Approximation of Functions by Means of the Modulus $\tau(f,\Delta)_{p,\mu}$

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تقريب الدوال بواسطة معدل القياس $\tau(f,\Delta)_{p,\mu}$.

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الخلاصة

في بحثنا استخدمنا برهنتي وتني و رايز- تورن لإيجاد أفضل درجة تقريب للدوال بواسطة النماذج التكاملية في فضاء $L_{p,\mu}(X)$.

ABSTRACT

In this paper, we are used Whitney's and Riesz–Torin Theorems to find the degree of best approximation of functions by means of the averaged modulus of smoothness in space $L_{p,\mu}(X)$.

INTRODUCTION

Let $X=[a,b]$; $a,b\in R$ (the set of all real numbers). Then we define the space of all bounded measurable functions f on X , by norm a.e :

$$\|f\|_p=\left(\int_a^b|f(x)|^p\,d(x)\right)^{1/p}<\infty \quad , \tag{1.1}$$

and denoted by $L_p(X)$, $(1\le p<\infty)$, [1]. Also we denote by $L_{p,\mu}[a,b]$ $(1\le p<\infty)$, of the space of all bounded $\mu-$measurable functions f on $[a,b]$, and defined by:

$$\|f\|_{p,\mu}=\left(\int_a^b|f(x)|^p\,d\mu(x)\right)^{1/p}<\infty \tag{1.2}$$

where μ is the non-negative measure function on a countable set, [2] . For every function f we define the k- difference with step (h) at a point x as follows [3]:

$$\Delta_h^k f(x)=\sum_{m=0}^{k}(-1)^{m+k}\binom{k}{m}f\left(x+mh\right),\quad x,x+mh\in[a,b] \tag{1.3}$$

And the kth locally of smoothness for $f\in L_\infty[a,b]$, (the set of all essentially bounded functions on $[a,b]$) is defined by [4]:

$$w_k\left(f,\delta\right)=\sup_{|h|<\delta}\left\{\left|\Delta_h^k f(x)\right|:|h|\le\delta,\, x,x+kh\in[a,b]\right\} \tag{1.4}$$

Also, for every bounded function f the following trivial estimate holds:

$$w_k(f,[a,b])\le 2^k\|f\|_{C_{[a,b]}} \tag{1.5}$$

where $C[a,b]=\max_{x\in[a,b]}|f(x)|$,[5].

Since

$$\left|\Delta_h^k f(x)\right|=\left|\sum_{i=0}^{k}(-1)^{i+k}\binom{k}{i}f(x+ih)\right|$$

$$\le\sum_{i=0}^{k}\binom{k}{i}\|f\|_{C_{[a,b]}}=2^k\|f\|_{C_{[a,b]}}$$

In [5] the anther proved if f is a measurable bounded function on $[a,b]$,

then : $w_k(f,\delta)_p \le \tau_k(f,\delta)_p \le w_k(f,\delta)(b-a)^{1/p}$, $(1 \le p < \infty)$

where

$$\tau_k(f,\delta)_p = \|w(f,\delta)\|_p = \left\| \sup_{|h|<\delta} \left\{ \left| \Delta_h^k f(x) \right| : |h| \le \delta, x, x+kh \in [a,b] \right\} \right\|_p$$

Also proved the following theorem which is now classical in approximation theory and numerical analysis .This theorem gives additional conditions which allow us to invert the above inequality.

Theorem 1, [5]:

For each integer $n \ge 1$ there is a number W_n with the following property, for any interval Δ and for any continuous function f on Δ there is a polynomial P of degree at most $n-1$ such that

$$\left| f(x) - P(x) \right| \le W_n w_n(f,\Delta) \;,\; x \in \Delta \tag{1.6}$$

where W_n is called Whitney's constant, $\Delta = [a,b]$.

Definition (Riesz – Torin Theorem), [5]:

Let T be a linear operator from the spaces $L_p[a,b]$ in to the spaces $L_q[a,b]$, if there exists a constant k , for which

$$\|T f(x)\|_{q[a,b]} \le k \, \|f(x)\|_{p[a,b]} \quad , 1 \le p < q < \infty \tag{1.7}$$

For every function f in $L_p[a,b]$, we say that the operator T is of the type (p,q) . The smallest number k with this property is called the (p,q)- norm of the operator T.

Theorem 2, [5]:

For each $n \in Z^+$, there is a number W_n and there is a polynomial p_n for each Lebesgue integral function f on $[a,b]$, such that

$$|f - p_n| < W_n w_n(f,\Delta), \tag{1.8}$$

where W_n Whitney's constant ,$\Delta=[a,b]$.

Lemma 1, [5]:

Let L_n be a linear operator and $\sum_n=\{x_i : a=x_0<...<x_{n+1}=b\}$. If $f\in M[a,b]$. Then $L_n(f)\in L_p[a,b]$, ($1\le p\le\infty$)

and $$\|L_n f\|_p \le K\|f\|_{p\sum_n} , \qquad (1.9)$$

where K is an absolute constant and $M[a,b]$ the space of all measurable functions bounded on interval $[a,b]$.

Lemma 2, [2]:

Let μ be a non – decreasing function on P, satisfying:

$\mu(y)-\mu(x)$ = Constant and $1<p<\infty$, we put

$w_\mu(\delta)=\sup\limits_{0<y-x\le\delta}\left(\mu(y)-\mu(x)\right)$, $\delta>0$, and

$$\left(\frac{1}{n}\sum_{k=o}^{n-1}\max_{x\in I_k}|P_n|^p\right)^{1/p} \le C(p)\|P_n\|_p ,$$

where P_n is an algebraic polynomial of degree at most n and $I_k=\left[\frac{k}{n},\frac{k+1}{n}\right]$. Then $$\|P_n\|_{p,\mu} \le C(p)\left(nw_\mu\left(\frac{1}{n}\right)\right)^{1/p} \|P_n\|_p \qquad (1.10)$$

Lemma 3, [2]:

Let f be a bounded μ- measurable function and $1\le p<\infty$. Then

$$\|f\|_p \le C(p)\|f\|_{p,\mu} , \qquad (1.11)$$

where $C(p)$ is a constant depends only on p.

2-Main Results

Now we are using the interpolation results of the Whitney's theorem and the Riesz- Torin theorem [4], [5] to obtain interpolation theorems which are using of the averaged modulus of smoothness.

Lemma 4:

Let f be a $2\pi-$ periodic bounded μ- measurable function then:

$$\tau_k(f,n\delta)_{p,\mu} \le (2n)^{k+1}\ \tau_k(f,\delta)_{p,\mu} \quad , \ 1 \le p < \infty$$

Proof:

We use the identity $\quad \Delta^k_{nh} f(t) = \sum_{i=0}^{(n-1)} A_i^{n,k}\ \Delta^k_h f(t+ih) \qquad (2.1)$

where $A_i^{n,k}$ are defined by

$$\left(1+t+\ldots+t^{n-1}\right)^k = \sum_{i=0}^{(n-1)k} A_i^{n.k}\, t^i = n^k \qquad (2.2)$$

since $\tau_k(f,n\delta)_{p,\mu} = \|w_k(f,x,n\delta)\|_{p,\mu}$

we get $\qquad \tau_k(f,n\delta)_{p,\mu} \le \tau_k(f,[n]\delta)_{p,\mu}$

$$= \left\| \sup_{|h|\le\delta} \left[\left| \Delta^k_{[n]\delta} f(t) \right| : t, t+k[n]h \in \left[x - \frac{k[n]\delta}{2}, x + \frac{k[n]\delta}{2} \right] \cap [a,b] \right] \right\|_{p,\mu}$$

$$= \Bigg\| \sup_{|h|\le\delta} \Bigg[\left| \sum_{i=0}^{[n-1]k} A_i^{[n],k}\ \Delta^k_{[n]\delta} f(t+ih) \right| :$$

$$t+ih, t+ih+nh \ \in \left[x - \frac{k[n]\delta}{2}, x + \frac{k[n]\delta}{2} \right] \cap [a,b] \Bigg] \Bigg\|_{p,\mu}$$

$$w_k(f,x,n\delta) \le \sum_{i=0}^{(2n-1)k} A_i^{2n,k} \sum_{j=1}^{2n-1} w_k\left(f, x-(n-j)\frac{k\delta}{2}, \delta\right) \qquad (2.3)$$

since,

$$t+ih\ , t+ih+nh \in \bigcup_{j=1}^{2(n)-1} \left[x - \frac{k[n]\delta}{2} + (j-1)\frac{k\delta}{2}, x + \frac{k[n]\delta}{2} + (j+1)\frac{k\delta}{2} \right]$$

So that by using definition of local modulus of smoothness and (2.2),(1.5) we obtain

$$\tau_k(f,n\delta)_{p,\mu} \le \left\| \sum_{i=0}^{(n-1)k} A_i^{[2n],k} \sum_{j=1}^{2n-1} w_k\left(f, x-(n-j)\frac{k\delta}{2},\delta\right) \right\|_{p,\mu}$$

$$\le 2n^k \left[\int_a^b \left| \sum_{j=1}^{2n-1} w_k\left(f, x-(n-j)\frac{k\delta}{2},\delta\right) \right|^p d_\mu(x) \right]^{1/p}$$

$$\le 2n^k \sum_{j=1}^{2n-1} \left[\int_a^b \left| w_k\left(f, x-(n-j)\frac{k\delta}{2},\delta\right) \right|^p d_\mu(x) \right]^{1/p}$$

$$\le 2n^k\,(2n-1) \left[\int_{a-(n-j)\frac{k\delta}{2}}^{b-(n-j)\frac{k\delta}{2}} \left| w_k(f,x,\delta) \right|^p d_\mu(x) \right]^{1/p}$$

$$= 2n^k\,(2n-1)\,.\,\tau_k(f,\delta)_{p,\mu}$$

$$= (2n^k\,.2n - 2n^k)\ \tau_k(f,\delta)_{p,\mu}$$

$$= (2n^{k+1} - 2n^k)\ \tau_k(f,\delta)_{p,\mu}$$

$$\le 2n^{k+1}\ \tau_k(f,\delta)_{p,\mu}$$

Lemma 5:

Let $\sum_n = \{ x_i, a = x_0 < \ldots < x_{n+1} = b \}$ be a partition of the interval $[a,b]$ into $n+1$ subintervals and let $k \ge 1$ be an integer .Using the notation $\Delta_i = |x_{i+1} - x_{i-1}|$, $i = 1,2,\ldots,n$, $d_n = \max\{ \Delta_i, 1 \le i \le n \}$

Then $$\|w_k(f,x_i,2h)\|_{p,\mu\sum} \le 2^{1/p+2(k+1)}\ \tau_k\left(f,h+\frac{d_n}{k}\right)_{p,\mu} \qquad (2.4)$$

Proof:

From(1.9) and (1.10),(1.5) we have

$$\|w_k(f,x_i,2h)\|_{p,\mu\sum} = \left\{ \sum_{i=1}^{n} | w_k(f,x_i,2h) |^p \Delta_i \right\}^{1/p}$$

$$= \left\{ \sum_{i=1}^{n} \int_{x_{i-1}}^{x_{i+1}} | w_k(f,x_i,2h) |^p d_\mu(x_i) \right\}^{1/p}$$

$$\le 2^{1/p} c_1(p) \left\{ \sum_{i=1}^{n} \int_{x_{i-1}}^{x_{i+1}} \left| w_k(f, x, 2(h + \frac{d_n}{k}) \right|^p d(x) \right\}^{1/p}$$

$$\le 2^{1/p} c_2(p) \tau_k(f, 2(h + \frac{d_n}{k}))_p$$

$$\le 2^{1/p} c_3(p) \tau_k(f, 2(h + \frac{d_n}{k}))_{p,\mu}$$

$$\le 2^{1/p+2(k+1)} c_3(p) \tau_k(f, \frac{d_n}{k})_{p,\mu} \qquad \sqcup$$

Now, the Whitney's theorem for $f \in L_{p,\mu}(\Delta)$ spaces, have been proved.

Theorem 2.1:

For each $n \in Z^+$ there is a number W_n and there is a polynomial p_n for each Lebesgue Integral function f on $[a,b]$ such that,

$$\|f - p_n\|_{p,\mu} \le W_n \tau_k(f, [a,b])_{p,\mu} \tag{2.5}$$

where W_n is Whitney's constant.

Proof:

Let $g = f\, d_\mu(x)$

From (1.8), (1.6), (1.11) and (1.4) there is a polynomial p_n of degree $n-1$ such that

$$|g - p_n| \le W_n w_k(g, [a,b])$$

$$= W_n . \sup\left\{ \left| \Delta_h^k g(t) \right| : |h| \le \delta,\ t, t+kh \in [a,b] \right\},\ h > 0$$

$$\|g - p_n\|_p \le W_n \|f - p_n\|_{p,\mu} = W_n \left(\int_\Delta \left| \sup \Delta_h^k f(t) \right|^p d_\mu(t) : t, t+kh \in [a,b] \right)^{1/p}$$

$$= C(p) \tau_k(f, [a,b])_{p,\mu} \qquad \sqcup$$

We shall call the polynomial $p = p_n(f)$ for which theorem 2 is valid Whitney's polynomial for the function $f \in L_{p,\mu}$ of degree $(n-1)$.

Theorem 2.2:

Let L be a bounded linear operator on $L_{p,\mu}[a,b]$ and let $L(P)=P$, for every polynomial $P\in H_{n-1}$, where H_{n-1} is the set of all algebraic polynomials of degree $n-1$. Then for every function $f\in L_{p,\mu}[a,b]$, we have $\|f-L(f)\|_{p,\mu}\le C(p)\,W_n\,\tau_k\left(f,\frac{b-a}{n}\right)_{p,\mu}$ (2.6)

where W_n is Whitney's constant.

Proof:

Let $P_n(f)$ be polynomial for f of degree $n-1$ .Then using (1.8), (1.10) and (1.11) we obtain

$$
\begin{aligned}
\|f-L(f)\|_{p,\mu} &\le \|f-P_n(f)\|_{p,\mu}+\|P_n(f)-L(P_n(f))\|_{p,\mu} \\
&\quad +\|L(P_n(f))-L(f)\|_{p,\mu} \\
&\le \|f-P_n(f)\|_{p,\mu}+\|P_n(f)-L(P_n(f))\|_{p,\mu} \\
&\quad +\|L\|_{p,\mu}.\|f-P_n(f)\|_{p,\mu} \\
&\le (1+\|L\|_{p,\mu})\,\|f-P_n(f)\|_{p,\mu}+\|P_n(f)-L(P_n(f))\|_{p,\mu} \\
&\le C_1(p)(1+\|L\|_p).\|f-P_n(f)\|_p \\
&\le C_1(p)\,W_n(1+\|L\|_p)\;w_k(f,[a,b])_p \\
&\le C_2(p)W_n(1+\|L\|_p)\,\tau_k(f,\frac{b-a}{n})_p \\
&\le C_3(p)W_n(1+\|L\|_{p,\mu})\,\tau_k(f,\frac{b-a}{n})_{p,\mu} \\
&= C_4(p)\,W_n\;\;\tau_k(f,\frac{b-a}{n})_{p,\mu}
\end{aligned}
$$

where $C_4(p)=C_3(p)\,(1+\|L\|_{p,\mu})$

Theorem 2.3:

Let F be a bounded linear functional on $L_{p,\mu}[a,b]$, let $F(P)=0$ for every $P\in H_{n-1}$. Then for every $f\in L_{p,\mu}[a,b]$,

$$\| F(f) \|_{p,\mu} \le M \; W_n \;\; \tau_k (f, b-a)_{p,\mu} \tag{2.7}$$

where M is a constant.

Proof:

By using (1.8), we have

$$\begin{aligned} \|F(f)\|_{p,\mu} &\le \|F(f-p)\|_{p,\mu} + \|F(p)\|_{p,\mu} \\ &= \|F(f-p)\|_{p,\mu} \\ &\le \|F\|_{p,\mu} . \|f-p\|_{p,\mu} \\ &\le M \; W_n \;\; \tau_k \big(f, [a,b]\big)_{p,\mu} \qquad \sqcup \end{aligned}$$

Now, we shows Riesz – Torin Theorem in the spaces of all functions belongs to $L_{p,\mu}[a,b]$.

Theorem 2.4:

Let T be a linear operator from the spaces $L_{p,\mu}[a,b]$ into the spaces $L_{p,\mu}[a,b]$, if there exists a constant K , for which

$$\|T f(x)\|_{q,\mu} \le K \|f(x)\|_{p,\mu} \tag{2.8}$$

For every $f \in L_{p,\mu}[a,b]$, then the operator T is of type (p,q) .

Proof:

By using (1.10), (1.7) and (1.11) we get

$$\begin{aligned} \|T_n f\|_{q,\mu} \le C_1(q) \|T_n f\|_q \le K\, C_1(q) \|f\|_p & \\ = C_2(p) \|f\|_p & \quad , \quad C_2(p) = K\, C_1(q) \\ \le K\, C_3(p) \|f\|_{p,\mu} & \qquad \sqcup \end{aligned}$$

Theorem 2.5:

Let L_n be a linear operator, If $f \in L_{p,\mu}[a,b]$,

then $L_n(f) \in L_{p,\mu}[a,b]$, $1 \le p < \infty$ and $\|L_n(f)\|_{p,\mu} \le k \|f\|_{p,\mu \sum_n}$ (2.9)

where k is an absolute constant .

Proof:

By (1.9) and (1.11) we have

$$\| L_n f \|_{p,\mu} \le C_1(p) \| L_n f \|_p$$
$$\le C_2(p) \| f \|_p$$
$$\le C_3(p) \| f \|_{p,\mu \sum_n} \quad \sqcup$$

CONCLUSIONS

1- The best approximation of bounded μ - measurable function in $L_{p,\mu}$ spaces by using Whitney's constant have been found.

2- The best approximation of bounded μ - measurable function in $L_{p,\mu}$ spaces by using Riesz – Torin Theorem, have also been found.

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