Fuzzy Bounded and Continuous Linear Operators on Standard Fuzzy Normed Spacesa

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ABSTRACT

In this paper we introduce the definition of standard fuzzy normed space then we discuss several properties after we give an example to illustrate this notion. Then we define F-bounded operator as an introduction to define a standard fuzzy norm of an operator and if T is a linear operator from standard fuzzy normed space X into a standard fuzzy normed space Y we prove that T is continuous if and only if T is F-bounded.

المؤثرات الخطية المستمرة و المقيدة ضبابيا على الفضاءات القياس الضبابية القياسية

الخلاصة

في هذا البحث قدمنا تعريف فضاء القياس الضبابي القياسي ثم ناقشنا عدة خواص بعد ان اعطينا مثالا لتوضيح هذا المفهوم بعد ذلك عرفنا المؤثر المقيد ضبابيا كمقدمة لتعريف القياس الضبابي للمؤثرات و اذا كانت T هي مؤثر خطي من فضاء القياس الضبابي القياسي X الى فضاء القياس الضبابي القياسي اخر Y برهنا ان T تكون مستمرة اذا و فقط اذا كانت مقيدة ضبابيا.

Key Words: standard fuzzy normed space, F-bounded linear operator, a fuzzy norm of an operator

INTRODUCTION

The theory of fuzzy sets was introduced by Zadeh in 1965**[1]**. Many authors have introduced the concept of fuzzy norm in different ways [2,3,4,5,6,7,11,12]. Cheng and Mordeson in 1994**[8]** defined fuzzy norm on a linear space whose associated fuzzy metric is of Kramosil and Mickalek type**[9]** as follows:

The order pair (X,N) is said to be a fuzzy normed space if X is a linear space and N is a fuzzy set on $X \times [0,\infty)$ satisfying the following conditions for every $x,y \in X$ and $s,t \in [0,\infty)$

(i) $N(x,0) = 0$, for all $x \in X$.

(ii)For all $t>0$, $N(x,t) = 1$ if and only if $x = 0$

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(iii) N(αx,t) = N(x,$\frac{t}{|\alpha|}$), for all $\alpha \neq 0$ and For all t>0 .
(iv)For all s, t>0, N(x+y,t+s) ≥ N(x,t) ∧ N(y,s) where a∧b = min{a,b}
(v) $\lim_{t\to\infty}$N(x,t) = 1.
George and Veeramani in [10] introduced the definition of continuous t-norm. Bag and Samanta in [2] modified the definition of Cheng and Mordeson of fuzzy norm as follows:

The triple (X,N,∗) is said to be a fuzzy normed space if X is a linear space, ∗ is a continuous t-norm and N is a fuzzy set on X ×[0,∞) satisfying the following conditions for every x,y ∈ X and s,t ∈ [0,∞)
(i) N(x,0) = 0, for all x ∈ X.
(ii)For all t>0, N(x,t) = 1 if and only if x = 0
(iii) N(αx,t) = N(x,$\frac{t}{|\alpha|}$), for all $\alpha \neq 0$
(iv)For all s, t>0, N(x,t) ∗ N(y,s) ≤ N(x+y,t+s)
(v)For x≠0, N(x, ·):(0,∞) → [0,1] is continuous.
(vi) $\lim_{t\to\infty}$N(x,t) = 1.
In this paper we introduce the definition of standard fuzzy normed space as a modification of the notion of fuzzy normed space duo to Bag and Samanta. In section one we recall the definition of t-norm then we introduce the definition of standard fuzzy normed space after that we give an example then we prove that every ordinary norm induced a standard fuzzy norm define open ball, convergent sequence, open set, Cauchy sequence, F-bounded set and a continuous operator between two standard fuzzy normed spaces. Also we prove several properties for F-bounded operator.

Standard fuzzy normed space
Definition 1.1:[1]
Let X be a nonempty set of elements, a fuzzy set A in X is characterized by a membership function, μ_A(x): X→ [0,1]. Then we can write A = {(x, μ_A(x)): x∈X, 0 ≤ μ_A(x) ≤ 1}. Then A is a continuous fuzzy set.
We now give an example of continuous fuzzy set
Example 1.2:[4]
Let X = ℝ and let A be a fuzzy set in ℝ with membership function defined by : $\mu_A(x) = \frac{1}{1+10x^2}$.
Definition 1.3:[10]
A binary operation ∗: [0,1] × [0,1]→ [0,1] is a continuous triangular norm (or simply t-norm) if for all a, b, c, e∈[0, 1] the following conditions hold:
1- a∗b = b∗a
2- a∗1 = a
3- (a∗b) ∗c = a∗(b∗c)
4- If a ≤ c and b ≤ e then a∗b ≤ c∗e
Example 1.4:[10]
Define a∗b = a.b, for all a, b∈[0,1], where a.b is the usual multiplication in [0,1] then ∗ is a continuous t-norm.
Example 1.5:[10]
Define a∗b = min{a,b} for all a, b∈[0,1], it follows that ∗ is a continuous t-norm.
Remark 1.6:[10]

For any $a > b$, we can find c such that $a*c \geq b$ and for any d we can find q such that $q*q \geq d$, where a, b, c, d and q belong to (0,1).
Now we introduce the basic definition in this paper

Definition 1.7:

Let X be a linear space over field $\mathbb{K}$ and $*$ is a continuous t-norm and N is a fuzzy set on X satisfying:

(FN_1) $N(x) > 0$ for all $x \in X$.

(FN_2) $N(x) = 1$ if and only if $x = 0$.

(FN_3) $N(\alpha x) = \frac{1}{|\alpha|} N(x)$ for all $x \in X$ and $\alpha \neq 0 \in \mathbb{K}$.

(FN_4) $N(x+y) \geq N(x) * N(y)$ for all $x, y \in X$.

(FN_5) N(x) is a continuous fuzzy set.

Then the triple (X,N, $*$) is called standard fuzzy normed space.

Definition 1.8:

Let (X,N,$*$) be a standard fuzzy normed space. N is called continuous fuzzy set if whenever $x_n \to x$ in X then $N(x_n) \to N(x)$, that is $\lim_{n\to\infty} N(x_n) = N(x)$.

Example 1.9:

Let $X = \mathbb{Z}$, the set of integers, $a*b = a.b$ for all $a,b \in [0,1]$

$$\text{Define } N(x) = \begin{cases} \frac{1}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then (X,N,$*$) is standard fuzzy normed space.

Proposition 1.10:

Let (X,||.||) be an ordinary normed space with $\| x \|$ is an integer for all x $\in$ X. Define

$$N_{\|.\|}(x) = \begin{cases} \frac{1}{\|x\|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

and $a*b= a.b$ for all $a,b \in [0,1]$. Then $(X, N_{\|.\|},*)$ is standard fuzzy normed space induced by ||.||.

Proof:

(FN_1) Since $\|x\| > 0$ for all $x \in X$ then $N_{\|.\|}(x) > 0$ for all $x \in X$.

(FN_2) $N_{\|.\|}(x) = 1$ if and only if $x = 0$.

(FN_3) Let $\alpha \neq 0 \in K$ then for all $x \in X$ we have

$$N_{\|.\|}(\alpha x) = \frac{1}{\|\alpha x\|} = \frac{1}{|\alpha|.\|x\|} = \frac{1}{|\alpha|} N_{\|.\|}(x)$$

(FN_4) $N_{\|.\|}(x+y) = \frac{1}{\|x+y\|} > \frac{1}{\|x\|} . \frac{1}{\|y\|} = N_{\|.\|}(x) * N_{\|.\|}(y)$

(FN_5) Let (x_n) be a sequence in X such that $x_n \to x$ that is $\lim_{n\to\infty} x_n = x$.

Now, $\lim_{n\to\infty} N_{\|.\|}(x_n) = \lim_{n\to\infty} \frac{1}{\|x_n\|} = \frac{1}{\|x\|} = N_{\|.\|}(x)$.

Therefore $N_{\|.\|}$ is continuous fuzzy set. Hence $(X, N_{\|.\|},*)$ is standard fuzzy normed space.■

Definition 1.11:

Let (X,N,$*$) be a standard fuzzy normed space, we define $B(x,r) = \{y \in X : N(x) > (1-r) \}$ then B(x,r) is called an open ball with center $x \in X$ and radius $0 < r < 1$.

Definition 1.12:
A sequence (x_n) in a standard fuzzy normed space $(X,N,*)$ is said to be converge to a point $x \in X$ if $0 < \varepsilon < 1$ is given, there exists a positive number K such that, $N(x_n - x) > (1- \varepsilon)$ for all $n \geq K$.
Theorem 1.13:
A sequence (x_n) in a standard fuzzy normed space $(X,N,*)$ is converge to a point $x \in X$ if and only if $\lim_{n\to\infty} N(x_n - x) = 1$.
Proof:
Suppose that the sequence (x_n) converges to x then for given any $0 < r < 1$ there is a positive number K such that $N(x_n - x) > (1- r)$ for all $n \geq K$ and hence $1-N(x_n - x) < r$. Therefore $N(x_n - x)$ converges to 1 as n tends to ∞.The proof of the converse is similar hence is omitted.■
Lemma 1.14:
Let $(X,N, *)$ be a standard fuzzy normed space. Then $N(x-y) = N(y-x)$ for all $x,y \in X$
Proof:
$N(x-y) = N[(-1)(y-x)] = \frac{1}{|-1|} N(y-x) = N(y-x)$. ■
Definition 1.15:
A subset A of a standard fuzzy normed space $(X,N,*)$ is said to be open if it contains a ball about each of its points. A subset B of X is said to be closed if its complement is open that is $B^c = X - B$ is open.
The proof of the following theorem is easy, hence it is omitted.
Theorem 1.16:
Every open ball in a standard fuzzy normed space $(X,N,*)$ is an open set.
Definition 1.17:
Let $(X,N,*)$ be a standard fuzzy normed space and let $A \subset X$ then the closure of A is denoted by $\overline{A}$ or cL(A) and is defined to be the smallest closed set contains A.
Lemma 1.18:
Let A be a subset of a standard fuzzy normed space $(X,N,*)$. Then $a \in \overline{A}$ if and only if there is a sequence (a_n) in A such that $a_n \to a$.
Proof:
Let $a \in \overline{A}$, if $a \in A$ then we take sequence of that type $(a,a,a,\ldots,a,\ldots)$. If $a \notin A$, then it is a limit point of A. Hence we construct the sequence $(a_n) \in A$ by $N(a_n - a) > 1- \frac{1}{n}$ for each n = 1, 2, 3,…..
The ball $B(a, \frac{1}{n})$ contains $a_n \in A$ and $a_n \to a$ because $\lim_{n\to\infty} N(a_n - a) = 1$.
Conversely if (a_n) in A and $a_n \to a$ then $a \in A$, or every neighborhood of a contains points $a_n \neq a$, so that a is a point of accumulation of A, hence $a \in \overline{A}$ by using the definition of the closure.■
Definition 1.19:
A sequence (x_n) in a standard fuzzy normed space $(X,N,*)$ is said to be Cauchy if for each $0 < \varepsilon < 1$ there is a positive number K such that $N(x_n - x_m) > (1- \varepsilon)$ for all n, m $\geq$ K.
The proof of the following theorem is easy, hence it is omitted.
Theorem 1.20:
In a standard fuzzy normed space every convergent sequence is Cauchy.
Definition 1.21:

Let $(X,N,*)$ be a standard fuzzy normed space. A subset A of X is said to be F-bounded if there exists a real number r, $0 < r < 1$ such that, $N(x) > (1- r)$, for all $x \in A$.

Definition 1.22:

Let $(X,N_X,*)$ and $(Y,N_Y,*)$ be two standard fuzzy normed spaces and $A \subset X$. The operator $f:A \to Y$ is said to be continuous at $a \in A$, if for every $0 < \varepsilon < 1$, there exists some $0 < \delta < 1$, such that $N_Y(f(x) - f(a)) > (1- \varepsilon)$ whenever $x \in A$ satisfying $N_X(x-a) > (1 - \delta)$. If f is continuous at every point of A, then it is said to be continuous on A.

Fuzzy Bounded and Continuous Linear Operator

Definition 2.1:

Let $(X,N_X,*)$ and $(Y,N_Y,*)$ be two fuzzy normed spaces and $T: D(T) \to Y$ be a linear operator, where $D(T) \subset X$. The operator T is said to be F-bounded if there is a real number c, $0 < c < 1$ such that for all $x \in D(T)$,

$N_Y (Tx) \geq (1-c) N_X (x)$ 2.1

Remark 2.2

1-Formula (2.1) shows that F-bounded linear operator maps F-bounded sets in D(T) onto F- bounded sets in Y.

2-What is the largest possible (1- c) such that equation (2.1) still holds for all $x \in D(T)$?. By division we have $\frac{N_Y(Tx)}{N_X(x)} \geq (1-c)$ and this shows that (1- c) must be at least as big as the infinimum of the expression on the left taken over $D(T) - \{0\}$.Hence the answer to our question is that the smallest possible (1- c) in (2.1) is that infinimum. This quantity is denoted by N(T). Thus

$N(T) = \inf_{x \in D(T)} \frac{N_Y(Tx)}{N_X(x)}$ ... (2.2)

N(T) is called the fuzzy norm of the operator T. If T = 0, we define

$N(T) = 1$.

3- From (2.1) with (1- c) = N(T) we have

$N_Y(Tx) \geq N(T) * N_X(x)$ ... (2.3)

Lemma 2.3

Let $T: D(T) \to Y$ be fuzzy bounded linear operator from a standard fuzzy normed space $(X,N_X,*)$ with $D(T) \subset X$ into a standard fuzzy normed space $(Y,N_Y,*)$ then

(i) An alternative formula for the norm of T is

$N(T) = \inf_{x \in D(T)} N_Y(Tx)$ ... (2.4)

(ii) The norm defined by (2.2) is a standard fuzzy normed space

Proof :

(i) We put $a = N_x(x)$ and set $y = ax$ .Then

$N_X(y) = N_X(ax) = \frac{1}{|a|} N_X(x) = \frac{1}{N_x(x)} N_X(x) = 1$ and since T is linear equation (2.2) gives

$\inf_{x \in D(T)} \frac{N_Y(Tx)}{N_X(x)} = \inf_{x \in D(T)} \frac{N_Y(Tx)}{a} = \inf_{x \in D(T)} N_Y (T(ax))$

$= \inf_{x \in D(T)} N_Y(Ty)$

Writing x for y on the right, we have (2.4).

(ii) (FN_1) $N_Y(Tx) > 0$ and $N_X(x) > 0$ implies $\frac{N_Y(Tx)}{N_X(x)} > 0$. Hence $N(T) > 0$.

(FN_2) $N(T) = 1 \Leftrightarrow \inf_{x \in D(T)} N_Y(Tx) = 1 \Leftrightarrow N_Y(Tx) = 1 \Leftrightarrow Tx = 0$

$\Leftrightarrow T = 0$.

(FN_3) $N(\alpha T) = \inf_{x\in D(T)} N_Y(\alpha Tx) = \frac{1}{|\alpha|} \inf_{x\in D(T)} N_Y(Tx) = \frac{1}{|\alpha|} N(T)$.

(FN_4) $N(T_1+T_2) = \inf_{x\in D(T_1)\cap D(T_2)} N_Y[(T_1+T_2)(x)]$

$= \inf_{x\in D(T_1)\cap D(T_2)} N_Y(T_1(x)+T_2(x))$

$\geq \inf_{x\in D(T_1)} N_Y(T_1(x) * \inf_{x\in D(T_2)} N_Y(T_2(x))$

$\geq N(T_1) * N(T_2)$

(FN_5) Since N_Y is continuous so N(T) is continuous.■

Before we consider general properties of F- bounded linear operators, let us take a look at some typical examples, so that we get a better feeling for the concept of a F-bounded linear operator.

Example 2.4:

Let X be the vector space of all polynomials on J = [0,1] with norm given by $\|x\| = \max |x(t)|$, $t \in J$ where $|x(t)|$ is an integer then $(X, N_{\|.\|}, *)$ is a standard fuzzy normed space where $N_{\|.\|}(x) = \frac{1}{\|x\|}$ if $x \neq 0$ and $N_{\|.\|}(0) = 1$ also $a * b = a \cdot b$ $\forall a, b \in J = [0,1]$.

Let T: X→ X defined by :

$T(x(t)) = x'(t)$.

T is linear but not F-bounded. Indeed $x_n(t) = t^n$,n=1, 2,… so $\|x_n\| = 1$, then $N_{\|.\|}(x_n) = 1$ where $n \in \mathbb{N}$. Now, $Tx_n(t) = nt^{n-1}$ so $\|Tx_n\| = n$ which implies that $N_{\|.\|}(Tx_n) = \frac{1}{n}$ so $\frac{N(Tx_n)}{N(x_n)} = \frac{1}{n}$. Since $n \in \mathbb{N}$ is arbitrary, this shows that there is no r, 0 < r < 1 such that $\frac{N(Tx_n)}{N(x_n)} \geq (1-r)$. From this we conclude that T is not F-bounded.■

Example 2.5:

Consider C[0,1] with $\|x\| = \max | x(t) |$, $t \in J = [0,1]$ with $|x(t)|$ is an integer.Then $(C[0,1], N_{\|.\|}, *)$ is a standard fuzzy normed space where

$N_{\|.\|}(x) = \frac{1}{\|x\|}$, $N_{\|.\|}(0) = 1$, and $a * b = a \cdot b$ for all a,b ∈ [0,1]. Define T: C[0,1] → C[0,1] by T(x) = y where $y(t) = \int_0^1 k(t,s)x(s)ds$ where k(t,s) is continuous on J×J. and k(t,s) is bounded say $|k(t,s)| \leq k_o$ for all $(t,s) \in J\times J$ where $k_o \in \mathbb{R}$ .This operator is linear and F-bounded. Now

$|x(t)| \leq \max|x(t)| = \|x\|$, $t \in J$.

Hence , $\|y\| = \|Tx\| = \max | \int_0^1 k(t,s)x(s)ds |$

$\leq \max \int_0^1 | k(t,s)| \, |x(s)| ds$

$\leq k_0 \|x\|$.

Therefore

$N_{\|.\|}(Tx) = \frac{1}{\|Tx\|} \geq \frac{1}{k_o} \cdot \frac{1}{\|x\|} = \frac{1}{k_o} N_{\|.\|}(x)$. Put $\frac{1}{k_o} = (1-r)$ for some r, 0 < r < 1, we get $N_{\|.\|}(Tx) \geq (1-r) N_{\|.\|}(x)$. Hence T is F-bounded.■

Operators are mappings, so that the definition of continuity applies to them as follows:

Let T: D(T) → Y be any operator, not necessarily linear, where D(T) ⊂ X and X and Yare standard fuzzy normed spaces. The operator T is called continuous at $x_o \in D(T)$ if for every 0 < ε < 1 there is a 0 < δ < 1 such that $N_Y(Tx - Tx_o) > (1 - \varepsilon)$ for all x ∈ D(T) satisfying $N_X(x - x_o) > (1 - \delta)$. T is continuous if T is continuous at every x ∈ D(T) .

Theorem 2.6:

Let T: D(T) → Y be a linear operator, where D(T) ⊂ X and $(X, N_X, *)$, $(Y, N_Y, *)$ are standard fuzzy normed spaces .Then:

(i) T is continuous if and only if T is F-bounded.
(ii) If T is continuous at a single point, it is continuous**.**
Proof of (i):
We assume T is F-bounded and consider any $x_o \in D(T)$. Let any $0 < \varepsilon < 1$ be given. Then, since T is linear, for every $x \in D(T)$ such that
$N_X(x - x_o) > (1 - \delta)$, $(1 - \delta) = \frac{(1-\varepsilon)}{N(T)}$, we obtain
$N_Y(Tx - Tx_o) = N_Y[T(x - x_o)] \geq N(T) \cdot N_X(x - x_o)$
$\geq N(T) * (1 - \delta) = (1- \varepsilon)$.
Since $x_o \in D(T)$ was arbitrary, this shows that T is continuous.
Conversely, assume that T is continuous at an arbitrary $x_o \in D(T)$ Then, given any $0 < \varepsilon < 1$, there is a $0 < \delta < 1$ such that $N_Y(Tx - Tx_o) \geq (1- \varepsilon)$ for all $x \in D(T)$ satisfying $N_X(x - x_o) > (1 - \delta)$ ⋯ (2.5)
We now take $y \neq 0$ in D(T) and set
$x = x_o + \frac{N_X(y)}{(1-\delta)} \cdot y$. Then $(x - x_o) = \frac{N_X(y)}{(1-\delta)} \cdot y$.
Hence $N_X(x - x_o) = \frac{(1-\delta)}{N_X(y)} N_X(y) = (1 - \delta)$. Now
$N_Y(Tx - Tx_o) = N_Y(T(x - x_o)) = N_Y[T(\frac{N_X(y)}{(1-\delta)} \cdot y)] = \frac{(1-\delta)}{N_X(y)} N_Y(Ty)$ and (2.5) implies
$\frac{(1-\delta)}{N_X(y)} N_Y((Ty) \geq (1- \varepsilon)$ implies $N_Y(Ty) \geq \frac{(1-\varepsilon)}{(1-\delta)} N_X(y)$. This can be written $N_Y((Ty) \geq (1- c) N_X(y)$, where $(1- c) = \frac{(1-\varepsilon)}{(1-\delta)}$ and this shows that T is F-bounded.
Proof of (ii):
Continuity of T at a point implies F-boundedness of T by the second part of the proof of (i), which in turn implies continuity of T by (i).
Corollary 2.7
Let $(X,N_X,*)$,$(Y,N_Y,*)$ be standard fuzzy normed spaces and let $T: D(T) \to Y$ be a F-bounded linear operator. Then:
(i) $x_n \to x$ [where $x_n , x \in D(T)$] implies $Tx_n \to Tx$.
(ii) The null space K(T) is closed, where $K(T) = \{x\in D(T) : T(x) = 0\}$.
Proof of (i):
Since T is F-bounded, $N(T) \geq (1-r)$ for some $0 < r < 1$,and since $x_n \to x$ given $0 < s < 1$ there is a positive number K such that $N_X(x_n - x) > (1-s)$. Now , by Remark (1.6) there is $(1- \varepsilon) \in (0,1)$ such that $(1-r) * (1-s) > (1- \varepsilon)$. Now
$N_Y(Tx_n - Tx) = N_Y(T(x_n - x)) \geq N(T) * N_X(x_n - x) > (1-r) * (1-s) > (1- \varepsilon)$ for all $n \geq k$.
Hence $Tx_n \to Tx$.
Proof of (ii):
For every $x \in \overline{K(T)}$ there is a sequence (x_n) in K(T) such that $x_n \to x$ by Lemma (1.18) . Hence $Tx_n \to Tx$ by part (i) of this corollary. Also $Tx = 0$ since $Tx_n = 0$. So that $x \in K(T)$. Since $x \in \overline{K(T)}$ was arbitrary K(T) is closed.■
Theorem 2.8
Let $(X,N_X,*)$ be a standard fuzzy normed space and let $(Y,N_Y,*)$be a Banach space. Let $T: D(T) \to Y$ be a F-bounded linear operator, where $D(T) \subset X$. Then T has an extension $\tilde{T}: \overline{D(T)} \to Y$ where $\tilde{T}$ is a F-bounded linear operator of norm $N(\tilde{T}) = N(T)$.
Proof:
We consider any $x \in \overline{D(T)}$. By Lemma (1.18) there is a sequence (x_n) in

D(T) such that $x_n \to x$. Since T is linear and F-bounded, we have $N(T) \geq (1\text{-}r)$ for some $0 < r < 1$ and since $x_n \to x$ for any given $0 < s < 1$ there is a number K such that $N_X(x_n - x) > (1\text{-}s)$ for all $n \geq K$. Hence by Remark (1.6), there is $(1\text{-}\varepsilon) \in (0,1)$ such that $(1\text{-}r) * (1\text{-}s) > (1\text{-}\varepsilon)$. Hence

$N_Y(Tx_n - Tx_m) = N_Y[T(x_n - x_m)] \geq N(T) * N_X(x_n - x_m) \geq (1\text{-}r) * (1\text{-}s) > (1\text{-}\varepsilon$, for all n,m $\geq$ k. This shows that (Tx_n) is Cauchy. By assumption Y is Banach so that (Tx_n) converges, say, $Tx_n \to y$, We define $\tilde{T}: \overline{D(T)} \to Y : \tilde{T}x = y$. We show that this definition is independent of the particular choice of a sequence in D(T) converging to x. Suppose $x_n \to x$ and $z_n \to x$. Then $v_m \to x$ where $(v_m) = (x_1, z_1, x_2, z_2, \ldots)$. Hence (Tv_m) converges by Corollary(2.7(i)), and the two subsequences (Tx_n) and (Tz_n) of (Tv_m) must have the same limit. This proves that $\tilde{T}$ is uniquely defined at every $x \in \overline{D(T)}$.

Clearly, $\tilde{T}$ is linear and $\tilde{T}x = Tx$ for every $x \in D(T)$, so that $\tilde{T}$ is an extension of T. We now use $N_Y(Tx_n) \geq N(T) * N_X(x_n)$ and let $n \to \infty$.

Then $Tx_n \to y = \tilde{T}x$. Since $x \to N_X(x)$ defines a continuous operator , we thus obtain $N_Y(\tilde{T}x) \geq N(T) * N_X(x)$. Hence $\tilde{T}$ is F-bounded and $N(\tilde{T}) \geq N(T)$. Of course, $N(\tilde{T}) \leq N(T)$ because the fuzzy norm being defined by a infinimum, cannot decrease in an extension. Together we have $N(\tilde{T}) = N(T)$.■

REFERANCES

[1] Zadeh, L. ,"Fuzzy sets", Inf. Control, Vol. 8(1965)338-452.

[2] Bag,T. and Samanta, S.,"Finite dimensional fuzzy linear spaces", Fuzzy Math.,Vol.11 ,No3(2003)678-705.

[3] Bag,T. and Samanta, S., " Fixed point theorems on Fuzzy normespaces", Inf. sci.vol.176(2006)2910-2931.

[4] Congxin, Wang Ming, M., "Continuity and boundness mappings between fuzzy normed spaces, Fuzzy Math, vol.1 (1993)13-24.

[5] Felbin, C., " Finite dimensional fuzzy normed linear spaces.", Fuzzy sets and Systems, vol. 48(1992) 239-248.

[6] Golet, I. , "On generalized fuzzy normed spaces and coincidence theorems", Fuzzy sets and Systems, vol.161(2010)1138-1144.

[7] Kider, J. ,"New fuzzy normed spaces", J. Baghdad Sci. , vol. 9(2012)559-564.

[8] Cheng, S. and Mordeson, J, " Fuzzy linear operators and fuzzy normed linear spaces", Ball. Cal. Math. Soc. vol. 86(1994)429 - 436.

[9] Kramosil, O. and Michalek, J. ,"Fuzzy metrics and statistical metric spaces", Kybernetika, vol. 11(1975)326-334.

[10] George, A. and Veeramani, P. , "On some results in fuzzy metric spaces" Fuzzy sets and Systems, vol. 64(1994)395-399.

[11] Kider, J. R.,"Completeness of the Cartesian product of two complete Fuzzy normed spaces", Eng. and Tech. Journal, Vol.31, No.3(2013) 310-315.

[12] Kider, J. K. and Hussain, Z. A.,"Continuous and uniform continuous Mapping on standard fuzzy metric spaces", Eng. and Tech. Journal, Vol.32, Part(B), No.6 (2014)1111-1119.