Existence of mild solution to the fractional order impulsive nonlinear control system

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وجود الحل المعلول لمعادلات السيطرة النبضية الغير خطية ذات الرتبة الكسرية

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المستخلص

في هذا البحث تم دراسة وجود الحل المعلول للمعادلات النبضة غير الخطية المختلطة (التفاضلية التكاملية) ذات الرتبة الكسرية مع تباطء غير منتهي و شروط غير محليه في فضاء بناخ . للتحقق من البرهان قمنا بتعريف بعض المفاهيم المهمة للعمل بها كالتفاضل الكسري , شبه زمرة المؤثرات , و نظرية النقطة الصامدة لكراسنوسلسكي.

Abstract

In this paper we study the existence of mild solution to the fractional impulsive nonlinear mixed-type integro-differential partial equation with neutral infinite delay and nonlocal conditions in a Banach space. As a prove of this investigation we address some important concepts of work such as fractional calculus, semigroup of operators and Krasnoselskii's fixed point theorem.

Introduction

A fractional calculus deals with the generalization of integrals and derivatives of noninteger order. Which involves a wide area of applications by bringing into a broader paradigm concepts of physics, mathematics and engineering (1,2). Additionally, fractional differential equation is considered as alternative model to nonlinear differential equations (3). The authors had proved the existence of solutions of abstract fractional differential equation by using fixed point techniques (4,5). In consequence, the subject of fractional differential equations is gaining much importance and attention (6, 7, 8) for more details therein. Subsequently, several authors have discussed the problem in different ways of nonlinear differential and integro- differential equations including functional differential equations in Banach spaces. The theory of impulsive differential equations has undergone, in a rapid development over the years and played a very important role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, chemical technology and economics. In (1,12), Benchohra et al. established sufficient condition for the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the Caputo fractional derivative of order $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$. In (22), Mophou proved the existence and uniqueness results of a mild solution to impulsive fractional semilinear differential equations. Benchohra, Henderson, Ntouyas and Quahab (10), proved existence results for fractional order functional differential equations with infinite delay. (2), Benchohra and Seba(11) studied the existence of fractional impulsive differential equations in Banach spaces while, Balachandran , Trujillo (14), studied the nonlocal Cauchy problem for nonlinear fractional integro-differential equation in Banach spaces. Balachandran and Kiruthika (15) discussed the existence of nonlocal Cauchy problem for semilinear evaluation equations. Arjunan and Selvi in (16), proved the Existence results for impulsive mixed Volterra-Fredholm integro-differential inclusions with nonlocal conditions. Chang, Anguraj and Karthikeyan in (17) proved the Existence results for initial value problems with integral condition for impulsive fractional differential equations. Bragdi and Hazi. in (18) Investigated Existence and uniqueness of solution of fractional equations with nonlocal condition in Banach spaces. Anguraj, Maheswari, in (19) investigated the Existence of solutions for fractional impulsive neutral functional infinite delay integro-differential equations with nonlocal conditions. Shaochun and Gang, in (20) Proved the existence and controllability results for fractional integro-differential equations with impulsive and nonlocal conditions. In this work, we study the existence of the fractional impulsive mixed – type integro-differential partial equation with neutral infinite delay and nonlocal conditions in Banach spaces established using fractional calculus, a semigroup of operators and krasnoselskii's fixed point theorem with the sum of completely continuous and contractive operators for the first time.

1. Main results

Let X and U be a pair of real Banach spaces, with norms $|\cdot|$ and $\|\cdot\|$, respectively. Considering the existance of the fractional impulsive mixed-type integro-differential partial functional equation with neutral infinite delay and nonlocal conditions.

$$D^{\alpha}\left[Ex(t)-g(t,x_t)\right]+(Ax)(t)=(Bu)(t)+f\left(t,x_t,\int_0^t h(s,\tau,x_\tau)d\tau,\int_a^b k(s,\tau,x_\tau)d\tau\right) \quad (2.1)$$

$$t\in J=[0,b],\quad t\neq t_k,\quad k=1,2,\dots,m\,.\ 0<\alpha<1. \quad (2.2)$$

$$\Delta x\big|_{t=t_k}=I_k(x(t_k^-)),\quad k=1,2,\dots,m\,. \quad (2.3)$$

$$x(0)+\tilde{g}(x)=\phi,\quad \phi\in B_V\,. \quad (2.4)$$

where as the state $x(\cdot)$ belongs to Banach space X and the control function $u(\cdot)$ take the value in a Banach space $L^2(J,U)$ of admissible control functions. Let A be a linear bounded operator with $D(A)\subset X$ and $B:U\to X$ is a linear bounded control operator from U into X, where $\Delta x\big|_{t=t_k}$ defined by :

$\Delta x \big|_{t=t_k} = x(t_k^+) - x(t_k^-)$ for all $k = 1, 2, \ldots, m$, $0 < t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = b$, with $x(t_k^+)$, $x(t_k^-)$ representing the right and left limit of x at t_k, respectively, $\tilde{g} : B_V \to X$ is a given function.

Let $x_t(.)$ denote $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$.

The domain $D(E)$ of E becomes a Banach space with norm $\|x\|_{D(E)} = \|Ex\|_X$, and $C(E) = C\left([-\infty, 0] : D(E)\right)$.

The following hypotheses of B_V constriction needed in description of piecewise continuous space PC($(-\infty, 0]$,X), see(24).

i. $V : (-\infty, 0] \to (0, +\infty)$ is a continuous function satisfy $\ell = \int_{-\infty}^{0} V(t)dt < +\infty$. The Banach space $(B_V, \|.\|_{B_V})$ induced by the function V is defined as follows

$B_V = \{\varphi : (-\infty, 0] \to X$: bounded and measurable function on $[-c, 0]$ and

$$\|\varphi\|_{B_V} = \int_{-\infty}^{0} V(s) \sup_{s \le \theta \le 0} |\varphi(\theta)| ds \}.$$

ii. Let $B_{V'} = \{\varphi : (-\infty, b] \to X : \varphi_k \in C(J_k, x),\ \ k = 0,1,2,\ldots,n$ and there exist $\varphi(t_k^-)$ and $\varphi(t_k^+)$ with $\varphi(t_k) = \varphi(t_k^-)$, $\varphi_0 = \varphi(0) + \tilde{g}(\varphi) = \varphi \in B_V\}$ where φ_k is the restriction of φ to $J_k, J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, n$. Denote by $\|.\|_{B_{V'}}$ a seminorm in space $B_{V'}$ as follows $\|\varphi\|_{B_{V'}} = \|\varphi\|_{B_V} + \max \|\varphi_k\|_{J_k}$, $k = 1, 2, \ldots, n$ where $\|\varphi_k\|_{J_k} = \sup_{s \in J_k} \|\varphi_k(s)\|$.

$B_{V''} = \{\varphi \in B_{V'}, 0 = \varphi_0 \in Bv$ with norm $\|\varphi\|_{B_{V'}} = \max |\varphi(s)|; s \in [0, b]\}$.

Definition (2.1), (25):

A real function $f(t)$ is said to be in the space C_α, $\alpha \in \mathbb{R}$, if there exists real number $p > \alpha$, such that: $f(t) = t^p g(t)$, where $g \in C[0, \infty)$ and it is said to be in the space C_α^n if $f^{(n)} \in C_\alpha$, $n \in N$.

Definition (2.2), (25):

If the function $f(t) \in C_{-1}^n$ and n is positive integer, then we can define the fraction derivative of $f(t)$ in the caputo sense as

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds , \quad n-1 < \alpha \le n .$$

If $0 < \alpha \le 1$ then

$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds$, where $f'(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in X .

Now we recall some definitions and properties which important to help our problem.

Definition (2.3),(7):

The fractional integral of order $\alpha > 0$ of a function $f \in C([0,\infty])$ is given by :

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds , t > 0 .$$

Now, we need the following definition and lemmas in order to prove main theorem(2.1) later on.

Lemma (2.1):(24):

For $\alpha, \beta > 0$ and f as a suitable function, we have:

i. $I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t)$.

ii. $I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t)$.

iii. $I^{\alpha} \left(f(t) + g(t) \right) = I^{\alpha} f(t) + I^{\alpha} g(t)$.

iv. $I^{\alpha}\, {}^{c}D^{\alpha} f(t) = f(t) - f(0), \quad 0 < \alpha < 1$.

v. ${}^{c}D^{\alpha} I^{\alpha} f(t) = f(t)$.

vi. ${}^{c}D^{\alpha} f(t) = I^{(1-\alpha)} Df(t) = I^{(1-\alpha)} f'(t) , \quad 0 < \alpha < 1$.

vii. ${}^{c}D^{\alpha}\, {}^{c}D^{\beta} f(t) \ne {}^{c}D^{(\alpha+\beta)} f(t)$.

viii. ${}^{c}D^{\alpha}\, {}^{c}D^{\beta} f(t) \ne {}^{c}D^{\beta}\, {}^{c}D^{\alpha} f(t)$.

Lemma(2.2),"Arezola-Ascoli's theorem",(5):

Let $\Psi \subset C([a,b];X)$ be a set satisfy :

(i) For any $t \in [a,b], \{f(t), f \in \Psi\}$ is relatively compact in X.

(ii) Ψ is equicontinous on [a,b] .

Then Ψ is a relatively compact subset of $C([a,b];X)$.

Remark (2.1), (21):

The Arzela-Ascoli theorem is the key to the following result Asubset F of C (X) is compact if and only if it is closed bounded and equicontinuous.

Lemma (2.3),"Krasnoselskii's fixed point theorem",(21):

Let M be a closed convex non-empty subset of a Banach space $(X, \|\cdot\|)$. Suppose that A and B maps M into X, such that the following hypotheses are satisfied.

i. $(Ax + By) \in M, \quad \forall x, y \in M$.

ii. A is continuous and $A(M)$ is contained in a compact set.

iii. B is a contraction with constant $\alpha < 1$. Then there is a $x \in M$ with $Ax + Bx = x$.

Lemma (2.4),(16):

Assume $x \in B_v^{'}$ then $t \in J, x_t \in B_v$ moreover $\ell \|x(t)\| \le \|\phi\|_{Bv} + \ell \sup_{s\in[0,t]} \|x(s)\|$.

Definition (2.4):

A function $x:(-\infty, b] \to X$ is called a mild solution of the problem (1-4) if $x(0) + \tilde{g}(t) = \phi \in B_V$, the impulsive condition $\Delta x\big|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,2,\ldots,m$ is verified, the restriction of $x(\cdot)$ to the interval $J_k = [t_k, t_{k+1}]$ is continuous and the following integral equation holds for $t \in J = [0,b]$.

$$
\begin{aligned}
x(t) &= E^{-1}g(t,x_t) - \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} E^{-1}AE^{-1}(t_k - s)^{\alpha-1}g(s,x_s)ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t} E^{-1}AE^{-1}(t-s)^{\alpha-1}g(s,x_s)ds + \frac{E^{-1}}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}T(t_k - s)Bu(s)ds \\
&+ \frac{E^{-1}}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)Bu(s)ds + E^{-1}T(t)Ex(0) - E^{-1}T(t)g(0,x_0) \\
&+ \frac{E^{-1}}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}T(t_k - s)f\left(s, x_s, \int_0^t h(s,\tau,x_\tau)d\tau, \int_0^t k(s,\tau,x_\tau)ds\right) \\
&+ \frac{E^{-1}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)f\left(s, x_s, \int_0^t h(s,\tau,x_\tau)d\tau, \int_a^b k(s,\tau,x_\tau)d\tau\right)ds \\
&+ \sum_{0<t_k<t} E^{-1}T(t-t_k)EI_k\left(x(t_k^-)\right).
\end{aligned}
$$

To investigate the existence of system (2.1)-(2.4), we assume the following conditions that which needed in theorem(2.1):

1. $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ are closed linear operators and $D(E) \subset D(A)$ E is bijective and $E^{-1} : X \to D(E)$ is compact, [23].
2. $\left|T(t)\right| \le M_1$ for $M_1 \ge 1$ where $t \ge 0$.
3. $g : J \times B_v \to X$ and there exist positive constants L_1, L_2, L_3, L_4 such that $\left|g(t_1,\phi_1) - g(t_2,\phi_2)\right| \le L_1\left(\left\|\phi_1 - \phi_2\right\|_{B_v} + \left|t_1 - t_2\right|\right)$;
   $\left|AE^{-1}T(t_1 - s)g(s,\phi_1) - AE^{-1}T(t_1 - s)g(s,\phi_2)\right| \le L_2\left(\left\|\phi_1 - \phi_2\right\|_{B_v} + \left|t_1 - t_2\right|\right)$,
   $L_3 = \sup_{t \in J}\left|g(t,0)\right|$, $L_4 = \sup_{(t,s)\in\Delta}\left|AE^{-1}T(t-s)g(s,0)\right|$
4. $f : J \times B_v \times X \times X \to X$ and there exists positive K_1, K_2 such that
   $\left|f(t_1,\phi_1,y_1,y_1') - f(t_2,\phi_2,y_2,y_2')\right|$
   $$\le K_1\left(\left\|\phi_1 - \phi_2\right\|_{B_v} + \left|y_1 - y_2\right| + \left|y_1' - y_1'\right| + \left|t_1 - t_2\right|\right),\ K_2 = \sup_{t\in J}\left|g(t,0,0,0)\right|.$$
5. $h : \Delta \times B_v \to X$, where $\Delta = \{(t,s) : 0 \le s \le t \le b\}$. equipped with positive constants $\varphi_1, \varphi_2, z_1$ and z_2 satisfying
   i. $\left\|h(t_1,s,\phi_1) - h(t_2,s,\phi_2)\right\| \le \varphi_1\left(\left\|\phi_1 - \phi_2\right\|_{B_v} + \left|t_1 - t_2\right|\right)$, $\varphi_2 = \sup_{(t,s)\in\Delta}\left\|h(t,s,0)\right\|$.
   ii. $\left\|k(t_1,s,\phi_1) - k(t_2,s,\phi_2)\right\| \le z_1\left(\left\|\phi_1 - \phi_2\right\|_{B_v} + \left|t_1 - t_2\right|\right)$, $z_2 = \sup_{(t,s)\in\Delta}\left\|h(t,s,0)\right\|$.
6. $I_k : X \longrightarrow X$, $\left|I_k(x_1) - I_k(x_2)\right| \le \gamma_k\left|x_1 - x_2\right|$, $\left|I_k(0)\right| \le \beta_k$, where constant $\gamma_k > 0$, $\beta_k > 0$, $k = 1,2,\ldots.m$.
7. Let :
   i. $$N = \frac{b^\alpha M_1 m}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left[k_1\left(r' + b(\varphi_1 r'' + \varphi_2) + c(z_1 r'' + z_2)\right) + k_2\right]$$
   $$+\frac{b^\alpha M_1}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left[k_1\left(r' + b(\varphi_1 r'' + \varphi_2) + c(z_1 r'' + z_2)\right) + k_2\right]$$
   $$+M_1\sum_{0<t_k<t}\left(\alpha_k\left(r + M_1\left|\phi(0) - h(x(0))\right|\right) + B_k\right) + \frac{b^\alpha M_1 m}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^* k_2^*$$
   $$+\frac{b^\alpha M_1}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^* k_2^* + M_1\left|E^{-1}\right|\left(\left|L_1\left\|\phi\right\|_{B_V} + L_3\right|\right) + \left|E^{-1}\right|\left(\left|L_1 r' + L_3\right|\right)$$
   $$+\frac{b^\alpha m}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left(\left|L_2 r'' + L_4\right|\right) + \frac{b^\alpha}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left(\left|L_2 r'' + L_4\right|\right) \le r$$

Where : $r' = \|\phi_{1\tau}\|_{B_V} + L\left(r + M_1\left|\phi - \tilde{g}(x)\right|\right),$

$$r'' = \|\phi_1\|_{B_V} + \left|\phi(t) + E^{-1}T(t)E(\phi - \tilde{g}(x))\right|.$$

ii. $|Bu(s)| \le |B||u| \le k_1^* k_2^*$ where k_1^*, k_2^* are positive constants.

8. $\gamma = \left(L_1\left|E^{-1}\right| + \dfrac{b^\alpha L_2}{\Gamma(\alpha+1)}(m+1)\right)\ell < 1.$

Let $B_r = \{\phi \in B_v'' : \|\phi\|_{B_v''} \le r\}$ for some $r > 0$ then B_r for is a bounded closed convex subset in X.

Definition (2.6):

Let $x(t) = y(t) + \hat{\phi}(t)$, and for $\phi \in B_V$, we define $\hat{\phi}$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t) & \text{for} \quad t \in (-\infty, 0] \\ T(t)E(\phi - \tilde{g}(x)) & \text{for} \quad t \in J. \end{cases}$$

Theorem (2.1):

If the problem formulation (1)-(4)

$$D^\alpha\left[Ex(t) - g(t, x_t)\right] + (Ax)(t) = (Bu)(t) + f\left(t, x_t, \int_0^s h(s,\tau,x_\tau)d\tau, \int_a^b k(s,\tau,x_\tau)d\tau\right)$$

$$t \in J = [0,b], \quad t \ne t_k, \quad k = 1,2,\ldots,m.$$

$$\Delta x\big|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,2,\ldots,m.$$

$$x(0) + \tilde{g}(x) = \phi, \quad \phi \in B_v.$$

Satisfied the conditions (1-9) has a mild solution (2.4).

Proof:

It suffices to show that the operator Ω defined as follow

$$(\Omega x)(t) = \phi(t), t \in (-\infty, 0]$$

$$(\Omega x)(t) = E^{-1}\left[g(t,x_t) + T(t)E(\phi - \tilde{g}(x)) + g(0,\phi)\right]$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}T(t_k - s)(Bu)(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t_k - s)Bu(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}E^{-1}T(t_k-s)f\left(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^b k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}E^{-1}T(t-s)f\left(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^b k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)ds$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}E^{-1}AE^{-1}T(t_k-s)g(s,x_s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}E^{-1}AE^{-1}T(t-s)g(s,x_s)ds$$

$$+\sum_{0<t_k<t}E^{-1}T(t-t_k)EI_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right),\ t\in J$$

has fixed point $x(\cdot)$ from which it follows that this fixed point is a mild solution of the system (2.1)-(2.4).

Define the operators Γ and θ by:

$$(\Gamma y)(t)=\begin{cases} 0 & t\in(-\infty,0] \\ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}E^{-1}\Big[T(t_k-s)f\Big(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau, \\ \int_0^b k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\Big)ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}E^{-1}\Big[T(t-s) \\ f\left(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^b k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)ds \\ +\sum_{0<t_k<t}E^{-1}T(t-t_k)EI_k(y(t_k^-)+\hat{\phi}(t_k^-)) \\ +\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}E^{-1}T(t_k-s)(Bu)(s)ds \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}E^{-1}T(t-s)(Bu)(s)ds, \quad where\ t\in J. \end{cases}$$

$$(\theta y)(t)=\begin{cases} 0 & t\in(-\infty,0] \\ E^{-1}\left[g(t,y_t+\hat{\phi}_t)+T(t)E(\phi-\tilde{g}(y+\hat{\phi}))+g(0,\phi)\right] \\ +\dfrac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}E^{-1}AET(t_k-s)g(s,y_s+\hat{\phi}_s)ds \\ +\dfrac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}E^{-1}AET(t-s)g(s,y_s+\hat{\phi}_s)ds, & t\in J. \end{cases}$$

Obviously, the operator Ω has a fixed point if and only if the operator $\Gamma+\theta$ has a fixed point, we shown that $(\Gamma+\theta)_{B_r}\subset B_r$, ($\forall\,\phi_1,\phi_2\in B_r$, implies that $\Gamma\phi_1+\theta\phi_2\in B_r$). It is easy from hypotheses (1-7) and lemma (2.4) , we get

$$\left|(\Gamma\phi_1)(t)+(\theta\phi_2)(t)\right|\le\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}M_1\left|E^{-1}\right|\Big|f\Big(s,\phi_{1_s}+\hat{\phi}_s,$$

$$\int_0^s h(s,\tau,\phi_{1_\tau}+\hat{\phi}_\tau)d\tau,\int_0^b k(s,\tau,\phi_{1_\tau}+\hat{\phi}_\tau)d\tau\Big)-f(s,0,0,0)+f(s,0,0,0)\Big|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}M_1\left|E^{-1}\right|\Big|f\Big(s,\phi_{1_s}+\hat{\phi}_s,\int_0^s h(s,\tau,\phi_{1_\tau}+\hat{\phi}_\tau)d\tau,$$

$$\int_0^b k(s,\tau,\phi_{1_\tau}+\hat{\phi}_\tau)d\tau\Big)-f(s,0,0,0)+f(s,0,0,0)\Big|ds$$

$$+\sum_{0<t_k<t}\left|E^{-1}\right|\left|T(t-t_k)\right|\left|E\right|\left|I_k(\phi_1(t_k^-)+\hat{\phi}(t_k^-))-I_k(0)+I_k(0)\right|$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}M_1\left|E^{-1}\right|\left|B\right|\left|u\right|ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}M_1\left|E^{-1}\right|\left|B\right|\left|u\right|ds$$

$$+M_1\left|E^{-1}\right|\left|g(0,\phi)-g(0,0)+g(0,0)\right|+\left|E^{-1}\right|\left|\tilde{g}(t,\phi_{2_t}+\phi_t)-\tilde{g}(t,0)+\tilde{g}(t,0)\right|$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\left|E^{-1}\right|\Big|AE^{-1}T(t_k-s)g(s,\phi_{2_s}+\phi_s)-AE^{-1}T(t_k-s)$$

$$g(s,0)+AE^{-1}T(t_k-s)g(t,0)\Big|ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|E^{-1}\right|$$

$$\left|AE^{-1}T(t-s)g(s,\phi_{2_s}+\phi_s)-AE^{-1}T(t-s)g(s,0)+AE^{-1}T(t-s)g(t,0)\right|ds$$

$$\leq \frac{M_1 b^{\alpha} m}{\Gamma(\alpha+1)} \left|E^{-1}\right| \left[k_1 \left(\left|(\phi_{1_s} + \hat{\phi}_s - 0)\right| + \left|\int_0^s h(s,\tau,\phi_{1_\tau} + \hat{\phi}_\tau) - 0\right| d\tau + \left|\int_0^b k(s,\tau,\phi_{1_\tau} + \hat{\phi}_\tau) - 0\right| d\tau \right) \right.$$

$$\left. + k_2 \right] + \frac{M_1 b^{\alpha}}{\Gamma(\alpha+1)} k_1 \left|E^{-1}\right| \left[\left(\left|(\phi_{1_s} + \hat{\phi}_s) - 0\right| + \left|\int_0^s h(s,\tau,\phi_{1_\tau} + \hat{\phi}_\tau) - 0\right| d\tau \right.\right.$$

$$\left.\left. \left|\int_0^b k(s,\tau,\phi_{1_\tau} + \hat{\phi}_\tau) - 0\right| d\tau \right) + k_2 \right] + M_1 \sum_{0<t_k<t} \left|E^{-1}\right| \left|E\right| \left(\alpha_k \left|\phi_1(t_k^-) + \hat{\phi}(t_k^-) - 0\right| + \beta_k \right)$$

$$+ \frac{b^{\alpha} M_1 m}{\Gamma(\alpha+1)} \left|E^{-1}\right| k_1^* k_2^* + \frac{b^{\alpha} M_1}{\Gamma(\alpha+1)} \left|E^{-1}\right| k_1^* k_2^* + M_1 \left|E^{-1}\right| \left| L_1 \left\|\phi\right\|_{B_V} + L_3 \right| + \left|E^{-1}\right| \left| L_1 \left\|\phi_{2_\tau} + \hat{\phi}_\tau\right\|_{B_V} \right.$$

$$\left. + L_3 \right| + \frac{b^{\alpha} m}{\Gamma(\alpha+1)} \left|E^{-1}\right| \left| L_2 \left\|\phi_{2_\eta} + \hat{\phi}_\eta\right\|_{B_V} + L_4 \right| + \frac{b^{\alpha}}{\Gamma(\alpha+1)} \left|E^{-1}\right| \left| L_2 \left\|\phi_{2_\eta} + \hat{\phi}_\eta\right\|_{B_V} + L_4 \right|$$

$$\leq \frac{M_1 b^{\alpha} m}{\Gamma(\alpha+1)} \left|E^{-1}\right| \left[k_1 \left(\left|\phi_{1_s}\right| + \left|\hat{\phi}_s\right| + \left|\int_0^s h(s,\tau,\phi_{1_\tau} + \hat{\phi}_\tau) - h(s,\tau,0) + h(s,\tau,0)\right| \right.\right.$$

$$\left.\left. + \left|\int_0^b k(s,\tau,\phi_{1_\tau} + \hat{\phi}_\tau) - k(s,\tau,0) + k(s,\tau,0)\right| + k_2 \right) \right] + \frac{M_1 b^{\alpha}}{\Gamma(\alpha+1)} k_1 \left|E^{-1}\right|$$

$$\left| \left[\left|\phi_{1_s}\right| + \left|\hat{\phi}_s\right| + \left|\int_0^s h(s,\tau,\phi_{1_\tau} + \hat{\phi}_\tau) - h(s,\tau,0) + h(s,\tau,0)\right| + \left|\int_0^b k(s,\tau,\phi_{1_\tau} + \hat{\phi}_\tau) \right.\right.\right.$$

$$\left.\left.\left. - k(s,\tau,0) + k(s,\tau,0)\right| \right) + k_2 \right] + M_1 \sum_{0<t_k<t} \left|E^{-1}\right| \left|E\right| \left(\alpha_k \left|\phi_1(t_k^-) + \hat{\phi}(t_k^-)\right| + \beta_k \right)$$

$$+ \frac{b^{\alpha} M_1 m}{\Gamma(\alpha+1)} \left|E^{-1}\right| k_1^* k_2^* + \frac{b^{\alpha} M_1}{\Gamma(\alpha+1)} \left|E^{-1}\right| k_1^* k_2^* + M_1 \left|E^{-1}\right| \left| L_1 \left\|\phi\right\|_{B_V} + L_3 \right| + \left|E^{-1}\right| \left| L_1 \left(\left|\phi_{2_\tau}\right| + \left|\hat{\phi}_\tau\right| \right) \right.$$

$$\left. + L_3 \right| + \frac{b^{\alpha} m}{\Gamma(\alpha+1)} \left|E^{-1}\right| \left| L_2 \left(\left|\phi_{2_\eta}\right| + \left|\hat{\phi}_\eta\right| \right) + L_4 \right| + \frac{b^{\alpha}}{\Gamma(\alpha+1)} \left|E^{-1}\right| \left| L_2 \left(\left|\phi_{2_\eta}\right| + \left|\hat{\phi}_\eta\right| \right) + L_4 \right|$$

$$\leq \frac{M_1 b^{\alpha} m}{\Gamma(\alpha+1)} \left|E^{-1}\right| \left[k_1 \left(\left\|\phi_{1_\tau}\right\|_{B_V} + \left\|\phi(t) + E^{-1} T(t) E(\phi - g(x))\right\| \right.\right.$$

$$\left.\left. + \left| \varphi_1 \left\|\phi_{1_\tau} + \hat{\phi}_\tau\right\| + \varphi_2 \right| + \left| z_1 \left\|\phi_{1_\tau} + \hat{\phi}_\tau\right\| + z_2 \right| \right) + k_2 \right]$$

$$\frac{M_1 b^{\alpha}}{\Gamma(\alpha+1)}\left|E^{-1}\right|\Big[k_1\Big(\left\|\phi_{1_\tau}\right\|_{B_V}+\left|\phi(t)+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|$$

$$+\left|\varphi_1\left\|\phi_{1_\tau}+\hat{\phi}_\tau\right\|+\varphi_2\right|+\left|z_1\left\|\phi_{1_\tau}+\hat{\phi}_\tau\right\|+z_2\right|\Big)+k_2\Big]$$

$$+M_1\sum_{0<t_k<t}\left|E^{-1}\right|\left|E\right|\left|\alpha_k\left(\left\|\phi_1(t_k^-)\right\|+\left|\phi(t_k^-)+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|\right)+\beta_k\right|$$

$$+\frac{b^{\alpha}M_1 m}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^* k_2^*+\frac{b^{\alpha}M_1}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^* k_2^*+M_1\left|E^{-1}\right|\left|L_1\|\phi\|_{B_V}+L_3\right|+\left|E^{-1}\right|$$

$$\left|L_1\left(\left\|\phi_{2_\tau}\right\|_{B_V}+\left|\phi(t)+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|+L_3\right|$$

$$+\frac{b^{\alpha}m}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left|L_2\left(\left\|\phi_{2_\eta}\right\|_{B_V}+\left|\phi_\eta+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|+L_4\right|$$

$$+\frac{b^{\alpha}}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left|L_2\left(\left\|\phi_{2_\eta}\right\|_{B_V}+\left|\phi_\eta+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|+L_4\right|$$

$$\le\frac{b^{\alpha}M_1 m}{\Gamma(\alpha+1)}\left|E^{-1}\right|\Big[k_1\Big(\left\|\phi_{1_\tau}\right\|_{B_V}+L(r+M_1\left|\phi-\tilde{g}(x)\right|+\left|\varphi_1\left(\left\|\phi_{1_\tau}\right\|\right.\right.$$

$$+\left|\phi(t)+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|+\varphi_2\Big|$$

$$+\left|z_1\left(\left\|\phi_{1_\tau}\right\|_{B_V}\left|\phi(t)+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|+z_2\right|\Big)+k_2\Big]$$

$$\frac{b^{\alpha}M_1}{\Gamma(\alpha+1)}\left|E^{-1}\right|\Big[k_1\Big(\left\|\phi_{1_\tau}\right\|_{B_V}+L(r+M_1\left|\phi-\tilde{g}(x)\right|+\left|\varphi_1\left(\left\|\phi_{1_\tau}\right\|_{B_V}\right.\right.$$

$$+\left|\phi(t)+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|+\varphi_2\Big|$$

$$+\left|z_1\left(\left\|\phi_{1_\tau}\right\|_{B_V}\left|\phi(t)+E^{-1}T(t)E(\phi-\tilde{g}(x))\right|+z_2\right|\Big)+k_2\Big]$$

$$+M_1\sum_{0<t_k<t}\left|E^{-1}\right|\left|E\right|\left|\left(\alpha_k\left(r+M_1\left|E^{-1}\right|\left|E(\phi-\tilde{g}(x))\right|\right)\right)\right.$$

$$+\frac{b^{\alpha}M_1 m}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^* k_2^*+\frac{b^{\alpha}M_1}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^* k_2^*+M_1\left|E^{-1}\right|\left|L_1\|\phi\|_{B_V}+L_3\right|+\left|E^{-1}\right|\left(L_1 r''+L_3\right)$$

$$+\frac{b^{\alpha}m}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left|L_2\left(\left\|\phi_{2_\eta}\right\|_{B_V}+\left|L\left(r+M_1(\phi-\tilde{g}(x))\right)\right|+L_4\right)\right|$$

$$+\frac{b^{\alpha}}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left|L_2\left(\left\|\phi_{2_\eta}\right\|_{B_V}+\left|L\left(r+M_1(\phi-\tilde{g}(x))\right)\right|+L_4\right)\right|$$

$$\leq\frac{b^{\alpha}M_1m}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left[k_1\left(r'+b\left(\varphi_1r''+\varphi_2\right)+c\left(z_1r''+z_2\right)\right)+k_2\right]$$

$$\frac{b^{\alpha}M_1}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left[k_1\left(r'+b\left(\varphi_1r''+\varphi_2\right)+c\left(z_1r''+z_2\right)\right)+k_2\right]$$

$$+M_1\sum_{0<t_k<t}\left(\alpha_k\left(r+M_1\left|\phi-h(x)\right|\right)+\beta_k\right)+\frac{b^{\alpha}M_1m}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^*k_2^*$$

$$+\frac{b^{\alpha}M_1}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^*k_2^*+M_1\left|E^{-1}\right|\left(\left|L_1\left\|\phi\right\|_{B_V}+L_3\right|\right)+\left|E^{-1}\right|\left(L_1r'+L_3\right)$$

$$+\frac{b^{\alpha}m}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left(\left|L_2r''+L_4\right|\right)+\frac{b^{\alpha}}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left(\left|L_2r''+L_4\right|\right)=N\leq r$$

then condition (i) in lemma (2.3) is verified.

Next, we shall show that Γ is an equicontinuous for $y\in B_r,\theta_1,\theta_2\in J$ and $0<s_1<s_2\leq b$, we have that

$$\left|(\Gamma y)(s_1)-(\Gamma y)(s_2)\right|\leq\left[\frac{E^{-1}}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t_k-s)+\frac{E^{-1}}{\Gamma(\alpha)}\int_{t_k}^{s_1}(s_1-s)^{\alpha-1}T(s_1-s)\right.$$

$$\left.-\frac{E^{-1}}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t_k-s)-\frac{E^{-1}}{\Gamma(\alpha)}\int_{t_k}^{s_2}(s_2-s)^{\alpha-1}T(s_2-s)\right]$$

$$f\left(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau,\int_0^b k(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)ds+\sum_{0<t_k<t}E^{-1}T(s_1-t_k)E$$

$$I_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right)+\left[\frac{E^{-1}}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t_k-s)+\frac{E^{-1}}{\Gamma(\alpha)}\int_{t_k}^{s_1}(\theta_1-s)^{\alpha-1}T(\theta_1-s)\right.$$

$$\left.-\frac{E^{-1}}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t_k-s)-\frac{E^{-1}}{\Gamma(\alpha)}\int_{t_k}^{s_2}(\theta_2-s)^{\alpha-1}T(\theta_2-s)\right]$$

$$(Bu)(s)ds - \sum_{0<t_k<t} E^{-1}T(s_2 - t_k)EI_k\left(y(t_k^-) + \hat{\phi}(t_k^-)\right)$$

$$\leq \left[\frac{1}{\Gamma(\alpha)}\int_{t_k}^{s_1}\left((s_1 - s)^{\alpha-1}E^{-1}T(s_1 - s) - (s_1 - s)^{\alpha-1}E^{-1}T(s_2 - s) + (s_1 - s)^{\alpha-1}E^{-1}T(s_2 - s)\right)\right]$$

$$f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau, \int_0^b k(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right)(s)ds - \left[\frac{1}{\Gamma(\alpha)}\int_{t_k}^{s_1}(s_2 - s)^{\alpha-1}\right.$$

$$\left. E^{-1}T(s_2 - s) + \int_{s_1}^{s_2}(s_2 - s)^{\alpha-1}E^{-1}T(s_2 - s)\right] f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau,\right.$$

$$\left.\int_0^b k(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right)(s)ds + \sum_{0<t_k<t}\left|E^{-1}\right|\left|T(s_1 - t_k) - T(s_2 - t_k)\right|\left|E\right|\left(I_k(y(t_k^-) + \hat{\phi}(t_k^-))\right)$$

$$+\left[\frac{1}{\Gamma(\alpha)}\int_{t_k}^{s_1}\left((s_1 - s)^{\alpha-1}E^{-1}T(s_1 - s) - (s_1 - s)^{\alpha-1}E^{-1}T(s_2 - s) + (s_1 - s)^{\alpha-1}E^{-1}T(s_2 - s)\right)\right]Bu(s)ds$$

$$-\left[\frac{1}{\Gamma(\alpha)}\int_{t_k}^{s_1}(s_2 - s)^{\alpha-1}E^{-1}T(s_2 - s) + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_2 - s)^{\alpha-1}E^{-1}T(s_2 - s)\right]Bu(s)ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_{t_k}^{s_1}\left((s_1 - s)^{\alpha-1}\left|E^{-1}\right|\left|T(s_1 - s) - T(s_2 - s)\right| + \left|(s_1 - s)^{\alpha-1} + (s_2 - s)^{\alpha-1}\right|\left|E^{-1}\right|\left|T(s_2 - s)\right|\right)$$

$$\left(k_1\left(r' + b(\varphi_1 r'' + \varphi_2) + c(z_1 r'' + z_2) + k_2\right)ds + \frac{M_1}{\Gamma(\alpha+1)}\left|E^{-1}\right|\left(k_1\left(r' + b(\varphi_1 r'' + \varphi_2)\right.\right.$$

$$\left.+c(z_1 r'' + z_2) + k_2\right)(s_2 - s_1)^{\alpha} + \sum_{0<t_k<t}\left|E^{-1}\right|\left|T(s_1 - t_k) - T(s_2 - t_k)\right|\left|E\right|$$

$$\left(\alpha_k\left(r + M_1\left|\phi(0) - h(x)\right|\right)\right) + \left[\frac{1}{\Gamma(\alpha)}\int_{t_k}^{s_1}\left((s_1 - s)^{\alpha-1}\left|E^{-1}\right|\left|T(s_1 - s) - T(s_2 - s)\right|\right)\right.$$

$$+\left|(s_1 - s)^{\alpha-1} + (s_2 - s)^{\alpha-1}\right|\left|E^{-1}\right|\left|T(s_2 - s)\right|)k_1^* k_2^* ds + \frac{M_1(s_2 - s_1)^{\alpha}}{\Gamma(\alpha+1)}\left|E^{-1}\right|k_1^* k_2^* \qquad ()$$

By hypotheses (1 - 6) and lemma (2.4) and the compactness of the semigroup $T(t)$ for $t > 0$ which implies the continuity in the uniform operator topology, the right-hand side tends to zero as $\theta_2 \to \theta_1 \to 0$. And hence ΓB_r is equicontinuous. For the case $\theta_1 < \theta_2 < 0$ or $\theta_1 < 0 < \theta_2$ is very simple, the proof is omitted.

We show that ΓB_r is precompact as follows:

Let $0 < t \le b$ be fixed and $0 < \varepsilon < t$. For $y \in B_r$, we define

$$(\Gamma_\varepsilon Y)(t) = \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} E^{-1} T(t_k - s) f\left(s, Y_s + \hat{\phi}_s, \int_0^s h(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau\right.$$

$$\left., \int_a^b k(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau\right)ds + \frac{T(\varepsilon)}{\Gamma(\alpha)} \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} E^{-1} T(t-s-\varepsilon) f\left(s, Y_s + \hat{\phi}_s,\right.$$

$$\left.\int_0^s h(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau, \int_a^b k(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau\right)ds + \sum_{0<t_k<t} E^{-1} T(t - t_k) E\left(I_k(y(t_k^-) + \hat{\phi}(t_k^-))\right)$$

$$+\left[\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} E^{-1} T(t_k - s) f\left(s, Y_s + \hat{\phi}_s, \int_0^s h(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau\right.\right.$$

$$\left., \int_a^b k(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau\right)ds + \frac{T(\varepsilon)}{\Gamma(\alpha)} \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} E^{-1} T(t-s-\varepsilon) f\left(s, Y_s + \hat{\phi}_s,\right.$$

$$\left.\left.\int_0^s h(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau, \int_a^b k(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau\right)ds\right](Bu)(s)$$

Since $T(t)$ is a compact operator, the set $Y_\varepsilon(t) = \{(\Gamma_\varepsilon y)(t) : y \in B_r\}$ is a relatively in X, for every ε, $0 < \varepsilon < t$. Moreover, for every $y \in B_r$, we have that $\left|(\Gamma y)(t) - (\Gamma_\varepsilon y)(t)\right| \le \frac{1}{\Gamma(\alpha)} \left\| \left[\sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} E^{-1} T(t_k - s) + \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} E^{-1} T(t-s) \right.\right.$

$$\left. + \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} E^{-1} T(t-s) - \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} E^{-1} T(t_k - s) - \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} E^{-1} T(t-s) \right]$$

$$\left. f\left(s, Y_s + \hat{\phi}_s, \int_0^s h(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau, \int_0^s k(s,\tau,Y_\tau + \hat{\phi}_\tau)d\tau\right)\right\| ds$$

$$+ \frac{1}{\Gamma(\alpha)} \left\| \left[\sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} E^{-1} T(t_k - s) + \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} E^{-1} T(t-s) \right.\right.$$

$$+ \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} E^{-1} T(t-s) - \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} E^{-1} T(t_k - s)$$

$$-\int_{t_k}^{t-\varepsilon}(t-s)^{\alpha-1}E^{-1}T(t-s)\Bigg]\Bigg|ds\,\big|(Bu)(s)\big|$$

Thus,

$$\big|(\Gamma Y)(t)-(\Gamma_\varepsilon Y)(t)\big|\le\frac{1}{\Gamma(\alpha)}\left|\int_{t-\varepsilon}^{t}(t-s)^{\alpha-1}E^{-1}T(t-s)f\left(s,Y_s+\hat{\phi}_s,\int_0^s h(s,\tau,Y_\tau+\hat{\phi}_\tau)d\tau\right.\right.$$

$$\left.\left.,\int_0^b k(s,\tau,Y_\tau+\hat{\phi}_\tau)d\tau\right)\right|ds+\frac{1}{\Gamma(\alpha)}\left|\int_{t-\varepsilon}^{t}(t-s)^{\alpha-1}E^{-1}T(t-s)\right|ds\,|B|\,\|u\|(s)$$

Therefor as :

$\varepsilon\to 0$. The sets $\{(\Gamma_\varepsilon y)(t):y\in B_r\}$ for every $\varepsilon>0$ are precompact close to

the set $\{(\Gamma y)(t):y\in B_r\}$ is precompact in X . Also ΓB_r is uniformly bounded.

From lemma (2.2), we get closure of ΓB_r is compact.

Now, we shown that the operator Γ is continuous as follows:

$$\big|(\Gamma\phi_1)(t)-(\Gamma\phi_2)(t)\big|\le\Bigg[\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\big|E^{-1}\big|\big|T(t_k-s)\big|+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}$$

$$\big|E^{-1}\big|\big|T(t-s)\big|\Bigg]\Bigg|f\left(s,\phi_{1_s}+\hat{\phi}_s,\int_0^s h(s,\tau,\phi_{1_\tau}+\hat{\phi}_\tau)d\tau,\int_0^b k(s,\tau,\phi_{1_\tau}+\hat{\phi}_\tau)d\tau\right)$$

$$-f\left(s,\phi_{2_s}+\hat{\phi}_s,\int_0^s h(s,\tau,\phi_{2_\tau}+\hat{\phi}_\tau)d\tau,\int_0^b k(s,\tau,\phi_{2_\tau}+\hat{\phi}_\tau)d\tau\right)\Bigg|ds$$

$$+\sum_{0<t_k<t}\big|E^{-1}\big|\big|T(t-s)\big|\big|E\big|\Big|I_k(\phi_1(t_k^-)+\hat{\phi}(t_k^{-1}))\Big|+\Bigg[\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\big|E^{-1}\big|$$

$$\big|T(t_k-s)\big|+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\big|E^{-1}\big|\big|T(t-s)\big|\Bigg]\big|Bu\big|(s)ds$$

$$\le\frac{b^\alpha M_1K_1\ell\big|E^{-1}\big|}{\Gamma(\alpha+1)}\big(m+b(\varphi_1+\varphi_2)\big)\|\phi_1-\phi_2\|_{B_V}+\big|E^{-1}\big|M_1\big|E\big|\sum_{0<t_k<t}\alpha_k\|\phi_1-\phi_2\|_{B_V}$$

$$+\frac{b^\alpha M_1\big|E^{-1}\big|}{\Gamma(\alpha+1)}[m+1]k_1^*k_2^*.$$

For $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|\phi_1 - \phi_2\| < \delta$ implies $\|\Gamma\phi_1 - \Gamma\phi_2\|_{B_v} < \varepsilon$. Thus the operator Γ is continuous and from above details, we have that is completely continuous which satisfies condition lemma (2.3) (ii).

Now, We show that θ is a contraction with constant γ as follows. We have

$$\left|(\theta\phi_1)(t) - (\theta\phi_2)(t)\right| \le \Big| E^{-1}\Big[(g(t,\phi_{1_t} + \hat{\phi}_t) - g(t,\phi_{2_t} + \hat{\phi}_t)) + T(t)E(\phi_1 - \tilde{g}(x))$$

$$-(T(t)E(\phi_2 - \tilde{g}(x)) + (g(0,\phi_1) - g(0,\phi_2))\Big]$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\left((t_k - s)^{\alpha-1}E^{-1}AET(t_k - s)(g(s,\phi_{1_s} + \hat{\phi}_s) - g(s,\phi_{2_s} + \hat{\phi}_s))\right)ds\Big|$$

$$\le L_1\left|E^{-1}\right| + \left[\frac{b^{\alpha}L_2 m}{\Gamma(\alpha+1)} + \frac{b^{\alpha}L_2}{\Gamma(\alpha+1)}\right]\|\phi_1 - \phi_2\|_{B_v^{\cdot}}$$

$$\le \left(L_1\left|E^{-1}\right| + \frac{bL_2}{\Gamma(\alpha+1)}(m+1)\right)\ell\,\|\phi_1 - \phi_2\|$$

$$\le \gamma\,\|\phi_1 - \phi_2\|_{B_v^{\cdot}}\,.$$

By hypotheses (3-8), and thus operator θ is a contractive operator. Therefore, all the conditions of Krasnoselskii's fixed point theorem are satisfied and thus operator $\Gamma + \theta$ has a fixed point in B_r. From this it follows that operator Ω has a fixed point and hence system (1-4) has a mild solution on J.

Conclusions

Sufficient conditions for the existence of The Fractional Impulsive Mixed-Type Integro-Differential Partial Equation with Neutral Infinite Delay and Nonlocal Conditions in a Banach space have been presented in details of Krasnoselskii's fixed point theorem supported by dynamical definition of semigroup operators.

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