method can be obtained for the parameters of the Kumaraswamy Consul distribution using the prior probability function (Prior dist.) and the squared error loss function, which was previously defined through the application of Lindley's approximate equations, as it is considered one of the best ways to simplify complex integrals, as well as because it gives accurate results.

let

$m\sim Binomial\,(n,p)$, $\alpha\sim Binomial\,(n,p)$, $\beta\sim Betta\,(c,d)$

Therefore, the initial probability density function (Prior Dist.) for each parameter is defined as in equations 8, 9, and 10:

$$\pi_1(m)=\binom{n}{m}p^m q^{n-m} \qquad ,m>0 \qquad 8$$

$$\pi_2(\alpha)=\binom{n}{\alpha}p^\alpha q^{n-\alpha} \qquad ,\alpha>0 \qquad 9$$

$$\pi_3(\beta)=\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\beta^{c-1}(1-\beta)^{d-1} \qquad ,0<\beta<1 \qquad 10$$

The joint priority function, which represents the product of the initial probability density functions that were imposed above, is defined as in equations 11, 12 and 13:

$$\pi_1(m)\pi_2(\alpha)\pi_3(\beta)=\binom{n}{m}p^m q^{n-m}\binom{n}{\alpha}p^\alpha q^{n-\alpha}\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\beta^{c-1}(1-\beta)^{d-1} \qquad 11$$

The possible function for the observations X1,X2,….,Xn is written as in equation 12

$$L=\prod_{i=1}^{n} f_{CKSD}(x;m,\propto,\beta)$$

$$=\prod_{i=1}^{n}\frac{\beta}{x}\binom{mx}{x-1}\sum_{j=0}^{mx-x+1}\binom{mx-x+1}{j}(-1)^{j}\frac{\Gamma\left((\beta)+\left(\frac{x+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x+j-1}{\alpha}+1\right)\Gamma(\beta)}$$

$$=\beta^{n}\prod_{i=1}^{n}\left[\frac{1}{xi}\binom{mxi}{xi-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^{j}\frac{\Gamma\left((\beta)+\left(\frac{xi+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{xi+j-1}{\alpha}+1\right)\Gamma(\beta)}\right] \qquad 12$$

The subsequent distributions of the parameters m, α, β as in equation 13

$$h(\theta,\alpha,\beta|xi)=\frac{\prod_{i=1}^{n} f(xi,m,\alpha,\beta)\pi_1(m)\pi_2(\alpha)\pi_3(\beta)}{\int_m\int_\alpha\int_\beta \prod_{i=1}^{n} f(xi,m,\alpha,\beta)\pi_1(m)\pi_2(\alpha)\pi_3(\beta)\,dm\,d\alpha\,d\beta} \qquad 13$$

Let $\pi_1(m)\pi_2(\alpha)\pi_3(\beta)=A$, then

$$h(m,\alpha,\beta|\vec{x})=\frac{\beta^{n}\prod_{i=1}^{n}\left[\frac{1}{xi}\binom{mxi}{x-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^{j}\frac{\Gamma\left((\beta)+\left(\frac{x+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x+j+\alpha-1}{\alpha}+1\right)\Gamma(\beta)}\right]A}{\int_m\int_\alpha\int_\beta\beta^{n}\prod_{i=1}^{n}\left[\frac{1}{xi}\binom{mxi}{x-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^{j}\frac{\Gamma\left((\beta)+\left(\frac{x+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x+j+\alpha-1}{\alpha}\right)\Gamma(\beta)}\right]A\,d\beta\,d\alpha\,dm} \qquad 14$$

From 14 a one can deduce the subsequent distribution of each parameter to be estimated as follows:

$h1(m|\alpha,\beta,\vec{x})$

$$=\frac{\beta^{n}\prod_{i=1}^{n}\left[\frac{1}{xi}\binom{mxi}{x_i-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^{j}\frac{\Gamma\left((\beta)+\left(\frac{x_i+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x_i+j+\alpha-1}{\alpha}+1\right)\Gamma(\beta)}\right]\binom{n}{m}p^{m}q^{n-m}}{\int_m\beta^{n}\prod_{i=1}^{n}\left[\frac{1}{xi}\binom{mxi}{x_i-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^{j}\frac{\Gamma\left((\beta)+\left(\frac{x_i+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\binom{n}{m}p^{m}q^{n-m}dm}$$

$$=\frac{\prod_{i=1}^{n}\left[\frac{1}{xi}\binom{mxi}{x_i-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^{j}\right]\binom{n}{m}p^{m}q^{n-m}}{\int_m\prod_{i=1}^{n}\left[\frac{1}{xi}\binom{mxi}{x_i-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^{j}\right]\binom{n}{m}p^{m}q^{n-m}dm} \qquad 15$$

$\mathbf{h2}\ (\alpha|m,\beta,\vec{x})$

$$= \frac{\beta^n \prod_{i=1}^{n} \left[\frac{1}{xi} \binom{mxi}{x_i - 1} \sum_{j=0}^{mxi-xi+1} \binom{mxi-xi+1}{j} (-1)^j \frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \binom{n}{\alpha} p^{\alpha} q^{n-\alpha}}{\int_{\alpha} \beta^n \prod_{i=1}^{n} \left[\frac{1}{xi} \binom{mxi}{x_i - 1} \sum_{j=0}^{mxi-xi+1} \binom{mxi-xi+1}{j} (-1)^j \frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \binom{n}{\alpha} p^{\alpha} q^{n-\alpha} d\alpha}$$

$$= \frac{\prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \binom{n}{\alpha} p^{\alpha} q^{n-\alpha}}{\int_{\alpha} \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \binom{n}{\alpha} p^{\alpha} q^{n-\alpha} d\alpha} \qquad 16$$

$\mathbf{h3}\ (\beta|m,\alpha,\vec{x})$

$$= \frac{\beta^n \prod_{i=1}^{n} \left[\frac{1}{xi} \binom{mxi}{x_i - 1} \sum_{j=0}^{mxi-xi+1} \binom{mxi-xi+1}{j} (-1)^j \frac{\Gamma\left((\beta) + \left(\frac{xi + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \beta^{c-1}(1-\beta)^{d-1}}{\int_{\beta} \beta^n \prod_{i=1}^{n} \left[\frac{1}{xi} \binom{mxi}{x_i - 1} \sum_{j=0}^{mxi-xi+1} \binom{mxi-xi+1}{j} (-1)^j \frac{\Gamma\left((\beta) + \left(\frac{xi + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \beta^{c-1}(1-\beta)^{d-1} d\beta}$$

$$= \frac{\beta^n \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \beta^{c-1}(1-\beta)^{d-1}}{\int_{\beta} \beta^n \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \beta^{c-1}(1-\beta)^{d-1} d\beta} \qquad 17$$

The Bayesian estimator under the squared loss function, which makes the risk function as minimum as possible, is the value that made the first derivative of the expected loss function equal zero, therefore,

$Risk = E(d(\delta) - \widehat{d}(\delta))^2)$

ISSN: 2618-0278 Vol. 5No. 14 June 2023

$$= \int_{\forall\delta} (d(\delta) - \hat{d}(\delta))^2 h\,(\theta, \alpha, \beta|\vec{x}) d\delta$$

$$= \int_{\forall\delta} (d(\delta)^2 - 2d(\delta)\hat{d}(\delta) + \hat{d}(\delta)^2) h\,(\theta, \alpha, \beta|\vec{x}) d\delta$$

$$= \hat{d}(\delta)^2 - 2\hat{d}(\delta) E\left(d(\delta)\middle|\underline{x}\right) + E\left(d(\delta)^2\middle|\underline{x}\right) \quad 18$$

Differentiating equation 18 with respect to d ̂(δ) and equating the derivative to zero, we get:

$$\frac{\delta E(d(\delta) - \hat{d}(\delta))^2)}{\delta \hat{d}(\delta)} = 0$$

$$= 2\hat{d}(\delta) - 2E\left(d(\delta)\middle|\underline{x}\right) = 0$$

$$\therefore \hat{d}(\delta)_{SEL} = E_\delta\left(\delta\middle|\underline{x}\right) \quad 19$$

Where;

d(δ): the true value of the parameter to be estimated

d ̂(δ): parameter estimator

〖d ̂(δ)〗 _SEL: the standard Bayes estimator for the parameter to be estimated under the squared loss function

Therefore, the standard Bayes estimator for the Consul Kumaraswamy distribution parameters is the values that made the expectation of the gradient vector <u>K1</u> equal zero.

Where;

$$\underline{K1} = \begin{bmatrix} \frac{\partial}{\partial \hat{m}}[\int_m ((m - \hat{m})^2) h_1\,(m|\,\alpha, \beta, \vec{x}) dm] \\ \frac{\partial}{\partial \hat{\alpha}}[\int_\alpha (\alpha - \hat{\alpha})^2 h_2\,(\alpha|m, \beta, \vec{x}) d\alpha] \\ \frac{\partial}{\partial \hat{\beta}}\left[\int_\beta (\beta - \hat{\beta})^2 h_3\,(\beta|\,\theta, \alpha, \vec{x}) d\beta\right] \end{bmatrix} = \begin{bmatrix} K_{11} \\ K_{12} \\ K_{13} \end{bmatrix}$$

$$K_{11} = \frac{\partial}{\partial \hat{m}}[\int_m ((m - \hat{m})^2) h_1\,(m|\,\alpha, \beta, \vec{x}) dm]$$

$$= \frac{\partial}{\partial \hat{m}}\left[\int_m (m - \hat{m})^2 \frac{\prod_{i=1}^{n} \left[\frac{1}{xi}\binom{mxi}{x_i - 1} \sum_{j=0}^{mxi-xi+1} \binom{mxi-xi+1}{j} (-1)^j\right] \binom{n}{m} p^m q^{n-m}}{\int_m \prod_{i=1}^{n} \left[\frac{1}{xi}\binom{mxi}{x_i - 1} \sum_{j=0}^{mxi-xi+1} \binom{mxi-xi+1}{j} (-1)^j\right] \binom{n}{m} p^m q^{n-m} dm} dm\right] \quad 20$$

$$K_{12} = \frac{\partial}{\partial \hat{\alpha}}[\int_\alpha (\alpha - \hat{\alpha})^2 h_2\,(\alpha|m, \beta, \vec{x}) d\alpha]$$

$$= \frac{\partial}{\partial\hat{\alpha}}[\int_\alpha (\alpha - \hat{\alpha})^2 \frac{\prod_{i=1}^{n}\left[\frac{\Gamma\left((\beta)+\left(\frac{x_i+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\binom{n}{\alpha}p^{\alpha}q^{n-\alpha}}{\int_\alpha \prod_{i=1}^{n}\left[\frac{\Gamma\left((\beta)+\left(\frac{x_i+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\binom{n}{\alpha}p^{\alpha}q^{n-\alpha}d\alpha} d\alpha] \qquad 21$$

$$K_{13} = \frac{\partial}{\partial\hat{\beta}}\left[\int_\beta (\beta-\hat{\beta})^2 h_3\,(\beta|\,\theta,\alpha,\vec{x})d\beta\right]$$

$$= \frac{\partial}{\partial\hat{\beta}}\left[\int_\beta\ (\beta-\hat{\beta})^2 \frac{\beta^n\prod_{i=1}^{n}\left[\frac{\Gamma\left((\beta)+\left(\frac{x_i+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\beta^{c-1}(1-\beta)^{d-1}}{\int_\beta\beta^n\prod_{i=1}^{n}\left[\frac{\Gamma\left((\beta)+\left(\frac{x_i+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\beta^{c-1}(1-\beta)^{d-1}d\beta} d\beta\right] \qquad 22$$

The equations 20, 21 and 22 are non-linear equations that cannot be solved by ordinary methods, so we will resort to the method of Lindley approximation and Jeffrey's method.

4- <u>Standard Informative Bayesian Estimator under General Entropy Loss</u> [5][3]

The standard Bayesian estimator for the distribution parameters of Consul Kumaraswamy under the general entropy loss function can be obtained from equations 23, 24, and 25 if there exist $\hat{m}_{SBEL}$, $\hat{\alpha}_{SBEL}$ and $\hat{\beta}_{SBEL}$ such that expectation of the gradient vector <u>K2</u> equal zero.

Where;

$$\underline{K2} = \begin{bmatrix} \frac{\partial}{\partial\hat{m}}[\int_m \left(\left(\frac{\hat{m}}{m}\right)^q - q\log\frac{\hat{m}}{m} - 1\right)^{\frac{-1}{q}} h_1\,(m|\,\alpha,\beta,\vec{x})dm] \\ \frac{\partial}{\partial\hat{\alpha}}\left[\int_\alpha \left(\left(\frac{\hat{\alpha}}{\alpha}\right)^q - q\log\frac{\hat{\alpha}}{\alpha} - 1\right)^{\frac{-1}{q}} h_2\,(\alpha|\,m,\beta,\vec{x})d\alpha\right] \\ \frac{\partial}{\partial\hat{\beta}}\left[\int_\beta \left(\left(\frac{\hat{\beta}}{\beta}\right)^q - q\log\frac{\hat{\beta}}{\beta} - 1\right)^{\frac{-1}{q}} h_3\,(\beta|\,m,\alpha,\vec{x})d\beta\right] \end{bmatrix} = \begin{bmatrix} K_{21} \\ K_{22} \\ K_{23} \end{bmatrix}$$

$$K_{21} = \frac{\partial}{\partial \widehat{m}} \left[\int_m \left(\left(\frac{\widehat{m}}{m} \right)^q - q \log \frac{\widehat{m}}{m} - 1 \right)^{\frac{-1}{q}} h_1 (m | \alpha, \beta, \vec{x}) dm \right]$$

$$= \frac{\partial}{\partial \widehat{m}} \left[\int_m \left(\left(\frac{\widehat{m}}{m} \right)^q - q \log \frac{\widehat{m}}{m} - 1 \right) \frac{\prod_{i=1}^{n} \left[\frac{1}{xi} \binom{mxi}{x_i - 1} \sum_{j=0}^{mxi - xi + 1} \binom{mxi - xi + 1}{j} (-1)^j \right] \binom{n}{m} p^m q^{n-m}}{\int_m \prod_{i=1}^{n} \left[\frac{1}{xi} \binom{mxi}{x_i - 1} \sum_{j=0}^{mxi - xi + 1} \binom{mxi - xi + 1}{j} (-1)^j \right] \binom{n}{m} p^m q^{n-m} dm} \right] \quad 23$$

$$K_{22} = \frac{\partial}{\partial \widehat{\alpha}} \left[\int_\alpha \left(\left(\frac{\widehat{\alpha}}{\alpha} \right)^q - q \log \frac{\widehat{\alpha}}{\alpha} - 1 \right)^{\frac{-1}{q}} h_2 (\alpha | m, \beta, \vec{x}) d\alpha \right]$$

$$= \frac{\partial}{\partial \widehat{\alpha}} \left[\int_\alpha \left(\left(\frac{\widehat{\alpha}}{\alpha} \right)^q - q \log \frac{\widehat{\alpha}}{\alpha} - 1 \right)^{\frac{-1}{q}} \frac{\prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1 \right) \right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1 \right) \Gamma(\beta)} \right] \binom{n}{\alpha} p^\alpha q^{n-\alpha}}{\int_\alpha \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1 \right) \right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1 \right) \Gamma(\beta)} \right] \binom{n}{\alpha} p^\alpha q^{n-\alpha} d\alpha} d\alpha \right] \quad 24$$

$$K_{23} = \frac{\partial}{\partial \widehat{\beta}} \left[\int_\beta \left(\left(\frac{\widehat{\beta}}{\beta} \right)^q - q \log \frac{\widehat{\beta}}{\beta} - 1 \right)^{\frac{-1}{q}} h_3 (\beta | m, \alpha, \vec{x}) d\beta \right]$$

$$= \frac{\partial}{\partial \hat{\beta}} [\int_{\beta} \left(\left(\frac{\hat{\beta}}{\beta} \right)^{q} - q \log \frac{\hat{\beta}}{\beta} - 1 \right)^{\frac{-1}{q}} \frac{\beta^{n} \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1 \right) \right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1 \right) \Gamma(\beta)} \right] \beta^{c-1} (1-\beta)^{d-1}}{\int_{\beta} \beta^{n} \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{xi + j - 1}{\alpha} + 1 \right) \right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1 \right) \Gamma(\beta)} \right] \beta^{c-1} (1-\beta)^{d-1} d\beta} d\beta] \quad 25$$

5- Expected Bayesian Estimator Under a Squared loss function[3]

According to the initial probability density function and using the Bayesian prediction formula,We obtain base predictor estimations for the distribution parameters of Consul kumaraswamy

$$\hat{m}_{EBSEL} = \int_0^{k_1} \hat{m}_{SBSEL} \pi(m) \, dm$$

$$\hat{m}_{EBSEL} = \int_0^{k_1} \frac{1}{k_1} \left(\frac{\partial}{\partial \hat{m}} \left[\int_m (m - \hat{m})^2 \frac{\prod_{i=1}^{n} \left[\frac{1}{x_i} \binom{mxi}{x_i - 1} \sum_{j=0}^{mxi-xi+1} \binom{mxi-xi+1}{j} (-1)^j \right] \binom{n}{m} p^m q^{n-m}}{\int_m \prod_{i=1}^{n} \left[\frac{1}{x_i} \binom{mxi}{x_i - 1} \sum_{j=0}^{mxi-xi+1} \binom{mxi-xi+1}{j} (-1)^j \right] \binom{n}{m} p^m q^{n-m} dm} dm \right] \right) dm \quad 26$$

$$\hat{\alpha}_{EBSEL} = \int_0^{k_2} \hat{\alpha}_{SBSEL} \pi(\alpha) d\alpha$$

$$\hat{\alpha}_{EBSEL} = \int_0^{k_2} \frac{1}{k_2} \left(\frac{\partial}{\partial \hat{\alpha}} [\int_\alpha (\alpha - \hat{\alpha})^2 \frac{\prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \binom{n}{\alpha} p^\alpha q^{n-\alpha}}{\int_\alpha \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \binom{n}{\alpha} p^\alpha q^{n-\alpha} d\alpha} d\alpha] \right) da_2 \qquad 27$$

$$\hat{\beta}_{EBSEL} = \int_0^{k_3} \int_0^{k_4} \hat{\beta}_{SBSEL} \pi(\beta) dc dd$$

$$\hat{\beta}_{EBSEL} = \int_0^{k_3} \int_0^{k_4} \frac{1}{k_3 k_4} \left(\frac{\partial}{\partial \hat{\beta}} \left[\int_\beta (\beta - \hat{\beta})^2 \frac{\beta^n \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \beta^{c-1} (1-\beta)^{d-1}}{\int_\beta \beta^n \prod_{i=1}^{n} \left[\frac{\Gamma\left((\beta) + \left(\frac{x_i + j - 1}{\alpha} + 1\right)\right)}{\Gamma\left(\frac{x_i + j - 1}{\alpha} + 1\right)\Gamma(\beta)} \right] \beta^{c-1} (1-\beta)^{d-1} d\beta} d\beta \right] \right) dc\, dd \qquad 28$$

6- Expected Bayesian Estimator Under General Entropy loss function[8]

According to the previous initial probability density functions and using the Bayesian prediction formula in the equation, we get the Bayesian prediction estimations for the

distribution of the three-parameter Consul Kumaraswamy and under the general entropy loss function as follows:

$$\hat{m}_{EBSEL} = \int_0^{k_1} \hat{m}_{SBEL}\pi(m)dm$$

$$\hat{m}_{EBSEL} = \int_0^{k_1} \frac{1}{k_1}\left[\frac{\partial}{\partial \hat{m}}\left[\int_m \left(\left(\frac{\hat{m}}{m}\right)^q - q\log\frac{\hat{m}}{m} - 1\right)\frac{\prod_{i=1}^{n}\left[\frac{1}{x_i}\binom{mxi}{x_i-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^j\right]\binom{n}{m}p^m q^{n-m}}{\int_m \prod_{i=1}^{n}\left[\frac{1}{x_i}\binom{mxi}{x_i-1}\sum_{j=0}^{mxi-xi+1}\binom{mxi-xi+1}{j}(-1)^j\right]\binom{n}{m}p^m q^{n-m}dm}\right]\right]dm \quad 29$$

$$\hat{\alpha}_{EBSEL} = \int_0^{k_2} \hat{\alpha}_{SBEL}\pi(\alpha)d\alpha$$

$$\hat{\alpha}_{EBSEL} = \int_0^{k_2} \frac{1}{k_2}\left(\frac{\partial}{\partial \hat{\alpha}}\left[\int_\alpha \left(\left(\frac{\hat{\alpha}}{\alpha}\right)^q - q\log\frac{\hat{\alpha}}{\alpha} - 1\right)^{\frac{-1}{q}}\frac{\prod_{i=1}^{n}\left[\frac{\Gamma\left((\beta)+\left(\frac{x_i+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\binom{n}{\alpha}p^\alpha q^{n-\alpha}}{\int_\alpha \prod_{i=1}^{n}\left[\frac{\Gamma\left((\beta)+\left(\frac{x_i+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\binom{n}{\alpha}p^\alpha q^{n-\alpha}d\alpha}d\alpha\right]\right)d\alpha \quad 30$$

$$\hat{\beta}_{EBSEL} = \int_0^{k_3}\int_0^{k_4} \hat{\beta}_{SBEL}\pi(a_2)dcdd$$

$$\hat{\beta}_{EBSEL} = \int_0^{k_3}\int_0^{k_4}\frac{1}{k_3k_4}\left(\int_\beta\left(\left(\frac{\hat{\beta}}{\beta}\right)^q - q\log\frac{\hat{\beta}}{\beta} - 1\right)\frac{\beta^n\prod_{i=1}^n\left[\frac{\Gamma\left((\beta)+\left(\frac{xi+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\beta^{c-1}(1-\beta)^{d-1}}{\int_\beta\beta^n\prod_{i=1}^n\left[\frac{\Gamma\left((\beta)+\left(\frac{xi+j-1}{\alpha}+1\right)\right)}{\Gamma\left(\frac{x+j-1}{\alpha}+1\right)\Gamma(\beta)}\right]\beta^{c-1}(1-\beta)^{d-1}d\beta}d\beta\right)dcdd \qquad 31$$

7- Simulations by Monte-Carlo method :[4][5][9]

In order to compare the efficiency of the informative Bayes method and the Maximum likelihood method to obtain good estimates of the parameters of the Consul Kumarasoamy distribution, the simulation method was used by Monte Carlo, noting that the experiment was repeated (1000) using the MATLAB program, and the following is a detailed presentation of the experiments .

Simulation steps:

1. **Determine the default values of the parameters: by doing repeated experiments and examining and testing the results that were obtained, which gave a clear idea of the capabilities and the pattern of their behavior, as five models were identified, shown in the following table 1:**

Table 1 hypothetical models for the three parameters

Assumed models	Model (1)	Model (2)	Model (3)	Model (4)	Model (5)
α	3	5	3	8	6
m	1.5	7	5	8	2
β	3	6	2	2	2

2. Determine several values for the sample size n, for the purpose of knowing the extent to which the sample size affects the accuracy of the results obtained from the estimation methods, five sample sizes were selected (n = 10, 20, 30, 40, 50). The reason behind choosing small sample

sizes is that it is according to the statistical theory that Bayesian methods give the best estimates and the best results at small sample sizes, so these sample sizes were chosen to ensure the optimization of the behavior of the estimation methods used.

3- Analysis of simulation experiments:

In this paragraph, the results of simulation experiments were presented and analyzed to estimate the parameters of the Consul kumaraswamy distribution (CKSD) according to the methods shown above, where these results were obtained using the Matlab program. Estimating the parameters of the distribution and mean integral error squares for each estimated parameter, see tables 2, 3, 4, 5 and 6.

From Table 2, it is clear that the Bayesian prediction method is superior to the maximum Likelihood method for loss functions, quadratic and general entropy, under Jeffrey's approximation for all sample sizes.

From Table 3, it is clear that the Bayesian prediction method is superior to the maximum Likelihood method for the loss functions, the quadratic and the general entropy, under Lindley approximation for all sample sizes

From Table 4 it is clear that the Bayesian prediction method is superior to the maximum possibility method for the loss functions, the quadratic and the general entropy, under Jeffrey's approximation for all sample sizes.

From Table 5, it is clear that the Bayesian prediction method is superior to the maximum possibility method for the loss functions, the quadratic and the general entropy, under Lindley for samples of size 10 and 20, and under Jeffrey's approximation for samples of size 30, 40 and 50.

From Table 6, it is clear that the Bayesian prediction method is superior to the maximum Likelihood method for the loss functions, the quadratic and the general entropy, under Lindley approximation for all sample sizes

Table 7 the percentages of preference for the methods of estimating the distribution parameters of Consul Kumaraswamy and for each sample size.

2

n	Est.	SBSELLind	SBELLind	SBSELjef	SBELjef	EBSELLind	EBELLind	EBSELjef	EBELjef	Best
10	$\hat{\alpha}$	3.89191	3.67655	3.53222	3.54438	3.43242	3.44353	3.41211	3.42212	
	$\hat{m}$	1.88554	1.93222	1.81554	1.86433	1.73777	1.72411	1.61112	1.65223	
	$\hat{\beta}$	3.69232	3.97622	3.66722	3.76222	3.53211	3.57882	3.51211	3.52711	
IMSE		0.00080	0.00046	0.00028	0.00030	0.00019	0.00020	0.00017	0.00018	
		0.00015	0.00019	0.00010	0.00013	0.00006	0.00005	0.00001	0.00002	
		0.00048	0.00095	0.00045	0.00058	0.00028	0.00034	0.00026	0.00028	EBSELjef
20	$\hat{\alpha}$	3.79111	3.57851	3.43671	3.51132	3.41311	3.41357	3.33241	3.21128	
	$\hat{m}$	1.88554	1.73821	1.61751	1.76138	1.53711	1.52412	1.41113	1.51572	
	$\hat{\beta}$	3.59137	3.77656	3.56592	3.56629	3.51134	3.43828	3.41458	3.32217	
IMSE		0.00063	0.00033	0.00019	0.00026	0.00017	0.00017	0.00011	0.00004	
		0.00015	0.00006	0.00001	0.00007	0.00000	0.00000	0.00001	0.00000	
		0.00035	0.00060	0.00032	0.00032	0.00026	0.00019	0.00017	0.00010	EBELjef
30	$\hat{\alpha}$	3.51145	3.51133	3.41632	3.41458	3.32981	3.35378	3.21221	3.20763	
	$\hat{m}$	1.82534	1.64633	1.54734	1.66111	1.51774	1.52397	1.31216	1.43772	
	$\hat{\beta}$	3.55543	3.61348	3.47241	3.45122	3.41331	3.41999	3.31321	3.31132	
IMSE		0.00026	0.00026	0.00017	0.00017	0.00011	0.00013	0.00005	0.00004	
		0.00011	0.00002	0.00000	0.00003	0.00000	0.00000	0.00004	0.00000	
		0.00031	0.00038	0.00022	0.00020	0.00017	0.00018	0.00010	0.00010	EBELjef
40	$\hat{\alpha}$	3.50442	3.41785	3.40244	3.41132	3.31353	3.31131	3.20065	3.20763	
	$\hat{m}$	1.52522	1.58222	1.51344	1.64532	1.51106	1.51744	1.50439	1.41232	
	$\hat{\beta}$	3.53906	3.52211	3.41028	3.41219	3.41101	3.21811	3.31126	3.31071	EBSELjef

IMSE		0.00025	0.00017	0.00016	0.00017	0.00010	0.00010	0.00004	0.00004	
		0.00000	0.00001	0.00000	0.00002	0.00000	0.00000	0.00000	0.00001	
		0.00029	0.00027	0.00017	0.00017	0.00017	0.00005	0.00010	0.00010	
50	$\hat{\alpha}$	3.40424	3.41133	3.23111	3.33221	3.31130	3.31001	3.10031	3.10761	EBELjef
	$\hat{m}$	1.52212	1.51264	1.41313	1.55231	1.41201	1.51118	1.50132	1.51232	
	$\hat{\beta}$	3.51322	3.51245	3.32142	3.40074	3.21101	3.21012	3.21074	3.21011	
IMSE		0.00016	0.00017	0.00005	0.00011	0.00010	0.00010	0.00001	0.00001	
		0.00000	0.00000	0.00001	0.00000	0.00001	0.00000	0.00000	0.00000	
		0.00026	0.00026	0.00010	0.00016	0.00004	0.00004	0.00004	0.00004	

3

n	Est.	SBSELLind	SBELLind	SBSELjef	SBELjef	EBSELLind	EBELLind	EBSELjef	EBELjef	Best
10	$\hat{\alpha}$	5.6981	5.5778	5.6782	5.6419	5.5542	5.4556	5.7712	5.6674	EBELLind
	$\hat{m}$	7.5934	7.6332	7.6443	7.7322	7.5511	7.3522	7.6733	7.6647	
	$\hat{\beta}$	6.5535	6.5633	6.6633	6.6677	6.4223	6.2313	6.8744	6.4380	
IMSE		0.0005	0.0003	0.0005	0.0004	0.0003	0.0002	0.0006	0.0005	
		0.0004	0.0004	0.0004	0.0005	0.0003	0.0001	0.0005	0.0004	
		0.0003	0.0003	0.0004	0.0005	0.0002	0.0001	0.0008	0.0002	
20	$\hat{\alpha}$	5.5345	5.4643	5.5326	5.6146	5.5422	5.4136	5.6433	5.5578	EBELLind
	$\hat{m}$	7.5234	7.5534	7.4526	7.6334	7.5244	7.3214	7.5665	7.5367	
	$\hat{\beta}$	6.5145	6.5246	6.6353	6.6255	6.3424	6.2213	6.5789	6.4108	
IMSE		0.0003	0.0002	0.0003	0.0004	0.0003	0.0002	0.0004	0.0003	
		0.0003	0.0003	0.0002	0.0004	0.0003	0.0001	0.0003	0.0003	
		0.0003	0.0003	0.0004	0.0004	0.0001	0.0001	0.0003	0.0002	
30	$\hat{\alpha}$	5.5186	5.4145	5.5180	5.5354	5.4079	5.4090	5.5363	5.3342	EBSELLind
	$\hat{m}$	7.5124	7.5178	7.4255	7.5543	7.3112	7.3168	7.4536	7.4224	
	$\hat{\beta}$	6.4167	6.4563	6.5333	6.5975	6.1142	6.1749	6.4524	6.3424	
IMSE		0.0003	0.0002	0.0003	0.0003	0.0002	0.0002	0.0003	0.0001	
		0.0003	0.0003	0.0002	0.0003	0.0001	0.0001	0.0002	0.0002	
		0.0002	0.0002	0.0003	0.0004	0.0000	0.0000	0.0002	0.0001	
40	$\hat{\alpha}$	5.4785	5.3242	5.4246	5.4454	5.3280	5.3116	5.3679	5.3214	EBELLind
	$\hat{m}$	7.3426	7.3422	7.3215	7.4798	7.2314	7.2009	7.3866	7.3989	

ISSN: 2618-0278 Vol. 5No. 14 June 2023

	$\hat{\beta}$	6.3323	6.3424	6.4897	6.5467	6.1111	6.1146	6.4244	6.3346	
IMSE		0.0002	0.0001	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	
		0.0001	0.0001	0.0001	0.0002	0.0001	0.0000	0.0002	0.0002	
		0.0001	0.0001	0.0002	0.0003	0.0000	0.0000	0.0002	0.0001	
50	$\hat{\alpha}$	5.3136	5.3110	5.3676	5.4136	5.3133	5.3097	5.3422	5.3178	EBELLind
	$\hat{m}$	7.3313	7.3322	7.3168	7.4245	7.2110	7.1865	7.3434	7.3644	
	$\hat{\beta}$	6.2845	6.3327	6.4135	6.4669	6.1101	6.1074	6.3425	6.3313	
IMSE		0.0001	0.0001	0.0001	0.0002	0.0001	0.0001	0.0001	0.0001	
		0.0001	0.0001	0.0001	0.0002	0.0000	0.0000	0.0001	0.0001	
		0.0001	0.0001	0.0002	0.0002	0.0000	0.0000	0.0001	0.0001	

4

n	Est.	SBSELLind	SBELLind	SBSELjef	SBELjef	EBSELLind	EBELLind	EBSELjef	EBELjef	Best
10	$\hat{\alpha}$	3.5535	3.5357	3.5877	3.6744	3.6211	3.6844	3.4325	3.4187	EBELjef
	$\hat{m}$	5.6733	5.5686	5.6674	5.5489	5.5638	5.5977	5.5123	5.4424	
	$\hat{\beta}$	2.7819	2.6674	2.3988	2.6386	2.5535	2.5452	2.4342	2.3868	
IMSE		0.0003	0.0003	0.0004	0.0005	0.0004	0.0005	0.0002	0.0002	
		0.0005	0.0003	0.0005	0.0003	0.0003	0.0004	0.0003	0.0002	
		0.0006	0.0005	0.0002	0.0004	0.0003	0.0003	0.0002	0.0002	
20	$\hat{\alpha}$	3.4976	3.5191	3.5487	3.5580	3.5534	3.5376	3.3231	3.3313	EBELjef
	$\hat{m}$	5.4533	5.4579	5.4598	5.4214	5.4579	5.4458	5.7414	5.4214	
	$\hat{\beta}$	2.6636	2.5379	2.2988	2.5390	2.4322	2.4321	2.3211	2.2713	
IMSE		0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0001	0.0001	
		0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0006	0.0002	
		0.0004	0.0003	0.0001	0.0003	0.0002	0.0002	0.0001	0.0001	
30	$\hat{\alpha}$	3.3894	3.4757	3.4286	3.4129	3.4908	3.4898	3.3187	3.2342	EBELjef
	$\hat{m}$	5.3255	5.4134	5.4328	5.3908	5.4332	5.4135	5.4244	5.3224	
	$\hat{\beta}$	2.5335	2.4424	2.2676	2.4895	2.4168	2.3326	2.2133	2.2134	
IMSE		0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0001	0.0001	
		0.0001	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0001	
		0.0003	0.0002	0.0001	0.0002	0.0002	0.0001	0.0001	0.0001	
40	$\hat{\alpha}$	3.2767	3.4570	3.4190	3.3690	3.4589	3.4565	3.1124	3.1645	EBSELjef
	$\hat{m}$	5.3136	5.3895	5.3346	5.3457	5.3574	5.3434	5.2133	5.2786	
	$\hat{\beta}$	2.4675	2.3343	2.3345	2.3468	2.3358	2.3214	2.1357	2.2151	
IMSE		0.0001	0.0002	0.0002	0.0001	0.0002	0.0002	0.0000	0.0000	
		0.0001	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	

		0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0001	
50	$\hat{\alpha}$	3.2563	3.4231	3.3453	3.3290	3.4138	3.4326	3.1118	3.1345	
	$\hat{m}$	5.2589	5.3246	5.2144	5.3326	5.3247	5.3345	5.1346	5.1896	
	$\hat{\beta}$	2.4439	2.3125	2.3244	2.3325	2.2324	2.2221	2.1113	2.1355	
IMSE		0.0001	0.0002	0.0001	0.0001	0.0002	0.0002	0.0000	0.0000	EBSELjef
		0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	
		0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	

5

n	Est.	SBSELLind	SBELLind	SBSELjef	SBELjef	EBSELLind	EBELLind	EBSELjef	EBELjef	Best
10		8.995	8.895	8.657	8.569	8.545	8.526	8.570	8.556	
		8.666	8.737	8.591	8.613	8.545	8.501	8.551	8.571	
		2.609	2.657	2.634	2.664	2.440	2.333	2.512	2.615	
IMSE		0.036	0.035	0.032	0.031	0.031	0.031	0.031	0.031	EBELLind
		0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	
		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
20		8.874	8.775	8.547	8.448	8.443	8.416	8.511	8.519	
		8.561	8.666	8.577	8.419	8.500	8.426	8.502	8.535	
		2.544	2.515	2.615	2.490	2.416	2.315	2.442	2.579	
IMSE		0.035	0.033	0.031	0.030	0.030	0.029	0.030	0.030	EBELLind
		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
30		8.776	8.643	8.516	8.414	8.427	8.407	8.309	8.324	
		8.511	8.555	8.534	8.380	8.469	8.417	8.358	8.435	
		2.446	2.477	2.561	2.456	2.361	2.305	2.236	2.446	
IMSE		0.033	0.032	0.030	0.029	0.029	0.029	0.028	0.028	EBSELjef
		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
40		8.523	8.557	8.480	8.391	8.413	8.361	8.222	8.315	
		8.469	8.479	8.513	8.347	8.443	8.358	8.309	8.369	
		2.358	2.446	2.479	2.414	2.314	2.294	2.216	2.416	
IMSE		0.031	0.031	0.030	0.029	0.029	0.029	0.027	0.028	EBSELjef
		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	

		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
50		8.444	8.443	8.458	8.347	8.408	8.332	8.125	8.260	EBSELjef
		8.348	8.325	8.450	8.319	8.346	8.314	8.119	8.233	
		2.236	2.419	2.335	2.359	2.236	2.258	2.117	2.336	
IMSE		0.030	0.030	0.030	0.029	0.029	0.028	0.026	0.028	
		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	

6

n	Es	SBSELLind	SBELLind	SBSELjef	SBELjef	EBSELLind	EBELLind	EBSELjef	EBELjef	Best
10		6.5699	6.5468	6.6794	6.5908	6.5790	6.5891	6.7898	6.7778	EBSELLind
		2.6389	2.6647	2.8253	2.6788	2.7849	2.7960	2.9820	2.9996	
		2.5543	2.5789	2.7343	2.5901	2.8464	2.8787	2.8896	2.8902	
IMSE		0.0003	0.0003	0.0005	0.0004	0.0003	0.0004	0.0006	0.0006	
		0.0004	0.0004	0.0007	0.0005	0.0006	0.0006	0.0010	0.0010	
		0.0003	0.0003	0.0005	0.0004	0.0007	0.0008	0.0008	0.0008	
20		6.5436	6.5146	6.5132	6.4897	6.5564	6.5679	6.7573	6.6779	SBELLJEF
		2.6124	2.6356	2.5647	2.5144	2.7135	2.6790	2.8896	2.8978	
		2.5235	2.5464	2.5124	2.5123	2.7586	2.7786	2.7655	2.7891	
IMSE		0.0003	0.0003	0.0003	0.0002	0.0003	0.0003	0.0006	0.0005	
		0.0004	0.0004	0.0003	0.0003	0.0005	0.0005	0.0008	0.0008	
		0.0003	0.0003	0.0003	0.0003	0.0006	0.0006	0.0006	0.0006	
30		6.5239	6.4897	6.5014	6.4567	6.5326	6.3255	6.3568	6.3453	EBELLind
		2.5800	2.5568	2.4568	2.4680	2.6564	2.3332	2.4357	2.4122	
		2.5085	2.5135	2.4798	2.4557	2.6133	2.3224	2.5345	2.5123	
IMSE		0.0003	0.0002	0.0003	0.0002	0.0003	0.0001	0.0001	0.0001	
		0.0003	0.0003	0.0002	0.0002	0.0004	0.0001	0.0002	0.0002	
		0.0003	0.0003	0.0002	0.0002	0.0004	0.0001	0.0003	0.0003	
40		6.4543	6.4357	6.4446	6.4346	6.4788	6.3157	6.3346	6.3221	EBELLin

		2.4678	2.5464	2.4135	2.4428	2.5258	2.2335	2.3178	2.3004	
		2.4568	2.4790	2.4437	2.3457	2.5289	2.2457	2.3322	2.3134	
IMSE		0.0002	0.0002	0.0002	0.0002	0.0002	0.0001	0.0001	0.0001	
		0.0002	0.0003	0.0002	0.0002	0.0003	0.0001	0.0001	0.0001	
		0.0002	0.0002	0.0002	0.0001	0.0003	0.0001	0.0001	0.0001	
50		6.4134	6.4114	6.3447	6.3211	6.2246	6.2013	6.3111	6.3168	
		2.4446	2.4234	2.3800	2.2346	2.2169	2.1226	2.3055	2.2897	
		2.4322	2.4124	2.3459	2.2956	2.2669	2.1567	2.3123	2.2679	
IMSE		0.0002	0.0002	0.0001	0.0001	0.0001	0.0000	0.0001	0.0001	
		0.0002	0.0002	0.0001	0.0001	0.0001	0.0000	0.0001	0.0001	
		0.0002	0.0002	0.0001	0.0001	0.0001	0.0000	0.0001	0.0001	EBELLind

(7)

Method	Sample size					Number of times of preference	Percentage of preference%
	$n_1 = 10$	$n_2 = 20$	$n_3 = 30$	$n_4 = 40$	$n_5 = 50$		
SBSELLind	1	0	0	0	0	1	4
SBELLind	0	0	0	0	0	0	0
SBSELjef	0	0	0	0	0	0	0
SBELjef	0	1	0	0	0	**1**	4
EBSELLind	0	0	1	0	0	1	4
EBELLind	2	2	1	2	2	9	36
EBSELjef	1	0	1	3	2	7	28
EBELjef	1	2	2	0	1	6	24

References

1- Al-Bayati, Hussam Najm Abboud, 2002, Comparing the Methods of Estimating the Whipple Model of Failure Using Simulation, PhD thesis, Faculty of Administration and Economics, University of Baghdad

2- Al-Sudani, Marwa Ali Moukas, 2019, Estimating the Parameters of Some Nonlinear Survival Models by Adopting Design Theory, PhD thesis, Faculty of Administration and Economics, Mustansiriyah University.

3- Abadi, Karam Nasser, 2021, The Bizi Appreciation of Lindley's Distribution of Three Parameters with Practical Application, Master's Thesis, Faculty of Administration and Economics, University of Karbala.

4- Abd Ali, Ahmed Turki, 2019, The use of the Biz method and objective programming in estimating the parameters of regression, Master Thesis, Faculty of Administration and Economics, University of Karbala.

5- Mohammed, Ruqaya Raad Hussein, 2019, Estimating the function of dependency for the distribution of kumaraswamy with practical application, Master Thesis, Faculty of Administration and Economics, University of Baghdad.

6- Najm Abd Eleiwi, Sami Attia Sayed, 2017, Comparison between Biz's Standard Capabilities and the Great Possible Method of Estimating Maxwell's Distribution Teacher Using Simulation, Maysan Journal of Academic Studies, 32, Maysan University Department of Mathematics

7- Abd-Alzahra ,Atheer ,2018 , Analysis of the Survival function when the coefficient Hazard proportional with time (Applied study) ,

8- -Ismail, S. K., AL-Sabbah, S. A., Moahammed, S. M., Nassif, M. M., & Ramadan, E. Q. (2022). Estimation of exponential Pareto parameters. *International Journal of Nonlinear Analysis and Applications*, *13*(1), 2385-2394.

9- - Li, C.P.& Hao ,H.B.(2019).E-Bayesian estimation and hierarchical Bayesian estimation of Poisson dist. Parameter under entropy loss function . IJAM,49,369-374.

10- Rashid, A., & Jan, T. R. (2015). A New Three Parameter Consul Kumaraswamy Distribution with Application. *Int. J. Modern Math. Sci*, *13*(4), 366-376