# Application of Local Fractional Variational Iteration Method for Solving Fredholm integral equations Involving Local Fractional Operators

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**الملخص**

في هذا البحث طريقة التغاير التكراريه الكسريه المحليه استخدمت لحل المعادلات التكاملية الخطية وغير الخطية نوع فردهلم من الصنف الثاني ضمن المؤثرات التفاضلية الكسريه المحليه. لتوضيح قدرة وبساطة الطريقة، قدمت بعض الأمثلة. بينت النتائج أن الطريقة المقترحة هي فعالة جدا وبسيطة وتؤدي إلى الحل الدقيق.

**Abstract**

In this paper, the local fractional variational iteration method (LFVIM) is used for solving linear and nonlinear Fredholm integral equations of the second kind within local fractional derivative operators. To illustrate the ability and simplicity of the method, some examples are provided. The results reveal that the proposed method is very effective and simple and it leads to the exact solution.

**Keywords:** Linear and nonlinear Fredholm integral equation, Local fractional variational Iteration Method, Local fractional operator.

## 1. Introduction

Integral equation is encountered in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, medicine, mathematical problems of radioactive equilibrium, the particle transport problems of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, and radioactive heat transfer problems. Fredholm integral equation is one of the most important integral equations (Ray and Sahu, 2013 and Wazwaz , 2011).

The theory of local fractional derivatives and integrals was dealing with fractal functions, and was successfully used in local fractional Fokker Planck equation (Kolwankar and Gangal,1997 and Kolwankar and Gangal,1998), local fractional diffusion equation (Fan et al, 2015, Xu et al, 2015), local fractional integral equations (Yang (11th reference), 2011), local fractional ordinary differential equations (Yang (10th reference), 2011 and Yang (11th reference), 2011), local fractional partial differential equation (Jassim , 2015, Wang et al, 2014, Yan et al, 2014 and Yang, 2012). In this paper, we investigate the application of local fractional variational iteration methods for solving the local fractional Fredholm integral equations of the second kind. The general form of this integral equation is given by

$$u(x) = f(x) + \frac{1}{\Gamma(1+\alpha)} \int_a^b K(x,t) F(u(t))(dt)^{\alpha} \ ,0 < \alpha \le 1 \, , \, a \le x \le b \tag{1.1}$$

where $k(x,t)$ is the kernel of the local fractional integral equation, $f(x)$ and $F(u)$ are known functions. The limits of integration $a$ and $b$ are constants and $u(x)$ is the unknown solution of integral equation, which we are going to find, via local fractional variational iteration method. The paper has been organized as follows. In Section 2, we give the concept of local fractional calculus. In Section 3, we give analysis of the methods used. In Section 4, we consider several illustrative examples. Finally, in Section 5, we present our conclusions.

## 2. Preliminaries

In this section we present some basic definitions and notations of the local fractional operators (see(Wang et al, 2014,Yan et al, 2014, Yang (10th reference), 2011 and Yang, 2012)).

**Definition 1**. The local fractional derivative of $\psi(x)$ of order $\alpha$ at $x = x_0$ is given by

$$\frac{d^{\alpha}}{dx^{\alpha}} \psi(x)\Big|_{x=x_0} = \psi^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^{\alpha}(\psi(x) - \psi(x_0))}{(x - x_0)^{\alpha}} \, , \tag{2.1}$$

where $\Delta^{\alpha}(\psi(x) - \psi(x_0)) \cong \Gamma(\alpha+1)(\psi(x) - \psi(x_0))$.

The formulas of local fractional derivatives of special functions used in the paper are as follows:

$$D_x^{(\alpha)} a\psi(x) = aD_x^{(\alpha)} \psi(x), \tag{2.2}$$

$$\frac{d^{\alpha}}{dx^{\alpha}} \left( \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \frac{x^{(n-1)^{\alpha}}}{\Gamma(1+(n-1)\alpha)} \, , \ n \in N \tag{2.3}$$

**Definition 2**. The local fractional integral of $\psi(x)$ of order $\alpha$ in the interval $[a,b]$ is given by

$${}_aI_b^{(\alpha)} \psi(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b \psi(t)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} \psi(t_j)(\Delta t_j)^{\alpha} \, , \tag{2.4}$$

where the partitions of the interval $[a,b]$ are denoted as $(t_j, t_{j+1})$, with $\Delta t_j = t_{j+1} - t_j$, $t_0 = a$, $t_N = b$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, .....\}$, $j = 0, \ldots, N-1$.

The formulas of local fractional integrals of special functions used in the paper are as follows:

$${}_0I_x^{(\alpha)} a\psi(x) = a \, {}_0I_x^{(\alpha)} \psi(x), \tag{2.5}$$

$${}_0I_x^{(\alpha)} \left( \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \frac{x^{(n+1)^{\alpha}}}{\Gamma(1+(n+1)\alpha)} \, , \ n \in N \tag{2.6}$$

**Definition 3.** The Mittage Leffler function is defined as

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, \quad 0 < \alpha \le 1 \tag{2.7}$$

## 3. Analysis of the Method

For solving equation (1.1) by local fractional variational iteration method, first we differentiate once from both sides of equation (1.1) with respect to $x$:

$$u^{(\alpha)}(x) = f^{(\alpha)}(x) + \frac{1}{\Gamma(1+\alpha)} \int_a^b \frac{\partial^\alpha K(x,t)}{\partial x^\alpha} F(u(t))(dt)^\alpha \tag{3.1}$$

Now, we apply local fractional variational iteration method for equation (3.1). According to this method correction functional can be written in the following form (Xu et al, 2015) :

$$u_{n+1}(x) = u_n(x) + {}_0I_x^{(\alpha)}\left( \frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)} \left[ u_n^{(\alpha)}(\zeta) - f^{(\alpha)}(\zeta) - \frac{1}{\Gamma(1+\alpha)} \int_a^b \frac{\partial^\alpha K(\xi,t)}{\partial \xi^\alpha} F(\tilde{u}_n(t))(dt)^\alpha \right]\right), \tag{3.2}$$

where $\frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)}$ is a general fractal Lagrange's multiplier. To make the above correction functional stationary with respect to $u_n$ , we have :

$$\delta^\alpha u_{n+1}(x) = \delta^\alpha u_n(x) + \delta^\alpha \, {}_0I_x^{(\alpha)}\left( \frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)} \left[ u_n^{(\alpha)}(\zeta) - f^{(\alpha)}(\zeta) - \frac{1}{\Gamma(1+\alpha)} \int_a^b \frac{\partial^\alpha K(\xi,t)}{\partial \xi^\alpha} F(\tilde{u}_n(t))(dt)^\alpha \right]\right)$$

$$= \delta^\alpha u_n(x) + {}_0I_x^{(\alpha)}\left( \frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)} \delta^\alpha \left(u_n^{(\alpha)}(\xi)\right)\right)$$

$$= \delta^\alpha u_n(x) + \frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} \delta^\alpha u_n(x) + {}_0I_x^{(\alpha)}\left( \frac{\lambda^{(\alpha)}(\zeta)^\alpha}{\Gamma(1+\alpha)} \delta^\alpha u_n(\xi)\right) = 0 .$$

From the above relation for any $\delta^\alpha u_n$ (variation of $u_n$), we obtain

$$1 + \left.\frac{\lambda_i(\xi)^\alpha}{\Gamma(1+\alpha)}\right|_{\xi=t} = 0 , \quad \left.\left(\frac{\lambda_i(\xi)^\alpha}{\Gamma(1+\alpha)}\right)^{(\alpha)}\right|_{\xi=t} = 0.$$

This in turn gives

$$\frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)} = -1 . \tag{3.3}$$

Substituting the identified Lagrange multiplier (3.3) into equation (3.2), result in the following iterative formula:

$$u_{n+1}(x) = u_n(x) - {}_0I_x^{(\alpha)}\left( u_n^{(\alpha)}(\xi) - f^{(\alpha)}(\xi) - \frac{1}{\Gamma(1+\alpha)} \int_a^b \frac{\partial^\alpha K(\xi,t)}{\partial \xi^\alpha} F(u_n(t))(dt)^\alpha \right). \tag{3.4}$$

Finally, we obtain the exact solution or an approximate solution of the equation (1.1) as follows:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) . \tag{3.5}$$

## 4. Illustrative examples

To illustrate the ability and simplicity of the proposed method, some examples are provided here.

**Example 1.** Consider the following linear local fractional Fredholm integral equation

$$u(x)=E_\alpha(x^\alpha)-\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{x^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}u(t)(dt)^\alpha\,, \tag{4.1}$$

with the exact solution, $u(x)=E_\alpha(x^\alpha)$.

Differentiating both sides Eq. (4.1) with respect to $x$ yields

$$u^{(\alpha)}(x)=E_\alpha(x^\alpha)-1+\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{t^\alpha}{\Gamma(1+\alpha)}u(t)(dt)^\alpha \tag{4.2}$$

The correction functional for (4.2) is given by

$$u_{n+1}(x)=u_n(x)-{}_0I_x^{(\alpha)}\left(u_n^{(\alpha)}(\zeta)-E_\alpha(\zeta^\alpha)+1-\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{r^\alpha}{\Gamma(1+\alpha)}u_n(r)(dr)^\alpha\right) \tag{4.3}$$

when we put $\frac{\lambda(\zeta)^\alpha}{\Gamma(1+\alpha)}=-1$.

Notice that the initial condition $u(0)=1$ is obtained by substituting $x=0$ into (4.1).

Therefore, we have

$$u_0(x)=1 \tag{4.4}$$

$$u_1(x)=u_0(x)-{}_0I_x^{(\alpha)}\left(u_0^{(\alpha)}(\zeta)-E_\alpha(\zeta^\alpha)+1-\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{r^\alpha}{\Gamma(1+\alpha)}u_0(r)(dr)^\alpha\right)$$

$$=E_\alpha(x^\alpha)-\frac{1}{2}\frac{x^\alpha}{\Gamma(1+\alpha)} \tag{4.5}$$

$$u_2(x)=u_1(x)-{}_0I_x^{(\alpha)}\left(u_1^{(\alpha)}(\zeta)-E_\alpha(\zeta^\alpha)+1-\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{r^\alpha}{\Gamma(1+\alpha)}u_1(r)(dr)^\alpha\right)$$

$$=E_\alpha(x^\alpha)-\frac{1}{6}\frac{x^\alpha}{\Gamma(1+\alpha)} \tag{4.6}$$

$$u_3(x)=u_2(x)-{}_0I_x^{(\alpha)}\left(u_2^{(\alpha)}(\zeta)-E_\alpha(\zeta^\alpha)+1-\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{r^\alpha}{\Gamma(1+\alpha)}u_2(r)(dr)^\alpha\right)$$

$$=E_\alpha(x^\alpha)-\frac{1}{18}\frac{x^\alpha}{\Gamma(1+\alpha)} \tag{4.7}$$

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$$u_n(x)=E_\alpha(x^\alpha)-\frac{1}{2\times 3^{n-1}}\frac{x^\alpha}{\Gamma(1+\alpha)},n\geq 1. \tag{4.8}$$

Thus, we have

$$u(x)=\lim_{n\to\infty}u_n(x)$$

$$= \lim_{n\to\infty}\left[E_\alpha(x^\alpha) - \frac{1}{2\times 3^{n-1}}\frac{x^\alpha}{\Gamma(1+\alpha)}\right] = E_\alpha(x^\alpha),$$

which is the exact solution.

**Example 2.** Consider the following nonlinear local fractional Fredholm integral equation

$$u(x) = \frac{7}{8}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{2}\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}u^2(t)(dt)^\alpha, \tag{4.9}$$

with the exact solution, $u(x) = \frac{x^\alpha}{\Gamma(1+\alpha)}$.

In view (3.4) and (4.9), the local fractional iteration algorithm can be written as follows:

$$u_{n+1}(x) = u_n(x) - {}_0I_x^{(\alpha)}\left(u_n^{(\alpha)}(\zeta) - \frac{7}{8} + \frac{1}{2}\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{t^\alpha}{\Gamma(1+\alpha)}u_n^2(t)(dt)^\alpha\right). \tag{4.10}$$

Consider the initial approximation $u_0(x) = \frac{7}{8}\frac{x^\alpha}{\Gamma(1+\alpha)}$ .

Therefore other terms of the sequence are computed as follows:

$$u_0(x) = \frac{7}{8}\frac{x^\alpha}{\Gamma(1+\alpha)}, \tag{4.11}$$

$$u_1(x) = u_0(x) - {}_0I_x^{(\alpha)}\left(u_0^{(\alpha)}(\zeta) - \frac{7}{8} + \frac{1}{2}\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{t^\alpha}{\Gamma(1+\alpha)}u_0^2(t)(dt)^\alpha\right)$$

$$= \frac{497}{512}\frac{x^\alpha}{\Gamma(1+\alpha)}, \tag{4.12}$$

$$u_2(x) = u_1(x) - {}_0I_x^{(\alpha)}\left(u_1^{(\alpha)}(\zeta) - \frac{7}{8} + \frac{1}{2}\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{t^\alpha}{\Gamma(1+\alpha)}u_1^2(t)(dt)^\alpha\right)$$

$$= \frac{2082017}{2097152}\frac{x^\alpha}{\Gamma(1+\alpha)}, \tag{4.13}$$

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and so on. The sequence tends to $\frac{x^\alpha}{\Gamma(1+\alpha)}$ as $n\to\infty$, which is the exact solution.

**Example 3.** Consider the following nonlinear local fractional Fredholm integral equation

$$u(x) = \cos_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)}\left(u^2(t) + \sin_\alpha^2(t^\alpha)\right)(dt)^\alpha, \tag{4.14}$$

with the exact solution, $u(x) = \cos_\alpha(x^\alpha)$.

In the same procedure, the iterative formula can be expressed as the following:

$$u_{n+1}(x) = u_n(x) - {}_0I_x^{(\alpha)}\left(u_n^{(\alpha)}(\zeta) + \sin_\alpha(\xi^\alpha) + 1 - \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left(u_n^2(t) + \sin_\alpha^2(t)\right)(dt)^\alpha\right). \tag{4.15}$$

By using this iterative formula and taking $u_0(x) = \cos_\alpha(x^\alpha)$ , we have:

$$u_0(x) = \cos_\alpha(x^\alpha), \tag{4.16}$$

$$u_1(x) = u_0(x) - {}_0I_x^{(\alpha)}\left( u_0^{(\alpha)}(\zeta) + \sin_\alpha(\xi^\alpha) + 1 - \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left(u_0^2(t) + \sin_\alpha^2(t)\right)(dt)^\alpha \right) = \cos_\alpha(x^\alpha)\,, \quad (4.17)$$

$$u_2(x) = u_1(x) - {}_0I_x^{(\alpha)}\left( u_1^{(\alpha)}(\zeta) + \sin_\alpha(\xi^\alpha) + 1 - \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left(u_1^2(t) + \sin_\alpha^2(t)\right)(dt)^\alpha \right) = \cos_\alpha(x^\alpha)\,, \quad (4.18)$$

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$$u_n(x) = \cos_\alpha(x^\alpha). \quad (4.19)$$

Thus, we have

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= \lim_{n\to\infty} \cos_\alpha(x^\alpha) \\ &= \cos_\alpha(x^\alpha)\,, \end{aligned}$$

which is the exact solution.

## 5. Conclusion

In this work local fractional variational iteration method has been used successfully for solving the linear and nonlinear Fredholm integral equations on local fractional derivative operators. The present method reduces the computational difficulties of the other methods and all the calculations can be made simply. On the other hand the results are quite reliable. The present study has confirmed that the local fractional variational iteration method offers great advantages of straightforward applicability, computational efficiency and high accuracy.

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