Discrete ADM for Solving Fisher's equation

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في هذا البحث ، نستخدم أسلوبًا نسبيًا يسمى طريقة Adomian التحليلية المتقطعة (DADM) لبعض حالات معادلة فيشر التي تم حلها مسبقًا بحل دقيق بواسطة طريقة Homotopy المتخلخلة ، ثم قمنا بمقارنة النتائج .

Abstract

In this paper, the discrete Adomian decomposition method (DADM) is used for solving Fisher's equation . we solved some cases of Fisher's equation bt the proposed method, then we compared the obtained results with exact solution which found by the Homotopy perturbation.

Keywords: Discrete Adomian decomposition method, Partial Differential Equations; Fisher's Equations; Finite difference scheme.

1. Introduction

we consider the Fisher's equation in this work,

$$u_t(x,t) - u_{xx}(x,t) - u(x,t) + u^2(x,t) = 0 , \quad t > 0, \qquad \ldots(1)$$

And the initial condition

$$u(x,0) = f(x) , \quad x_0 < x < x_1 , \qquad \ldots(2)$$

With boundary conditions

$u(x_0,t) = f_1(t)$, $u(x,t) = f_2(t)$, $t > 0$, ...(3)

$u_t \rightarrow$ the first time derivative of $u(x,t)$,

$u_{xx} \rightarrow$ the second time derivative of $u(x,t)$.

in Eq. (1), the density of an advantageous is denoting by u. This equation as a model for the propagation of a mutant gene and it encountered in chemical kinetics [7] and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. Moreover, the same equation occurs in logistic population growth models [5], flame propagation, neurophysiology, autocatalytic chemical reactions, and branching Brownian motion processes. In this paper, we use the discrete Adomian decomposition method to find the analytic solution of the Fisher's equation. This obtained method was first proposed by Bratsos et al. [1] applied to discrete nonlinear Schrödinger equations. the DADM are developed to 2D Burgers difference equations by Zhu et al. [3]. Al-Rozbayani et al. [2] using discrete Adomian decomposition method to solve

Burger's-Huxley Equation. In this study, the DADM is tested on Fisher's equations.

The remaining part of the paper is structured as following:

In section 2, we review the DADM. Section 3 is on the application of the DADM to three cases of Fisher's equation to show its efficiency. the conclude of this paper in Section 4.

2. Discrete Adomian Decomposition Method(DADM)

we formulate the following fully implicit scheme to apply the DADM to Equation (1) with initial condition (2) :

$$u_t(x, t) - u_{xx}(x, t) - u(x, t) + u^2(x, t) = 0 \; ; \; t > 0,$$

$$\frac{١}{\tau}\left(u_i^{n+١} - u_i^n\right) - \frac{١}{h^٢}\left(u_{i+١}^{n+١} - ٢u_i^{n+١} + u_{i-١}^{n+١}\right) - u_i^{n+١} + \left(u_i^{n+١}\right)^٢ = ٠ \qquad ....(4)$$

the discrete approximation of u (x , t) is denoted by u_i^n at the grid point (ih , $n\tau$) (i = 0,1,2, … , N ; n = 0,1,2,…) , where τ represents time increment and h=1/N is the spatial step size .

the operator form of the above scheme is written as the follows

$$D_\tau^+ u_i^{n+١} - D_h^٢ u_i^{n+١} - u_i^{n+١} + \left(u_i^{n+١}\right)^٢ = ٠ \qquad ....(5)$$

with the initial condition $u_i^n = f_i$ ....(6)

D_τ^+ is the linear operator denoted to the forward difference approximation, i.e.,

$$D_\tau^+ u_i^n = \frac{١}{\tau}\left(u_i^{n+١} - u_i^n\right) \qquad ....(٧)$$

The second order central difference approximations, denoted by $D_h^٢$ is given by

$$D_h^٢ = \frac{١}{h^٢}\left(u_{i+١}^{n+١} - ٢u_i^{n+١} + u_{i-١}^{n+١}\right) \qquad ....(8)$$

$(D_\tau^+)^{-١}$ is the inverse discrete operator, given by [4]

$(D_\tau^+)^{-1} w^n = \tau \sum_{m=0}^{n-1} w^m$,

….(9) Using the above definition, we get

$(D_\tau^+)^{-1} D_\tau^+ u_i^n = u_i^n - u_i^0$

….(10)

Applying $(D_\tau^+)^{-1} u_i^n$ to Eq.(5) we get

$u_i^n = u_i^0 + (D_\tau^+)^{-1} D_h^2 u_i^{n+1} + (D_\tau^+)^{-1} u_i^{n+1} - (D_\tau^+)^{-1} \left(u_i^{n+1} \right)^2$

….(11)

the discrete approximation u_i^n can be decomposed into a sum of components by the series $u_i^n = \sum_{k=0}^{\infty} u_{i,k}^n$

….(12)

The nonlinear term $N u_i^{n+1}$ are related to the nonlinear terms and can be decomposed by the infinite series as follows

$$N u_i^{n+1} = \sum_{k=0}^{\infty} A_k$$

….(13)

Where A_k are the so-called Adomian polynomials that can be generated according to the following algorithms

$$A_k = \frac{١}{k!}\left[\frac{d^k}{d\lambda^k} N \sum_{l=٠}^{\infty}(\lambda^l u_{i,l}^{n+١})\right]_{\lambda=٠}\ , k \geq ٠ \qquad ...(١٤)$$

Substituting (12) and (13) into (11) yields

$$\sum_{k=٠}^{\infty} u_{i,k}^{n} = f_i + (D_\tau^+)^{-١} D_h^٢ \left(\sum_{k=٠}^{\infty} u_{i,k}^{n+١}\right) + (D_\tau^+)^{-١} \sum_{k=٠}^{\infty} u_{i,k}^{n+١} - (D_\tau^+)^{-١} \sum_{k=٠}^{\infty} A_k \qquad ...(١٥)$$

the recurrence relation that gives Each term of series (12), are given below

$$u_{i,٠}^{n} = f_i \qquad ....(16)$$

$$u_{i,k+١}^{n} = (D_\tau^+)^{-١} D_h^٢ u_{i,k}^{n+١} + (D_\tau^+)^{-١} u_{i,k}^{n+١} - (D_\tau^+)^{-١} A_k \qquad ....(17)$$

So, the practical solution for the *l−term* approximation and the exact solution are

$$\varphi_l = \sum_{l=٠}^{l-١} u_{i,k}^{n}\ , \quad \geq ١ \ , \qquad ....(١٨)$$

$$u_i^n = \lim_{l \to \infty} \varphi_l \qquad ....(19)$$

3. Application of DADM

In this section, we will be applying the DADM to the Fisher's equations with three test examples to measure the accuracy of solutions and verify the efficiency in comparison with the exact solution.

Example 1: in this case the Fisher's equation take the form **[4]** , **[6]**

$$u_t(x, t) - u_{xx}(x, t) - u(x, t)(1 - u(x, t)) = 0,$$

subject to the initial conditions $u(x, 0) = \beta$.

Applying the DADM of this problem The first few terms of the Adomian polynomial A_n can be obtained from the (14) equation:

$$A_0 = \left(u_{i,0}^n\right)^2$$

$$A_1 = 2\, u_{i,0}^n\, u_{i,1}^n$$

$$A_2 = \left(u_{i,1}^n\right)^2 + 2\, u_{i,0}^n\, u_{i,2}^n$$

$$A_3 = 2\, u_{i,1}^n\, u_{i,2}^n + 2\, u_{i,0}^n\, u_{i,3}^n$$

According to Eqs. (16) and (17), one can obtain:

$$u^{n}_{i,0} = f_i = \beta$$

$$u^{n}_{i,1} = n\,\tau\left(\beta - \beta^2\right)$$

$$u^{n}_{i,2} = \left(-\beta^2 + \beta\right) n\tau^2\, \beta(\beta - 1)(n + 1)\left(\beta - \frac{1}{2}\right);$$

$$u^{n}_{i,3} = \left(-\beta^2 + \beta\right)\left(n^2 + n\right)\frac{\tau^3}{3}\,\beta(\beta - 1)\left(n^2 + 2n\right)\left(\beta - \frac{1}{2}\right) - \left(n^3 + 2n^2 + n\right)\frac{\tau^3}{3}\left(-\beta^2 + \beta\right)^3 - 2\beta\left(-\beta^2 + \beta\right)\left(n^2 + n\right)\frac{\tau^3}{3}\,\beta(\beta - 1)\left(n^2 + 2n\right)\left(\beta - \frac{1}{2}\right)$$

Since, $u(x,t) = \sum_{k=0}^{\infty} u^{n}_{i,k} = u^{n}_{i,0} + u^{n}_{i,1} + u^{n}_{i,2} + \cdots$, Then

$$u(x,t) = \beta + n\tau\left(-\beta^2 + \beta\right) + \frac{\tau^2}{2}\left[\left(n^2 + n\right)\left(-\beta^2 + \beta\right) - 2\beta\left(n^2 + n\right)\right] + \frac{\tau^3}{3}\left[\left(-\beta^2 + \beta\right)\left(n^2 + n\right)\beta(\beta - 1)\left(n^2 + 2n\right)\left(\beta - \frac{1}{2}\right) - \left(n^3 + 2n^2 + n\right)\left(-\beta^2 + \beta\right)^3 - 2\beta\left(-\beta^2 + \beta\right)\left(n^2 + n\right)\beta(\beta - 1)\left(n^2 + 2n\right)\left(\beta - \frac{1}{2}\right)\right] + \ldots\ldots$$

Now, from comparing the results obtained in [5], [6] and our results where the exact solution in [5], is that implies

$$u(x,t) = \frac{\beta \exp t}{1 - \beta + \beta \exp t}$$

Example 2 : in second case the Fisher's equation take the form **[4]** , **[6]**

$$u_t = u_{xx} + u(1 - u^6),$$

with initial condition $u(x, ٠) = \frac{١}{\sqrt[٣]{١+e^{\frac{٣}{٢}x}}}$

Applying the DADM of this problem the first three components of Adomian polynomials read According to Eqs. (19), one can obtain:

$$A_0 = \left(u^n_{i,\,0}\right)^7$$

$$A_1 = 7\left(u^n_{i,\,0}\right)^6 u^n_{i,\,1}$$

$$A_2 = 21\left(u^n_{i,\,0}\right)^5 \left(u^n_{i,\,1}\right)^2 + 7\left(u^n_{i,\,0}\right)^6 u^n_{i,\,2}$$

$$A_3 = 35\left(u^n_{i,\,0}\right)^4 \left(u^n_{i,\,1}\right)^3 + 42\left(u^n_{i,\,0}\right)^5 u^n_{i,\,1}\, u^n_{i,\,2} + 7\left(u^n_{i,\,0}\right)^6 u^n_{i,\,3}$$

According to Eqs. (16) and (17), one can obtain:

$$u^n_{i,\,0} = f_i = \left(\frac{1}{\left(1 + e^{\frac{3}{2}ih}\right)^{1/3}}\right)$$

$$u_{i,1}^{n} = n\tau\left(\frac{1}{h^2}\left(\frac{1}{\left(1+e^{\frac{3}{2}(i+1)h}\right)^{1/3}} - \frac{2}{\left(1+e^{\frac{3}{2}ih}\right)^{1/3}} + \frac{1}{\left(1+e^{\frac{3}{2}(i-1)h}\right)^{1/3}}\right) + \frac{1}{\left(1+e^{\frac{3}{2}ih}\right)^{1/3}} - \frac{1}{\left(1+e^{\frac{3}{2}ih}\right)^{7/3}}\right)$$

$n \quad 1 \; (\; 2 \; + \;) \; ^{2}(\; 1 \; (\; 1 \; (\qquad 1$

we are not listed other components here. by using the 4−term approximations, we which evaluated the final numerical solutions and given below :

Since, $u(x,t) = \sum_{k=٠}^{\infty} u_{i,k}^{n} = u_{i,٠}^{n} + u_{i,١}^{n} + u_{i,٢}^{n} + u_{i,٣}^{n} \ldots$

that implies to the exact solution which given by

$$u(x,t) = \sqrt[٢]{\left(\frac{١}{٢}\tanh\left[\frac{-٣}{٤}\left(x - \frac{٥}{٢}t\right)\right] + \frac{١}{٢}\right)}$$

Example 3 : in third case the Fisher's equation take the form **[4]** , **[6]**

$$u_t - u_{xx} - 6u(1 - u) = 0,$$

the initial conditions $u(x,٠) = \frac{١}{(١+e^{x})^{٢}}$

$$u_{i,0}^{n} = f_i = \frac{1}{\left(1 + e^{ih}\right)^2}$$

$$u_{i,1}^{n} = n\tau\left(\frac{1}{h^2}\left(\frac{1}{\left(1 + e^{(i+1)h}\right)^2} - \frac{2}{\left(1 + e^{ih}\right)^2} + \frac{1}{\left(1 + e^{(i-1)h}\right)^2}\right) + \frac{6}{\left(1 + e^{ih}\right)^2} - \frac{6}{\left(1 + e^{ih}\right)^4}\right)$$

$$
\begin{aligned}
u_{i,2}^{n} = \frac{\left(n^2+n\right)\tau^2}{2}\Bigg(&\frac{1}{h^2}\cdot\Bigg(\frac{1}{h^2}\Bigg(\frac{1}{\left(1+e^{(i+2)h}\right)^2} \\
&-\frac{2}{\left(1+e^{(i+1)h}\right)^2}+\frac{1}{\left(1+e^{ih}\right)^2}\Bigg)+\frac{6}{\left(1+e^{(i+1)h}\right)^2} \\
&-\frac{6}{\left(1+e^{(i+1)h}\right)^4}-2\Bigg(\frac{1}{h^2}\Bigg(\frac{1}{\left(1+e^{(i+1)h}\right)^2}-\frac{2}{\left(1+e^{ih}\right)^2} \\
&+\frac{1}{\left(1+e^{(i-1)h}\right)^2}\Bigg)+\frac{6}{\left(1+e^{ih}\right)^2}-\frac{6}{\left(1+e^{ih}\right)^4}\Bigg) \\
&+\frac{1}{h^2}\Bigg(\frac{1}{\left(1+e^{ih}\right)^2}-\frac{2}{\left(1+e^{(i-1)h}\right)^2}+\frac{1}{\left(1+e^{(i-2)h}\right)^2}\Bigg) \\
&+\frac{6}{\left(1+e^{(i-1)h}\right)^2}-\frac{6}{\left(1+e^{(i-1)h}\right)^4}\Bigg) \\
&+6\Bigg(\frac{1}{h^2}\Bigg(\frac{1}{\left(1+e^{(i+1)h}\right)^2}-\frac{2}{\left(1+e^{ih}\right)^2} \\
&+\frac{1}{\left(1+e^{(i-1)h}\right)^2}\Bigg)+\frac{6}{\left(1+e^{ih}\right)^2}-\frac{6}{\left(1+e^{ih}\right)^4}\Bigg) \\
&-12\Bigg(\frac{1}{\left(1+e^{ih}\right)^2}\Bigg(\frac{1}{h^2}\Bigg(\frac{1}{\left(1+e^{(i+1)h}\right)^2}-\frac{2}{\left(1+e^{ih}\right)^2} \\
&+\frac{1}{\left(1+e^{(i-1)h}\right)^2}\Bigg)+\frac{6}{\left(1+e^{ih}\right)^2}-\frac{6}{\left(1+e^{ih}\right)^4}\Bigg)\Bigg)\Bigg)
\end{aligned}
$$

We can be compute more components in the decomposition series to enhance the approximation. the solutions are obtained in series form as

$$
\begin{aligned}
u(x,t) = \frac{1}{\left(1+e^{ih}\right)^4}+n\tau\Bigg(&\frac{1}{h^2}\Bigg(\frac{1}{\left(1+e^{(i+1)h}\right)^2}-\frac{2}{\left(1+e^{ih}\right)^2} \\
&+\frac{1}{\left(1+e^{(i-1)h}\right)^2}\Bigg)+\frac{6}{\left(1+e^{ih}\right)^2}-\frac{6}{\left(1+e^{ih}\right)^4}\Bigg)+\ldots\ldots
\end{aligned}
$$

That's will be gives the exact solution which take the form

$$u(x,t) = \frac{١}{(١+\exp(x-٥□))^{٢}}$$

4. Conclusions

In our work, we obtained by the DADM to fund an approximate solution for Fisher,s equation is where the exact solution is known by the HPM, we concluded that the DADM one of the best methods to find the approximate solution because it gives a better results to the exact solution.

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