Partitions by subgeometries of projective plane over a galois field of order 3^n , $n = 1 , 2 , 3$

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تجزئات بواسطة ألهندسة ألجزئية للمستوي ألإسقاطي على حقل كالوز من ألرتبة 3^n

$n = 1 , 2 , 3$

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المستخلص

الغرض من هذا البحث هو لتجزئة المستوي الاسقاطي من الرتبة التاسعة $PG(2,9)$ الى سبع مستويات جزئية $PG(2,3)$ اسقاطية منفصلة والى ثلاثة عشر من الاقواس الكاملة من الدرجة الثانية والحجم السابع المنفصلة. الخط الاسقاطي من الرتبة السابعة والعشرين $PG(1,27)$ تم تجزئته الى سبع خطوط جزئية $PG(1,3)$ منفصلة وعدد المجموعات من الرتبة الرابعة المختلفة تم تصنيفها . تأثير الزمر على الخطوط الاسقاطية $PG(1,3^n)$ والمستويات الاسقاطية $PG(2,3^n)$, $n = 1 ,2 ,3$ هي فسرت وقمنا بتقديم نظريات وامثلة والفضاءات الجزئية من $PG(1,3^n)$, $PG(2,3^n)$, $n = 1 ,2 ,3,$ تم توضيحها . كل من هذه التجزئات تعطي ارتفاع لتصحيح الخطاء الرمز والذي يصحح اكبر عدد ممكن من الأخطاء لأطوالها. تم تطبيق نظام حسابات برمجة خوارزميات الزمر GAP لحساب الجبر المنتهي.

ألكلمات ألمفتاحية : ألخط ألإسقاطي , ألمستوي ألإسقاطي , تأثير ألزمر, ألأقواس ألكاملة

Abstract

The purpose of this thesis is to split the projective plane of order nine $PG(2,9)$ into seven disjoint projective subplanes $PG(2,3)$ and thirteen disjoint complete arcs of degree two and size seven. The projective line of order twenty-seven $PG(1,27)$ has been partitioned into seven disjoint projective sublines $PG(1,3)$ and the number of inequivalent 4-sets which are unordered sets of four points is classified. The group action on projective lines

$PG(1, 3^n)$ and projective planes $PG(2, 3^n), n = 1 ,2 ,3$ is explained and we have introduced theorems and examples and the subspaces of $PG(1, 3^n)$ and $PG(2, 3^n), n = 1 ,2 ,3$ are shown. Each of these partitions gives rise to an error-correcting code that corrects the maximum possible number of errors for its length. GAP-Groups–Algorithms, Programming a system for computational discrete algebra has been applied.

Key words: Projective line, projective plane, groups' action, complete arcs.

Introduction

The basic concepts of this study depends on the subjects of Projective Geometry, Group Theory and Vector Space over Galois Field $F_q , q = 3^n , n = 1 ,2 ,3$. The following three notions are equivalent for $k \geq 1$:

1. An $(k ; 2)$-arc in $\mathrm{PG}(2, q)$, that is , a set of k points with at most 2 in any hyperplane ;
2. A set of k vectors in $\mathrm{V}(3, q)$ with any 3 linearly independent ;
3. A maximum distance separable linear code of length k , dimension 3 , and hence minimum distance $d = k - 2$, that is, $[k , 3 , k - 2]$ code.

The Projective space $\mathrm{PG}(k - 1, q)$ over a finite field F_q contains

- $(q^k - 1)/(q - 1)$ points , this is the number of 1- dimensional subspaces in $\mathrm{V}(k, q)$;
- $(q^k - 1)(q^{k-1} - 1)/(q^2 - 1)(q - 1))$ lines, this is the number of 2- dimensional subspaces in $\mathrm{V}(k, q)$;
- $q + 1$ points on a line, this is the number of 1- dimensional subspaces in a 2- dimensional subspace;
- $(q^{k-1} - 1)/(q - 1)$ Lines through a point, this is the number of 2- dimensional subspaces through a 1- dimensional subspace.

The points of projective plane $\mathrm{PG}(2, q)$ are $(x , y , z) \neq (0 ,0 ,0)$ where $(\lambda x , \lambda y , \lambda z) = (x , y , z)$ and the lines of PG $(2, q)$ are $uX + vY + wZ = \{[x , y , z] | ux + vy + wz = 0\}$. A Projective plane satisfying the following four axioms.

1. Any two distinct points lie on one and only one line.
2. Any two lines meet in at least one point.
3. There exist three non-collinear points, such that a set of points is said to be collinear if there exists a line containing them all.
4. Every line contains at least three points.

The brief history of this subject is given as follows: In 1976, Hirshfeld (1), (2) partitions PG (2 ,4) into three disjoint PG(2 ,2), also he splits PG(3 ,2) into 15 disjoint PG (2 ,2) . In 2008, Almuktar (3) shows that PG (2 ,5) is embedded in PG (3 ,5) . In 2011, Al–Seraji (4) partitions PG (2 ,16) into disjoint projective subplanes PG (2 ,2) and PG(2 ,4). The sets in projective line and projective plane of orders 2,3,4,5,7,8,9,11,13 have been described (1). In 2010, Al-Seraji (5) classifys the sets in projective line and projective plane of order 17. In 2011, Al-Zangana (6) shows the sets in projective line and plane of order 19. In 2014, Al-Seraji (7) explains the sets in PG$(1,16)$ and In 2015 he classfys the subsets in PG$(1,23)$ for more details see(8).we are looking at to partition the projective plane of order nine and the projective line of order twenty-seven and studying the group action of them as the next in the sequence.

The following definitions are interesting to our subject.

Definition 1: (9) An$(n\,;r)$-arc K or arc of degree r in PG$(K\,,q)$ with $n \geq r+1$ is aset of points with property that every hyperplane meets K in at most r points of K and there is some hyperplane meeting K in exactly r points. An $(n\,;2)$-arc is also called an r-arc.

Definition 2 : (9) An $(n\,;\,r)$ -arc is complete if it is maximal with respect to inclusion; that is, it is not contained in an $(n+1\,;r)$-arc.

Theorem 3: (5) **(The fundamental theorem of projective geometry)** If $\{\,P_0\,,\dots,P_{n+1}\}$ and $\{\,P'_0,\dots,P'_{n+1}\}$ are both subsets of PG$(n\,,q)$ of cardinality $n+2$ such that no $n+1$ points chosen from the same set lie in a hyperplane , then there exists a unique projectivity $\mathfrak{J}$ such that $P'_i = P_i\,\mathfrak{J}$ for $i = 0\,,1\,,\dots,n+1$.

Definition 4: (7) Let S and S^* be two spaces of PG $(n\,,K)$, A projectivity $\beta\colon S \to S^*$ is a bijection given by a matrix T, necessarily nonsingular, where $P(X^*) = P(X)\beta$ if $tX^* = XT$, with $t\epsilon K-\{0\}$. write $\beta = M(T)$; then $\beta = M(\lambda\mathrm{T})$ for any λ in $K-\{0\}$. The group of projectivities of PG$(n\,,K)$ is denoted by PGL$(n+1\,,K)$.

Definition 5: (1) A group G acts on a set Λ if there is a map $\Lambda \times G \to \Lambda$ such that given $g\,,\acute{g}$ elements in G and 1 is identity, then

1. $x1 = x$,
2. $(xg)\acute{g} = x(g\acute{g})$ for any x in Λ .

Definition 6: (1) The orbit of x in Λ under the action of G is the set

$$xG = \{xg \mid g\epsilon G\}\,.$$

Definition 7: (1) The stabilizer of x in Λ under the action of G is the group

$$G_x = \{g\epsilon G \mid xg = x\}\,.$$

Definition 8: (10) Let the group G act on the set Λ .

1. If $y = xg$, for x , y ϵ Λ , then
   - $yG = xG$;
   - $\mathrm{G_y} = g^{-1}G_x g$.
2. $|G_x| = \frac{|G|}{|xG|}$

Definition 9: (1) The action of G on Λ is transitive if given any two elements x , y in Λ there exists g in G such that $y = xg$. In that case, there is only one orbit. The action is regular if it is transitive and $G_x = \{1\}$ for all x in Λ .

Definition 10: (6) (Primitive and Subprimitive Polynomial)
Let $F(X) = X^n - a_{n-1}X^{n-1} - \cdots a_0$ be a monic polynomial of degree $n \geq 1$ over F_q . Let F be irreducible over F_q and $\alpha\epsilon F_{q^n}$ be root of F .

- It is called primitive if the smallest power s of α such that $\alpha^s = 1$ is $(q^n - 1)$; that is , α a primitive root over F_{q^n} .

- It is called subprimitive if the smallest power s of α such that $\alpha^s \epsilon F_q$ is $\theta(n-1\,,q) = (q^n-1)/(q-1)$.

The main results

In this section, we introduce the properties of the group action and we make partitions on projective line PG $(1\,,3^n)$ and projective plane PG$(2\,,3^n)$ n $= 1\,,2\,,3$.Where PG$(n\,,q)$ is n-dimensional projective space over F_q.

The group action on projective line of order three PG$(1\,,3)$

The polynomial of degree two $F(X) = X^2 - X - 1$ is irreducible over $F_3 = \{0\,,1\,,-1 = 2\}$, since $F(t) \neq 0$ for all t in F_3 . The points of PG $(1\,,3)$ are generated by a nonsingular matrix $T = C(F) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, such that $P(i) = (1\,,0)T^i, i = \ 0\,,1\,,2\,,3$.

The action of $\langle\, T\, \rangle$ on PG(1,3) is

$$P(0) \overset{T}{\rightarrow} P(1) \overset{T}{\rightarrow} P(2) \overset{T}{\rightarrow} P(3) \overset{T}{\rightarrow} P(0)$$

Therefore, T cycles of four points and

$$\langle\, T\, \rangle = \{T\,,T^2,T^3,T^4 = I_{2\times 2}\}$$

The action of $\langle T^2 \rangle$ on PG(1,3) is given as follows:

$$A_1 = \mathrm{P}(0) \overset{T^2}{\rightarrow} \mathrm{P}(2) \overset{T^2}{\rightarrow} \mathrm{P}(0)$$

$$A_2 = \mathrm{P}(1) \overset{T^2}{\rightarrow} \mathrm{P}(3) \overset{T^2}{\rightarrow} \mathrm{P}(1)$$

Therefore, T^2cycles of two points .This gives the following conclusion .

Theorem 1: On PG(1,3) , we have

i. The set $\langle\, T\, \rangle$ together with the usual multiplication of matrices is a cyclic group of order 4 ;
ii. The set $\langle T^2 \rangle$ together with the usual multiplication of matrices is a cyclic group of order 2 ;
iii. The action of $\langle\, T \rangle$ on PG(1,3) is transitive ;
iv. The action of $\langle T^2 \rangle$ on A_i , $i = 1{,}2$ is transitive.

Example 1: The different between $V(2{,}3)$ the 2-dimensional vector space over a finite field of order three and PG (1,3) the projective line over a finite field of order three is given as follows:

$$V(2\,,3) - \{(0\,,0)\} = \begin{cases} (1\,,0), (2\,,0) \\ (0\,,1), (0\,,2) \\ (1\,,1), (2\,,2) \\ (2\,,1), (1\,,2) \end{cases}$$

In general, $|\, V(n,q)| = q^n$ and $|\, PG(n,q)| = \theta(n) \ = \frac{q^{n+1}-1}{q-1}$

Therefore, $|\, V(2{,}3)\, | = 9$ and $\theta(1) = 4$

Thus, the elements of PG(1,3) are a first column of $V(2\,,3) - \{(0\,,0)\}$.

The stabilizer group of 3-set

The stabilizer of any 3-set is the group of six projectivities found by shifting the 3-set to its six permutations. The stabilizer (it is isomorphic to S_3) of the 3-set

$A = \{\infty, 0, 1\}$, where $\infty = (1,0)$, $0 = (0,1)$, $1 = (1,1)$ is generated by two projectivities marked $\alpha = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$ and $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The action of $S_3 =< \frac{2}{x+2}, \frac{1}{x} >$on PG(1,3) is $\{\infty, 0, 1\}, \{2\}$.

Example 2: To construct of F_9 in exponential and polynomial form take $F(X) = X^2 - X - 1 = 0$, which is irreducible over F_3 with $\alpha^2 - \alpha - 1 = 0$, the elements of F_9 are $0, 1, \alpha$

$\alpha^2 = \alpha + 1$

$\alpha^3 = \alpha^2 + \alpha = 2\alpha + 1$

$\alpha^4 = 2\alpha^2 + \alpha = 2 = -1$

$\alpha^5 = -\alpha$

$\alpha^6 = -\alpha^2 = -\alpha - 1$

$\alpha^7 = -\alpha^2 - \alpha = \alpha - 1$

$\alpha^8 = -2\alpha^2 - \alpha = 1$

From definition 10, we have F is primitive over F_3 .

Example 3: The polynomial of degree two $F(X) = X^2 - X - \alpha^7$ is irreducible and primitive over F_9 . $F(0) = \alpha^3, F(1) = \alpha^3, F(\alpha) = \alpha^6, F(\alpha^2) = \alpha^7$, $F(\alpha^3) = \alpha^6, F(\alpha^4) = \alpha^5$, $F(\alpha^5) = \alpha^7$, $F(\alpha^6) = 1$, $F(\alpha^7) = 1$.

So, $F(t) \neq 0$ for all t in F_9 , therefore, F is irreducible over F_9 .

Now, let α is a primitive root over F_{81} , where $F_{81} = \{0, 1, \alpha, \alpha^2, \dots, \alpha^{79}, \alpha^{80} = 1, 3 = 0\}$ take $F(\alpha) = \alpha^2 - \alpha - \alpha^7 = 0$, $\alpha = \alpha^2 - \alpha^7$, $\alpha^{80} = (\alpha^2 - \alpha^7)^{80}$ by help the computer and (11), we have $\alpha^{80} = 1$. From definition 10, F is a primitive over F_9 .

The group action on projective line of order nine PG(1,9)

The points of PG(1,9) are generated by a nonsingular matrix $M = C(F) = \begin{pmatrix} 0 & 1 \\ \alpha^7 & 1 \end{pmatrix}$, where $F(X) = X^2 - X - \alpha^7$ such that, $P(i) = (1,0)\, M^i$, $i = 0, 1, \dots 9$.

The action of $\langle M \rangle$ on PG (1,9) with $P(i) = i$, $i = 0, 1, \dots, 9$ is given as follows:

$$0 \xrightarrow{M} 1 \xrightarrow{M} 2 \dots \xrightarrow{M} 9 \xrightarrow{M} 0$$

Therefore, M is cycles of ten points and

$$\langle M \rangle = \{M, M^2, \dots, M^9, M^{10} = I_{2\times 2}\}$$

The action of $\langle M^2 \rangle$ on PG(1,9) is given as follows:

$$B_1 = 0 \xrightarrow{M^2} 2 \dots \xrightarrow{M^2} 0$$

$$B_2 = 1 \xrightarrow{M^2} 3 \dots \xrightarrow{M^2} 1$$

Therefore, M^2 cycles of five points. The action of $\langle M^5 \rangle$ on PG(1,9) is given as follows :

$$C_1 = 0 \xrightarrow{M^5} 5 \xrightarrow{M^5} 0 ,$$

$$C_2 = 1 \xrightarrow{M^5} 6 \xrightarrow{M^5} 1 ,$$

$$C_3 = 2 \xrightarrow{M^5} 7 \xrightarrow{M^5} 2 ,$$

$$C_4 = 3 \xrightarrow{M^5} 8 \xrightarrow{M^5} 3 ,$$

$$C_5 = 4 \xrightarrow{M^5} 9 \xrightarrow{M^5} 4$$

This gives the following conclusion.

Theorem2: On PG(1,9) , we have

i. The set $\langle M \rangle$ together with the usual multiplication of matrices is a cyclic group of order 10 ;
ii. The set $\langle M^2 \rangle$ together with the usual multiplication of matrices is a cyclic group of order 5 ;
iii. The set $\langle M^5 \rangle$ together with the usual multiplication of matrices is a cyclic group of order 2 ;
iv. The action of $\langle M \rangle$ on PG (1, 9) is transitive ;
v. The action of $\langle M^2 \rangle$ on B_i, $i = 1,2$ is transitive ;
vi. The action of $\langle M^5 \rangle$ on C_i, $i = 1,2,\dots,5$ is transitive.

The action of $S_3 =< \frac{2}{x+2}, \frac{1}{x} >$on PG(1,9) is $\{\infty, 0, 1\}, \{\alpha^4\}, \{\alpha, \alpha^2, \alpha^3, \alpha^5, \alpha^6, \alpha^7\}$.

The group action on projective line of order twenty- seven PG(1,27)

The points of PG(1,27) are generated by a nonsingular matrix $Q = C(F) = \begin{pmatrix} 0 & 1 \\ \beta^6 & 1 \end{pmatrix}$,where $F(X) = X^2 - X - \beta^6$, such that $P(i) = (1,0)\, Q^i$, $i = 0,1,2, \dots 27$.

The action of $\langle Q \rangle$ on PG(1,27) is

$$P(0) \xrightarrow{Q} P(1) \xrightarrow{Q} \dots \xrightarrow{Q} P(27) \xrightarrow{Q} P(0)$$

Therefore, Q is cycles of twenty- eight point and $\langle Q \rangle = \{Q, Q^2, \dots, Q^{27}, Q^{28} = I_{2\times 2}\}$

The action of $\langle Q^2 \rangle$ on PG (1,27) with $P(i) = i$, $i = 0,1,\dots,27$ is given as follows:

$$D_1 = 0 \xrightarrow{Q^2} 2 \xrightarrow{Q^2} 4 \xrightarrow{Q^2} 6 \dots \xrightarrow{Q^2} 0$$

$$D_2 = 1 \xrightarrow{Q^2} 3 \xrightarrow{Q^2} 5 \xrightarrow{Q^2} 7 \dots \xrightarrow{Q^2} 1$$

The stabilizer group of $D_i, i = 1, 2$ is given as follows:

$$G(D_i) \cong Z_1 =< \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} >, i = 1,2.$$

The action of $\langle Q^4 \rangle$ on PG(1,27) is given as follows:

$$E_1 = 0 \xrightarrow{Q^4} 4 \dots \xrightarrow{Q^4} 0$$

$$E_2 = 1 \xrightarrow{Q^4} 5 \dots \xrightarrow{Q^4} 1$$

$$E_3 = 2 \xrightarrow{Q^4} 6 \dots \xrightarrow{Q^4} 2$$

$$E_4 = 3 \xrightarrow{Q^4} 7 \dots \xrightarrow{Q^4} 3$$

$$G(E_i) \cong D_7, \mathrm{i} = 1,2,3,4.$$

The dihedral group D_7 satisfies the following properties:

- $|D_7| = 14$;
- D_7 contains 7 matrices of order 2 ;
- D_7 contains 6 matrices of order 7 ;
- $D_7 =< a, b : a^7 = b^2 = (ab)^2 = I_2 >$;
- D_7 is a nonabelian group .

The action of $\langle Q^7 \rangle$ on PG(1,27) is given as follows:

$$F_1 = 0 \xrightarrow{Q^7} 7 \xrightarrow{Q^7} 14 \xrightarrow{Q^7} 21 \xrightarrow{Q^7} 0 ,$$

$$F_2 = 1 \xrightarrow{Q^7} 8 \xrightarrow{Q^7} 15 \xrightarrow{Q^7} 22 \xrightarrow{Q^7} 1 ,$$

$$F_3 = 2 \xrightarrow{Q^7} 9 \xrightarrow{Q^7} 16 \xrightarrow{Q^7} 23 \xrightarrow{Q^7} 2,$$

$$F_4 = 3 \xrightarrow{Q^7} 10 \xrightarrow{Q^7} 17 \xrightarrow{Q^7} 24 \xrightarrow{Q^7} 3 ,$$

$$F_5 = 4 \xrightarrow{Q^7} 11 \xrightarrow{Q^7} 18 \xrightarrow{Q^7} 25 \xrightarrow{Q^7} 4 ,$$

$$F_6 = 5 \xrightarrow{Q^7} 12 \xrightarrow{Q^7} 19 \xrightarrow{Q^7} 26 \xrightarrow{Q^7} 5 ,$$

$$F_7 = 6 \xrightarrow{Q^7} 13 \xrightarrow{Q^7} 20 \xrightarrow{Q^7} 27 \xrightarrow{Q^7} 6$$

Let λ_i be the cross-ratio of $F_i , i = 1, 2, \dots , 7$, then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \beta^{13}$.

$$G(F_i) \cong S_4, i = 1,2, \dots , 7.$$

The action of $\langle Q^{14} \rangle$ on PG (1,27) is given as follows:

$$G_1 = 0 \xrightarrow{Q^{14}} 14 \xrightarrow{Q^{14}} 0 ,$$

$$G_2 = 1 \xrightarrow{Q^{14}} 15 \xrightarrow{Q^{14}} 1 ,$$

$$G_3 = 2 \xrightarrow{Q^{14}} 16 \xrightarrow{Q^{14}} 2 ,$$

$$G_4 = 3 \xrightarrow{Q^{14}} 17 \xrightarrow{Q^{14}} 3,$$

$$G_5 = 4 \xrightarrow{Q^{14}} 18 \xrightarrow{Q^{14}} 4 ,$$

$$G_6 = 5 \xrightarrow{Q^{14}} 19 \xrightarrow{Q^{14}} 5 ,$$

$$G_7 = 6 \xrightarrow{Q^{14}} 20 \xrightarrow{Q^{14}} 6 ,$$

$$G_8 = 7 \xrightarrow{Q^{14}} 21 \xrightarrow{Q^{14}} 7,$$

$$G_9 = 8 \xrightarrow{Q^{14}} 22 \xrightarrow{Q^{14}} 8 ,$$

$$G_{10} = 9 \xrightarrow{Q^{14}} 23 \xrightarrow{Q^{14}} 9 ,$$

$$G_{11} = 10 \xrightarrow{Q^{14}} 24 \xrightarrow{Q^{14}} 10 ,$$

$$G_{12} = 11 \xrightarrow{Q^{14}} 25 \xrightarrow{Q^{14}} 11 ,$$

$$G_{13} = 12 \xrightarrow{Q^{14}} 26 \xrightarrow{Q^{14}} 12 ,$$

$$G_{14} = 13 \xrightarrow{Q^{14}} 27 \xrightarrow{Q^{14}} 13$$

$G(G_i) \cong D_{26}, i = 1,2, \dots ,14.$ see (12)

The dihedral group D_{26} satisfies the following properties:

- $|D_{26}| = 52$;
- D_{26} contains 12 matrices of order 26 ;
- D_{26} contains 12 matrices of order 13 ;
- D_{26} contains 27 matrices of order 2 ;
- $D_{26} = < a, b : a^{26} = b^2 = (ab)^2 = I_2 >$;
- D_{26} is a nonabelian group .

This gives the following conclusion.

Theorem 3: On PG(1,27) , we have

i. The set $\langle$ Q $\rangle$ together with the usual multiplication of matrices is a cyclic group of order 28 ;
ii. The set $\langle Q^2 \rangle$ together with the usual multiplication of matrices is a cyclic group of order 14 ;
iii. The set $\langle Q^4 \rangle$ together with the usual multiplication of matrices is a cyclic group of order 7 ;
iv. The set $\langle Q^7 \rangle$ together with the usual multiplication of matrices is a cyclic group of order 4 ;

v. The set $\langle Q^{14} \rangle$ together with the usual multiplication of matrices is a cyclic group of order 2 ;
vi. The action of $\langle Q \rangle$ on PG $(1, 27)$ is transitive ;
vii. The action of $\langle Q^2 \rangle$ on $D_i = 1{,}2$ is transitive ;
viii. The action of $\langle Q^4 \rangle$ on $E_i = 1\,,2\,,3\,,4$ is transitive ;
ix. The action of $\langle Q^7 \rangle$ on $F_i = 1\,,2\,,3\,,4\,,5\,,6\,,7$ is transitive ;
x. The action of $\langle Q^{14} \rangle$ on $G_i = 1\,,2\,,3\,\,,\ldots,14$ is transitive;
xi. The orbit $F_i\,, i = 1, 2, 3, 4, 5, 6, 7$ represents a subline $PG(1, 3)$ in $PG(1, 27)$;
xii. There are precisely one projectively subline $PG(1, 3)$ in $G(1, 27)$.

The action of $S_3 = <\frac{2}{x+2}\,,\frac{1}{x}>$ on $\mathrm{PG}(1{,}27) - A$ is given as follows:

$$\Gamma_1 = \{\beta\,, \beta^2, \beta^{10}, \beta^{16}, \beta^{24}, \beta^{25}\}$$

$$\Gamma_2 = \{\beta^3, \beta^4, \beta^6, \beta^{20}, \beta^{22}, \beta^{23}\}$$

$$\Gamma_3 = \{\beta^5, \beta^7, \beta^{11}, \beta^{15}, \beta^{19}, \beta^{21}\}$$

$$\Gamma_4 = \{\beta^8, \beta^9, \beta^{12}, \beta^{14}, \beta^{17}, \beta^{18}\}$$

$$\Gamma_5 = \{\beta^{13}\}$$

The partitions of PG$(1{,}27)$

The line of order twenty-seven consists of seven disjoint line of order three. According to the action of S_3 on PG $(1{,}27)$ and the cross-ratio $\lambda = (P_1 - P_3)(P_2 - P_4)/(P_1 - P_4)(P_2 - P_3)$, where $P_1\,, P_2\,, P_3\,, P_4$ in PG$(1{,}27)$, we have five types of 4-set (unordered set of four distinct points) are given as follows

1. The 4-sets of type one, when $\lambda \in \Gamma_1$.

The 4-sets and their cross-ratio are given in Table 1 as follows:

Table (1) : The 4-sets of type one

Number	The 4-set	The cross-ratio
1	$\{\infty, 0, 1, \beta\}$	$\lambda = \beta$
2	$\{\beta^2, \beta^3, \beta^4, \beta^9\}$	$\lambda = \beta$
3	$\{\beta^5, \beta^6, \beta^7, \beta^{12}\}$	$\lambda = \beta$
4	$\{\beta^8, \beta^{10}, \beta^{11}, \beta^{23}\}$	$\lambda = \beta^{24}$
5	$\{\beta^{13}, \beta^{14}, \beta^{20}, \beta^{21}\}$	$\lambda = \beta$
6	$\{\beta^{15}, \beta^{16}, \beta^{22}, \beta^{25}\}$	$\lambda = \beta^{24}$
7	$\{\beta^{17}, \beta^{18}, \beta^{19}, \beta^{24}\}$	$\lambda = \beta$

The transformations between the 4-sets in above Table are given in Table 2 as following:

Table (2): The transformations between 4-sets of type one

Number	The transformation	Projective equation
1	$1 \rightarrow 2$	$\frac{x+\beta^{15}}{\beta^{17}x+\beta^{11}}$
2	$1 \rightarrow 3$	$\frac{x+\beta^{15}}{\beta^{14}x+\beta^{8}}$
3	$1 \rightarrow 4$	$\frac{x+\beta^{9}}{\beta^{15}x+\beta^{25}}$
4	$1 \rightarrow 5$	$\frac{x+\beta^{7}}{\beta^{6}x+\beta^{12}}$
5	$1 \rightarrow 6$	$\frac{x+\beta^{15}}{\beta x+1}$
6	$1 \rightarrow 7$	$\frac{x+\beta^{15}}{\beta^{2}x+\beta^{22}}$

Therefore, on PG $(1,27)$, there are precisely one projectively 4-set of type one.

2. The 4-sets of type two, when $\lambda \epsilon \Gamma_2$.

The 4-sets and their cross-ratio are given in Table 3 as follows:

Table (3): The 4-sets of type two

Number	The 4-set	The cross-ratio
1	$\{\infty, 0, 1, \beta^3\}$	$\lambda = \beta^3$
2	$\{\beta, \beta^4, \beta^5, \beta^7\}$	$\lambda = \beta^3$
3	$\{\beta^2, \beta^6, \beta^8, \beta^{10}\}$	$\lambda = \beta^6$
4	$\{\beta^9, \beta^{12}, \beta^{13}, \beta^{20}\}$	$\lambda = \beta^{23}$
5	$\{\beta^{14}, \beta^{15}, \beta^{19}, \beta^{22}\}$	$\lambda = \beta^6$
6	$\{\beta^{23}, \beta^{18}, \beta^{21}, \beta^{25}\}$	$\lambda = \beta^{20}$
7	$\{\beta^{16}, \beta^{17}, \beta^{11}, \beta^{24}\}$	$\lambda = \beta^4$

The 4-sets in above Table are equivalent, therefore, on PG $(1,27)$, there are precisely one projectively 4-set of type two.

3. The 4-sets of type three, when $\lambda \epsilon \Gamma_3$.

The 4-sets and their cross-ratio are given in Table 4 as follows:

Table (4): The 4-sets of type three

Number	The 4-set	The cross-ratio
1	$\{\infty, 0, 1, \beta^5\}$	$\lambda = \beta^5$
2	$\{\beta^2, \beta^3, \beta^4, \beta^6\}$	$\lambda = \beta^{11}$
3	$\{\beta^7, \beta^8, \beta^{11}, \beta^{12}\}$	$\lambda = \beta^{11}$
4	$\{\beta^9, \beta^{10}, \beta^{17}, \beta^{23}\}$	$\lambda = \beta^{11}$
5	$\{\beta^{13}, \beta^{14}, \beta^{15}, \beta^{19}\}$	$\lambda = \beta^{19}$
6	$\{\beta^{16}, \beta^{17}, \beta^{20}, \beta^{21}\}$	$\lambda = \beta^{11}$
7	$\{\beta^{18}, \beta^{22}, \beta^{24}, \beta^{25}\}$	$\lambda = \beta^{25}$

Therefore, on PG $(1,27)$, there are precisely one projectively 4-set of type three.

4. The 4-sets of type four, when $\lambda \epsilon \Gamma_4$.

The 4-sets and their cross-ratio are given in Table 5 as follows:

Table (5): The 4-sets of type four

Number	The 4-set	The cross-ratio
1	$\{\infty, 0, 1, \beta^8\}$	$\lambda = \beta^8$
2	$\{\beta^2, \beta^3, \beta^6, \beta^{10}\}$	$\lambda = \beta^{17}$
3	$\{\beta, \beta^5, \beta^{12}, \beta^{13}\}$	$\lambda = \beta^{18}$
4	$\{\beta^4, \beta^7, \beta^{22}, \beta^{20}\}$	$\lambda = \beta^8$
5	$\{\beta^{14}, \beta^{15}, \beta^{19}, \beta^{25}\}$	$\lambda = \beta^{18}$
6	$\{\beta^9, \beta^{17}, \beta^{18}, \beta^{24}\}$	$\lambda = \beta^8$
7	$\{\beta^{11}, \beta^{16}, \beta^{21}, \beta^{23}\}$	$\lambda = \beta^{14}$

Therefore, on PG $(1,27)$, there are precisely one projectively 4-set of type four.

5. The 4-sets of type five, when $\lambda \epsilon \Gamma_5$.

The 4-sets and their cross-ratio are given in Table 6 as follows:

Table (6): The 4-sets of type five

Number	The 4-set	The cross-ratio
1	$\{\infty, 0, 1, \beta^{13}\}$	$\lambda = \beta^{13}$
2	$\{\beta^{1}, \beta^{2}, \beta^{3}, \beta^{15}\}$	$\lambda = \beta^{13}$
3	$\{\beta^{16}, \beta^{18}, \beta^{21}, \beta^{23}\}$	$\lambda = \beta^{13}$
4	$\{\beta^{12}, \beta^{17}, \beta^{20}, \beta^{24}\}$	$\lambda = \beta^{13}$
5	$\{\beta^{4}, \beta^{5}, \beta^{7}, \beta^{14}\}$	$\lambda = \beta^{13}$
6	$\{\beta^{6}, \beta^{8}, \beta^{9}, \beta^{25}\}$	$\lambda = \beta^{13}$
7	$\{\beta^{10}, \beta^{11}, \beta^{19}, \beta^{22}\}$	$\lambda = \beta^{13}$

Therefore, on PG$(1,27)$, there are precisely one projectively 4-set of type five.

Theorem 4: On PG$(1,27)$, there are precisely five distinct projectively 4-sets.

From Tables 1,2,3 ,4,5 ,6, we have the following Theorem .

The 4-sets and their stabilizer group are given in Table 7 as follows:

Table 7: The inequivalent 4-sets on PG$(\mathbf{1}, \mathbf{27})$

Number	The 4-set	The stabilizer group
1	$\{\infty, 0, 1, \beta\}$	$Z_2 \times Z_2 = < \dfrac{x+\beta^{14}}{x+\beta^{13}} > \times < \dfrac{x+\beta^{13}}{\beta^{25}x+\beta^{13}} >$
2	$\{\infty, 0, 1, \beta^{3}\}$	$Z_2 \times Z_2 = < \dfrac{x+\beta^{13}}{\beta^{23}x+\beta^{13}} > \times < \dfrac{x+\beta^{16}}{x+\beta^{13}} >$
3	$\{\infty, 0, 1, \beta^{5}\}$	$Z_2 \times Z_2 = < \dfrac{x+\beta^{18}}{x+\beta^{13}} > \times < \dfrac{x+\beta^{13}}{\beta^{21}x+\beta^{13}} >$
4	$\{\infty, 0, 1, \beta^{8}\}$	$Z_2 \times Z_2 = < \dfrac{x+\beta^{21}}{x+\beta^{13}} > \times < \dfrac{x+\beta^{13}}{\beta^{18}x+\beta^{13}} >$
5	$\{\infty, 0, 1, \beta^{13}\}$	$S_4 = < \dfrac{1}{x}, \dfrac{1}{x+1} >$

The group action on projective plane of order three PG$(2,3)$

The projective plane PG$(2,3)$ contains thirteen points, thirteen lines, four points on line and four lines through a point. The points of PG$(2,3)$ are generated by a nonsingular matrix $E = C(F) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, where$F(X) = X^3 - X + 1$, such that $P(i) = (1,0,0)\,E^{i}$, i = 0,1,…,12 .

The action of $\langle E \rangle$ on PG(2,3) is
$P(0) \xrightarrow{E} P(1) \xrightarrow{E} \dots \xrightarrow{E} P(12) \xrightarrow{E} P(0)$

Therefore, E is cycles of thirteen point and $\langle E \rangle = \{E, E^2, \dots, E^{13} = I_{3\times 3}\}$

This gives the following conclusion.

Theorem5: On PG(2,3) , we have

i. The set $\langle E \rangle$ together with the usual multiplication of matrices is a cyclic group of order 13 ;
ii. The action of $\langle E \rangle$ on PG $(2,3)$ is transitive.

The group action on projective plane of order nine PG(2,9)

The projective plane PG(2 ,9) contains 91 points, 91 lines, ten points on line and ten lines through a point.The points of PG (2,9) are generated by a nonsingular matrix $W = C(F) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha^7 & 1 & 0 \end{pmatrix}$, where $F(X) = X^3 - X - \alpha^7$, such that $P(i) = (1,0,0)\, W^i$, $i = 0,1,\dots,90$.

The action of $\langle W \rangle$ on PG(2 ,9) is
$P(0) \xrightarrow{W} P(1) \xrightarrow{W} \dots \xrightarrow{W} P(90) \xrightarrow{W} P(0)$

Therefore, W cycles of 91 points and $\langle W \rangle = \{W, W^2, \dots, W^{91} = I_{3\times 3}\}$

This gives the following conclusion.

Theorem 6: on PG(2,9) , we have

i. The set $\langle W \rangle$ together with the usual multiplication of matrices is a cyclic group of order 91 ;
ii. The action of $\langle W \rangle$ on PG (2,9) is transitive.

The group action on projective plane of order twenty-seven PG(2,27)

The plane PG (2,27) contains 757 points, 757 lines, 28 points on line and 28 lines through a point. The points of PG (2,27) are generated by a nonsingular matrix $U = C(F) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta^{25} & 1 & 0 \end{pmatrix}$, where $F(X) = X^3 - X - \beta^{25}$, such that $P(i) = (1,0,0)\, U^i$, $i = 0,1,\dots,756$.

The action of $\langle U \rangle$ on PG(2,27) is
$P(0) \xrightarrow{U} P(1) \xrightarrow{U} \dots \xrightarrow{U} P(756) \xrightarrow{U} P(0)$

Therefore, U cycles of 757 points and $\langle U \rangle = \{U, U^2, \dots, U^{757} = I_{3\times 3}\}$

This gives the following conclusion.

Theorem 7: On PG(2,27) , we have

i. the set $\langle U \rangle$ together with the usual multiplication of matrices is a cyclic group of order 757 ;
ii. The action of $\langle U \rangle$ on PG $(2,27)$ is transitive.

The partition of PG (2, 9)

The action of $\langle W^7 \rangle = \langle \begin{pmatrix} \alpha^7 & \alpha^5 & \alpha^3 \\ \alpha^2 & 0 & \alpha^5 \\ \alpha^4 & 1 & 0 \end{pmatrix} \rangle$ on PG $(2,9)$ with $P(i) = i$, $i = 0,1,\dots,90$ is given as follows:

$$\pi_1 = 0 \xrightarrow{W^7} 7 \xrightarrow{W^7} 14 \dots \xrightarrow{W^7} 0\,,$$

$$\pi_2 = 1 \xrightarrow{W^7} 8 \xrightarrow{W^7} 15 \dots \xrightarrow{W^7} 1\,,$$

$$\pi_3 = 2 \xrightarrow{W^7} 9 \xrightarrow{W^7} 16 \dots \xrightarrow{W^7} 2\,,$$

$$\pi_4 = 3 \xrightarrow{W^7} 10 \xrightarrow{W^7} 17 \dots \xrightarrow{W^7} 3 ,$$

$$\pi_5 = 4 \xrightarrow{W^7} 11 \xrightarrow{W^7} 18 \dots \xrightarrow{W^7} 4 ,$$

$$\pi_6 = 5 \xrightarrow{W^7} 12 \xrightarrow{W^7} 19 \dots \xrightarrow{W^7} 5 ,$$

$$\pi_7 = 6 \xrightarrow{W^7} 13 \xrightarrow{W^7} 20 \dots \xrightarrow{W^7} 6 .$$

Each one of above orbit represents a subplane PG(2,3) in PG(2,9) .

The lines of subplanes π_i , $i = 1,2,\dots,7$ are given in Table 8 as follows:

Table (8): The lines of subplanes in PG$(\mathbf{2},\mathbf{9})$

The subplane	The lines of subplane												
π_1	0	7	0	7	14	0	0	7	14	21	28	35	42
	49	56	14	21	28	21	7	14	21	28	35	42	49
	56	63	63	70	77	35	28	35	42	49	56	63	70
	77	84	70	77	84	84	42	49	56	63	70	77	84
π_2	1	8	1	8	15	1	1	8	15	22	29	36	43
	50	57	15	22	29	22	8	15	22	29	36	43	50
	57	64	64	71	78	36	29	36	43	50	57	64	71
	78	85	71	78	85	85	43	50	57	64	71	78	85
π_3	2	9	2	9	16	2	2	9	16	23	30	37	44
	51	58	16	23	30	23	9	16	23	30	37	44	51
	58	65	65	72	79	37	30	37	44	51	58	65	72
	79	86	72	79	86	86	44	51	58	65	72	79	86
π_4	3	10	3	10	17	3	3	10	17	24	31	38	45
	52	59	17	24	31	24	10	17	24	31	38	45	52
	59	66	66	73	80	38	31	38	45	52	59	66	73
	80	87	73	80	87	87	45	52	59	66	73	80	87
π_5	4	11	4	11	18	4	4	11	18	25	32	39	46
	53	60	18	25	32	25	11	18	25	32	39	46	53
	60	67	67	74	81	39	32	39	46	53	60	67	74
	81	88	74	81	88	88	46	53	60	67	74	81	88
π_6	5	12	5	12	19	5	5	12	19	26	33	40	47
	54	61	19	26	33	26	12	19	26	33	40	47	54
	61	68	68	75	82	40	33	40	47	54	61	68	75
	82	89	75	82	89	89	47	54	61	68	75	82	89
π_7	6	13	6	13	20	6	6	13	20	27	34	41	48
	55	62	20	27	34	27	13	20	27	34	41	48	55
	62	69	69	76	83	41	34	41	48	55	62	69	76
	83	90	76	83	90	90	48	55	62	69	76	83	90

Moreover, $\pi_1 \xrightarrow{W} \pi_2 \dots \xrightarrow{W} \pi_1$

This gives the following conclusion.

Theorem 8: On PG(2,9) , we have

i. The set $\langle W^7 \rangle$ together with the usual multiplication of matrices is a cyclic group of order 13 ;

ii. The action of $\langle W^{7} \rangle$ on π_i , $i = 1,2,\dots,7$ is transitive ;

iii. There are precisely one projectively subplane $\mathrm{PG}(2,3)$ in $\mathrm{PG}(2,9)$.

The action of $\langle W^{13} \rangle = \langle \begin{pmatrix} \alpha^7 & \alpha^6 & \alpha^2 \\ \alpha & \alpha^5 & \alpha^6 \\ \alpha^5 & \alpha^4 & \alpha^5 \end{pmatrix} \rangle$ on $\mathrm{PG}(2,9)$ is given as follows:

$$K_1 = 0 \xrightarrow{W^{13}} 13 \dots \xrightarrow{W^{13}} 0$$

$$K_2 = 1 \xrightarrow{W^{13}} 14 \dots \xrightarrow{W^{13}} 1$$

$$K_3 = 2 \xrightarrow{W^{13}} 15 \dots \xrightarrow{W^{13}} 2$$

$$K_4 = 3 \xrightarrow{W^{13}} 16 \dots \xrightarrow{W^{13}} 3$$

$$K_5 = 4 \xrightarrow{W^{13}} 17 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 4$$

$$K_6 = 5 \xrightarrow{W^{13}} 18 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 5$$

$$K_7 = 6 \xrightarrow{W^{13}} 19 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 6$$

$$K_8 = 7 \xrightarrow{W^{13}} 20 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 7$$

$$K_9 = 8 \xrightarrow{W^{13}} 21 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 8$$

$$K_{10} = 9 \xrightarrow{W^{13}} 22 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 9$$

$$K_{11} = 10 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 10$$

$$K_{12} = 11 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 11$$

$$K_{13} = 12 \xrightarrow{W^{13}} \dots \xrightarrow{W^{13}} 12$$

Each one of above orbits represents a complete $(7;2)$ -arc. Moreover,

$$K_1 \xrightarrow{W} K_2 \xrightarrow{W} K_3 \dots \xrightarrow{W} K_{13}$$

This gives the following conclusion.

Theorem 9: On $\mathrm{PG}(2,9)$, we have

i. The set $\langle W^{13} \rangle$ together with the usual multiplication of matrices is a cyclic group of order 7 ;

ii. The action of $\langle W^{13} \rangle$ on K_i , $i = 1,2,\dots,13$ is transitive ;

iii. There are precisely one projectively complete $(7;2)$ -arc.

Conclusions

1. Partition PG (1,27) into five types of seven disjoint 4-sets.
2. Partition PG (2,9) into seven disjoint subplanes of order three PG(2,3) .
3. Partition PG (2,9) into thirteen disjoint complete $(7;2)$-arc .
4. Classify the group action on PG $(1,3^n)$ and PG $(2,3^n)$, $n = 1,2,3$.
5. Classify the subspaces of PG(1,27) and PG(2,27) .
6. To give rise to an err-correcting code that corrects the maximum possible number of errors.

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