Structure topology of magnetic field close to a null point

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البناء التوبولوجي للمجال المغناطيسي بالقرب من النقطة الملغية

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المستخلص

إعادة الربط المغناطيسي هو عملية مهمة جدا وأساسية في نقل الطاقة من المجال المغناطيسي الى البلازما. النظريات السابقة، والمحاكاة العددية والملاحظات ركزت في الغالب على نموذج ثنائي الأبعاد. وان إعادة النظر في تكوين المجالات المغناطيسية بالقرب من نقطة فارغة عن طريق التحليل التبولوجي. ما يسمى نوع -X ونوع -Oمن الحقل المغناطيسي وعلى التوالي والتي تحتل مكانها الخاص بالجدول . ثم سيظهر وجود دوامة ومعقدة من المجالات المغناطيسية وسوف ندرسها تحليليا.

Abstract

Magnetic reconnection is a very important and fundamental plasma process in transferring energy from magnetic field into plasma. Previous theory, numerical simulations and observations mostly concentrate on 2-dimensional (2D) model.Configuration of magnetic fields near anull point is re-examined by a topological analysis. The so-called X-and O-type magnetic field respectively occupy their own seat in our classified table. Then the existence of the spiral and node of configuration will be shown by the analysis.

1- Introduction

Magnetic null points are classified by the number of field lines passing through them [8]. A bundle of field lines having a common tangent at a null point is counted as one line only. It has been known that there exist two kinds of null points [2]. If there is only one field line passing through a null point, it is called (by the shape of field lines in vicinity) an O-type null point. Alternatively, if there field lines pass through it, it is known as an X-type null point. All current-free null points belong to this category.These two types of null points have considerably been investigated by many workers, among whom Dungey [2] found out an excellent concept of the so-called reconnection of magnetic field lines. He showed that the energy of magnetic field may be effectively converted into kinetic and thermal energies of plasma by the reconnection at the X-type null point. His idea has been followed by Sweet [9], Parker [5],Petschek [7]Yeh&Axfored [10] and Fukao&Tsuda [3,4]. Now it may be worthwhile to call up the original of idea of Dungey in order to clarify this sort of energy conversion . We assure that reconsideration of the magnetic fields follows, the classification of magnetic fields in the vicinity of null point will be strictly re-examined in terms of the phase trajectories. Much of what will be said is the extension and the generalization of Dungey's work, but some types of the null points which have not been recognized by the pioneers are first proposed here.

2-Review of two-dimensional neutral points

In two dimensions the matrix M is simply

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_{ij}$are real constants. The solenoidal constraint$\nabla \quad B = 0$ gives a_{11}=a_{22} thus the trace of M iszero. The diagonal entries in the matrix are associatedwith the potential part of the field so we let $a_{11} = p$and since the current associated with the neutral pointis

$$\boldsymbol{J} = \frac{1}{\mu_0}(0,0,a_{21} - a_{12}).$$

We define,

$$a_{12} = \frac{1}{2}(q - j_z) \text{ and } a_{21} = \frac{1}{2}(q + j_z)$$

Clearly, for a current-free neutral point a_{21}=$a_{12} = \frac{q}{2}$and the parameter q is therefore also associated withthe potential field whilst $j_z$is the magnitude of thecurrent perpendicular to the plane of the null point.The matrix M may now finally be written as

$$M = \begin{bmatrix} p & \frac{1}{2}(q - j_z) \\ \frac{1}{2}(q + j_z) & -p \end{bmatrix}.$$

We will find it useful to define a threshold current,

$$j_{thresh} = \sqrt{4p^2 + q^2}, \qquad 1$$

which we note only depends on the parameters associated with the potential part of the field. It is equal to the square root of the discriminant of the characteristic equation of the symmetric part of M.We now calculate the flux function A, which satisfies,

$B_X = \frac{\partial A}{\partial Y}$ and $B_Y = -\frac{\partial A}{\partial X}$

So that

$$A = \frac{1}{4}[(q - j_z)Y^2 - (q + j_z)X^2] + pXY \qquad 2$$

If we rotate the XY-axes through an angle θ to givexy-axes using the relations

$$\left.\begin{array}{l} X = x\ \cos\theta - y\ \sin\theta \\ Y = x\ \sin\theta + y\ \cos\theta \end{array}\right\}, \qquad 3$$

and substitute (3) into (2) with

$$\tan 2\theta = -2\frac{p}{q},$$

with j_{thresh} as in (1) then A becomes

$$A = \frac{1}{4}[(j_{thresh} - j_z)y^2 - (j_{thresh} + j_z)x^2]. \qquad 4$$

We thus see that in two dimensions the two parameters j_{thresh} and j_z govern the magnetic configuration.

The eigenvalues of the matrix M are given by

$$\lambda = \pm\frac{1}{2}\sqrt{j^2{}_{thresh} - j^2{}_z};$$

hence, depending on whether the current j_z is greater or less than the threshold value j_{thresh} the eigenvalues will be real or imaginary and the field will have a different structure(Parnell et al. (1996)) [6].

3- Topology of magnetic field lines

Since the magnetic field must be a solution of Maxwell's equation, it must be differentiable, and hence expansible in Taylor series. Taking the null point as the origin, the magnetic field (magnetic flux density) $\boldsymbol{B}$ near a null point may be expressed by lowest order terms, that is,

$$\boldsymbol{B} = M.\boldsymbol{r}$$

Where M is a matrix with the elements of theJacobian of $\boldsymbol{B}$

$$M = \begin{bmatrix} \frac{\partial \mathrm{B_X}}{\partial \mathrm{X}} & \frac{\partial \mathrm{B_X}}{\partial \mathrm{Y}} & \frac{\partial \mathrm{B_X}}{\partial \mathrm{Z}} \\ \frac{\partial \mathrm{B_Y}}{\partial \mathrm{X}} & \frac{\partial \mathrm{B_Y}}{\partial \mathrm{Y}} & \frac{\partial \mathrm{B_Y}}{\partial \mathrm{Z}} \\ \frac{\partial \mathrm{B_Z}}{\partial \mathrm{X}} & \frac{\partial \mathrm{B_Z}}{\partial \mathrm{Y}} & \frac{\partial \mathrm{B_Z}}{\partial \mathrm{Z}} \end{bmatrix}$$

and $\boldsymbol{r}$ is the position vector $(X, Y, Z)^T$. Only one constraint is that a must have zero trace, since the magnetic field is solenoidal.

Here we consider of

$$\dot{\boldsymbol{r}}(\boldsymbol{\tau}) = M.\boldsymbol{r}(\boldsymbol{\tau}) \qquad 5$$

Where $\dot{\boldsymbol{r}}(\boldsymbol{\tau})$is the derivative with respect to an arbitrary$\boldsymbol{\tau}$**.** It is apparent that a solution $\boldsymbol{r}(\boldsymbol{\tau})$describes a field line, since $\dot{\boldsymbol{r}}(\boldsymbol{\tau})$indicates$\boldsymbol{B}$itself**.** Therefore we can show magnetic field lines in terms of phase trajectories, which satisfy (5). It is readily seen that M must have at least one real eigenvalue, and that the orthogonal coordinate system (x, y, z) can always be chosen such that the z-axis is in the direction of the eigenvector corresponding to the real eigenvale. In such a coordinate system, M may be describe as

$$M^* = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad 6$$

Where

$$\sum_{i=1}^{3} a_{ii} = 0 \ \ (zero\ trace) \qquad 7$$

In this case the current density **J** in the vicinity of a null point is given by

$$\mu_0 \boldsymbol{J} = \begin{bmatrix} a_{32} \\ -a_{31} \\ a_0 \end{bmatrix} \qquad 8$$

Where μ_0 is the magnetic permeability in vacuum and a_0 is defined as

$$a_0 \left(\equiv \mu_0 J_z\right) = a_{21} - a_{12}$$

The characteristic equation of M^* , $f_a{}^*(\lambda)$, becomes

$$f_a{}^*(\lambda) = f_A(\lambda).(\lambda - a_{33}) \qquad 9$$

Where

$$f_A(\lambda) = \lambda^2 + \lambda a_{33} - X$$

In terms of X define as

$$X = a_{12}a_{21} - a_{11}a_{22}.$$

The eigenvalues of M^* are the root of equation (9). Let the two roots of $f_A(\lambda)$ be α and β, which are not necessarily real (not that $\alpha+\beta+a_{33} = 0$). As already mentioned, three linearly independent vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$ and $\boldsymbol{k}_3$are chosen such that $\boldsymbol{k}_3$ is the eigenvector corresponding to a_{33}. It may be quite sufficient, if only we illustrate magnetic field lines in the orthogonal Cartesian system, since the affine transformation is always possible. Hereafter $\boldsymbol{k}_1$and$\boldsymbol{k}_2$ aare respectively projected onto the x and y-axis, and all figures are illustrated in the xyplane.

3.1. Real eigenvalues

Both α and β are real if

$$4X + a^2{}_{33} \geq 0.$$

In this case the roots of (9) can be classified into the following three cases.

3.1.1. The case of $\boldsymbol{\alpha = \beta = a_{33}(= 0)}$.

This is satisfied if

$$X = a_{33} = 0. \qquad 10$$

There are three cases according to the dimensions of the eigenspaceW_0.

(i) When **J**=0 (i.e., $a_{31} = a_{32} = a_0 = 0$), it is readily shown from (7) and (10) that M=0 and, therefore, the dim W_0=0, which means no field in the whole space.

(ii) If $\boldsymbol{J} \neq 0$ and both $\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = 0$ and $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = 0$ are simultaneously satisfied, dim W_0=2. Either $J_z = 0$ or $J_x = J_y = 0$ satisfies these conditions. In this case W_0 becomes a plane including the z-axis (e. g., $a_{31}x + a_{32}y = 0$ if $|a_{31}|$+$|a_{32}| \neq 0$). Then there exist three vectors $\boldsymbol{h}_3, \boldsymbol{h}_2and\boldsymbol{h}_1$each of which satisfies$\boldsymbol{M}^*\boldsymbol{h}_3 = 0\boldsymbol{h}_3, \boldsymbol{M}^*\boldsymbol{h}_2 = 0\boldsymbol{h}_2$($\boldsymbol{h}_3$ and $\boldsymbol{h}_2$ are in the plane W_0) and $\boldsymbol{M}^*\boldsymbol{h}_1 = \boldsymbol{h}_2$ (if $J_z = 0$, $\boldsymbol{h}_3$ is taken instead of $\boldsymbol{h}_3$) respectively. Therefore the general solution of (5) is expressed as

$$r(\tau) = C_3h_3 + C_2h_2 + C_1(h_1\tau + h_1) = C_1h_1 + \xi_2h_2 + C_3h_3,$$

Where $\xi_2 = C_1\tau + C_2$ and $C_i{}'s$ $(i = 1, 2, 3)$ are arbitrary (real) constants. Therefore phase trajectories describe a pair of antiparallel magnetic field lines directed to h_2. The plane $W_0$is the so-called magnetic neutral sheet where the magnetic field vanishes.

(iii) For the other values of $\boldsymbol{M}^*$, dim W_0 =1. In this case both $J_z \neq 0$ or $|J_x| + |J_y| \neq 0$ must hold. W_0 is the z-axis this was studied by Al-hachami et al.[*]. Then there exist three vectors $\boldsymbol{h}_3, \boldsymbol{h}_2and\boldsymbol{h}_1$each of which satisfies $\boldsymbol{M}^*\boldsymbol{h}_3 = 0\boldsymbol{h}_3, \boldsymbol{M}^*\boldsymbol{h}_2 = 0\boldsymbol{h}_2$ and $\boldsymbol{M}^*\boldsymbol{h}_1 = \boldsymbol{h}_2$ respectively. Therefore the magnetic field lines are written by

$$r(\tau) = C_3h_3 + C_2(h_3\tau + h_2) + C_1(h_1\frac{\tau^2}{2} + \tau h_2 + h_1) = C_1h_1 + \xi_2h_2 + \xi_3h_3,$$

Where $\xi_2 = C_1\tau + C_2$ and $\xi_3 = (C_1/2)\tau^2 + C_2\tau + C_3$. This shows that W_0 is the magnetic null line along which the magnetic field vanishes. In the plane defined by $\boldsymbol{h}_2$ and $\boldsymbol{h}_3$ **(**i.e.,$\boldsymbol{C}_1 = 0$**)** a pair of antiparallel fields are formed in the direction of $\boldsymbol{h}_3$. in the planes parallel to it the magnetic field lines are parabolic and formed by superposition of two sets of antiparallel fields shown in (ii).It is apparent that the field lines mentioned here are due to the current near the null point; there remains no field when $\boldsymbol{J} \to 0$.

3.1.2. The case of two equal eigenvalues.

This is the case, either if

$$X - 2a^2{}_{33} = 0 \quad (a_{33} \neq 0),$$

or if

$$4X - 2a^2{}_{33} = 0 \quad (a_{33} \neq 0).$$

The former condition leads to α(or β)=a_{33}, and latter, $\alpha = \beta$(=-$a_{33}/2$). The formerwe consider only the latter here. Let's denote the eigenspace corresponding to the eigenvalue , $\alpha = \beta$(=-$a_{33}/2$) by W_α. Then the following two cases should be considered according to the dimension of W_α. (i) If $J_z = 0$, dim $W_\alpha = 2$ and W_α becomes a plane given by $2a_{31}x + 2a_{32}y + 2a_{33}z = 0$. When $J_x = J_y = 0$, W_α coincides with xy-plane. The three vectors $\boldsymbol{h}_3, \boldsymbol{h}_2$ and $\boldsymbol{h}_1$ can be chosen such that $\boldsymbol{M}^*\boldsymbol{h}_3 = a_{33}\boldsymbol{h}_3$, $\boldsymbol{M}^*\boldsymbol{h}_2 = -(a_{33}/2)\boldsymbol{h}_2$ and that$\boldsymbol{M}^*\boldsymbol{h}_1 = -(a_{33}/2)\boldsymbol{h}_2$. Therefore the general solution of (5) is given by

$$r(\tau) = C_3h_3e^{a_{33}\tau} + C_2h_2e^{(-\frac{a_{33}}{2}\tau)} + C_1h_1e^{(-\frac{a_{33}}{2}\tau)}.$$

In the W_α plane all field lines are straight and go towards the null point if $a_{33} > 0$ (see Figure 1a), while outwards if $a_{33} < 0$. (ii) If $J_z \neq 0$, dim $W_\alpha = 1$. W_α is a straight line given by the intersection of the two planes, i. e., $(2a_{11} + a_{11})x + 2a_{12}y = 0$. When $J_x = J_y = 0$, W_α coincides with xy-plane. The three vectors $\boldsymbol{h}_3, \boldsymbol{h}_2$ and $\boldsymbol{h}_1$ can be chosen such that $\boldsymbol{M}^*\boldsymbol{h}_3 = -\left(\frac{a_{33}}{2}\right)\boldsymbol{h}_3$, $\boldsymbol{M}^*\boldsymbol{h}_2 = -\left(\frac{a_{33}}{2}\right)\boldsymbol{h}_2 + \boldsymbol{h}_1$ and that $\boldsymbol{M}^*\boldsymbol{h}_3 = \boldsymbol{a}_{33}\boldsymbol{h}_3$. Therefore the general solution of (5) is given by

$$r(\tau) = C_1h_1e^{\left(-\frac{a_{33}}{2}\tau\right)} + C_2(h_1\tau + h_2)e^{\left(-\frac{a_{33}}{2}\tau\right)} + C_3h_3e^{a_{33}\tau}$$

$$= \xi_1h_1e^{(-\frac{a_{33}}{2}\tau)} + C_2h_2 + h_2e^{(-\frac{a_{33}}{2}\tau)} + C_3h_3e^{a_{33}\tau}$$

where $\xi_1 = C_1 + C_2\tau$. The field lines in the plane defined by $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are shown in Figure 1b. when $\boldsymbol{J} \to 0$, there remain such magnetic fields as shown in (i), which is referred to some external currents.

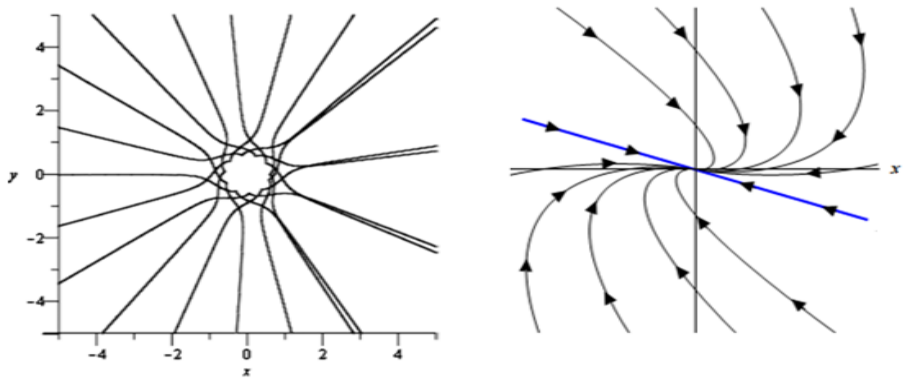

Figure 1: **Field lines for $= \beta < 0$. (a) dim $W_\alpha = 2$ and (b) dim $W_\alpha = 1$.**

3.1.3.The case of three different eigenvalues.

This is the case if

$$4X - a^2{}_{33} > 0 \quad (2a^2{}_{33} - X \neq 0).$$

Let's denote the eigenspaces corresponding to the eigenvalues α, β and a_{33} respectively by W_α, W_β and $W_{a_{33}}$(the z-axis). Both W_α and W_β are also straight lines and W_α, for instance, is given by the line of intersection between the planes$(a_{11} - \alpha)x + a_{12}y = 0$ and $a_{31}x + a_{32}y + (a_{33} - \alpha)z = 0$, if $|a_{11} - \alpha| + |a_{12}| \neq 0$. Otherwise $a_{21}x + (a_{32} - \alpha)y = 0$ is taken instead of the former plane.W_β is similarly determined. When $J_x = J_y = 0$, both W_α and W_β are in thexy-plane. Then the solution of (5) is expressed as

$$r(\tau) = C_1 h_1 e^{(\alpha\tau)} + C_2 h_2 e^{(\beta\tau)} + C_3 h_3 e^{a_{33}\tau}.$$

Three eigenspaces form the principal axes, i.e., field lines passing the null point. In the plane defined by $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$, a node is formed at the null point for $\alpha\,\beta > 0$. all field lines are directed towards the null point for $\alpha, \beta < 0$ (Figure 2a) andoutwards for $\alpha, \beta > 0$. A saddle, on the other hand, appears for $\alpha\,\beta < 0$ (Figure 2b), this was considered by Al-hachami et al.(2010)[1].

It may be worthwhile to note that, when $\boldsymbol{J} = 0$, the principal axes are orthogonal to each other (since M^* becomes symmetric). The special case that one eigenvalue vanishes will be discussed latter.

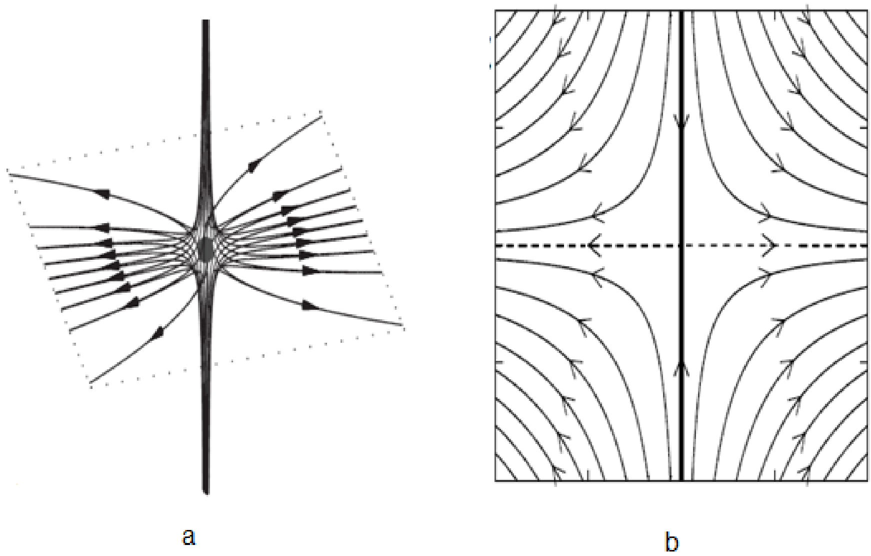

Figure.(2):Field lines for (a) $\alpha < \beta < 0$ and (b) $\alpha < 0 < \beta$.

3.2 comlex eigenvalues.

Both α and β are complex if

$$4X - a^2{}_{33} < 0.$$

The eigenvalues α and β may be written as

$\alpha = \mu + iv$ and $\beta = \bar{\alpha} = \mu - iv$ $\left(\mu = -\frac{a_{33}}{2}\right)$,

And corresponding eigenvectors can be chosen such that they are conjugate with each other, i.e., $\boldsymbol{h}_1$ and $\bar{\boldsymbol{h}}$. Then the general solution of (5) is expressed as

$$r(\tau) = C\boldsymbol{h}e^{(\alpha\tau)} + \overline{C\boldsymbol{h}}e^{(\bar{\alpha}\tau)} + C_3h_3e^{a_{33}\tau}, \qquad 11$$

Where C is an arbitrary complex number. We put

$$h = \frac{1}{2}(\boldsymbol{h}_1 - i\boldsymbol{h}_2),$$

where $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are in xy-plane if $J_x = J_y = 0$. Since $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are linearly independent, they compose the bases of the phase space together with $\boldsymbol{h}_3$. Putting

$$\xi(\tau) = \xi_1 + i\xi_3 = Ce^{\alpha\tau}, \qquad 12$$

Equation (11) is expressed as

$$r(\tau) = \xi_1\boldsymbol{h}_1 + \xi_2\boldsymbol{h}_2 + C_3h_3e^{a_{33}\tau},$$

If $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are projected onto (5) and i, respectively, the vector $\xi_1\boldsymbol{h}_1 + \xi_2\boldsymbol{h}_2$ corresponds to the complex number given by (12). Putting $C = Re^{i\theta}$, (12) is expressed as

$$\xi(\tau) = Re^{\mu\tau}.e^{i(\nu\tau+\theta)}.$$

Magnetic field lines in the plane defined by $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ therefore, become logarithmic spirals. They are inwards when $\mu < 0(a_{33} > 0)$, while outwards when$\mu < 0(a_{33} < 0)$. in the case of $\mu = 0(a_{33} = 0)$, each field line becomes a closed trajectory, forming a so-called center of spiral (therefore the z-axis is the magnetic null line). In the limit$\boldsymbol{J} \to 0$, the closed fileld lines tend to vanish, which means that these field lines are induced by the current near null, This was considered by Pontin et al.[8].Hence the spiral field ($a_{33} \neq 0$) are formed by the pointsuperposition of the closed field lines upon those due to the external source currents as shown in Figure 1a.

4. Conclusion

The classification of magnetic null points was stractully re-examind in terms of the phase trajectries. It may be the extension and generlisation of Dungey's work but the probable existence of some types of null points-spiral and node-was pointed out first. The result indicates that the ordinary two dimensional problem is only a very limited case. The magnetic volume force acting on the fluid may be discribed by the same topological analysis.

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