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Laplace Operator In Irregular Domain

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ABSTRACT

The aim of this paper is to prove that Laplace operator depending on nine points in irregular domains is of order two in addition, some examples as an applications for this operator are given.

Introduction

Milne[6] derived approximate Laplace operator depending on five points in regular domains in the forms:

$$h^2\nabla^2 u = Hu + O(h^4) \qquad (a)$$

and

$$h^2\nabla^2 u = Xu + O(h^4) \qquad (b)$$

and depending on nine points in regular domains of order $O(h^2)$ in the form: $K = 4H + 2X \qquad (c)$

and $N^2 = 2(X - H) \qquad (d)$

While in irregular domains he derived the following two formulas of order $O(h)$:

$$\bar{H} = F_{xx} + F_{yy} =$$
$$\frac{2}{h_1+h_3}\left\{\frac{F(x+h_1,y)-F(x,y)}{h_1} + \frac{F(x-h_3,y)-F(x,y)}{h_3}\right\} +$$
$$\frac{2}{h_2+h_4}\left\{\frac{F(x,y+h_2)-F(x,y)}{h_2} + \frac{F(x,y-h_4)-F(x,y)}{h_4}\right\} \ldots (e)$$

And

$$\bar{X} = F_{xx} + F_{yy} =$$
$$\frac{1}{h_5+h_7}\left\{\frac{F(x+h_5,y+h_5)-F(x,y)}{h_5} + \frac{F(x-h_7,y-h_7)-F(x,y)}{h_7}\right\} +$$
$$\frac{1}{h_6+h_8}\left\{\frac{F(x-h_6,y+h_6)-F(x,y)}{h_6} + \frac{F(x+h_8,y-h_8)-F(x,y)}{h_8}\right\} \ldots (f)$$

In eleven points near the boundary in non-rectangular domains Milne [6] derived second order Laplace operator for curved boundary using nine points of the two operators $\bar{H}$ and $\bar{X}$ to give new operator called $\bar{K} = 4\bar{H} + 2\bar{X}$. We shall give the prove that the last operator is of second order, moreover we introduce some examples as an applications of this operator. The operator $\bar{K} = 4\bar{H} + 2\bar{X}$ is defined as the following [1]:

$$L_h[u] = \bar{K} = \sum_{i=0}^{8}\alpha_i u_i = 6(\alpha^2 + \beta^2)u + \cdots.$$

where

$$\alpha_1 = \frac{8}{h^2 s_1(s_1+s_3)}, \alpha_2 = \frac{8}{h^2 s_2(s_2+s_4)},$$
$$\alpha_3 = \frac{8}{h^2 s_3(s_3+s_1)}, \alpha_4 = \frac{8}{h^2 s_4(s_4+s_2)},$$

$$\alpha_5 = \frac{2}{h^2 s_5(s_5+s_7)}, \alpha_6 = \frac{2}{h^2 s_6(s_6+s_8)},$$
$$\alpha_7 = \frac{2}{h^2 s_7(s_7+s_5)}, \alpha_8 = \frac{2}{h^2 s_8(s_8+s_6)},$$

$$\alpha_0 = -\sum_{i=1}^{8}\alpha_i, \textit{where}, h_i = s_i h, 0 < s_i \le 1, i = 1,2,3,\cdots.$$

Lemma 1[6]: Let $L_h[u]$ be a discrete operator of the form

$$L_h[u] = \sum_{i=0}^{8}\alpha_i u_i = G$$

where $-\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_8$, are positive functions such that

$$-\alpha_0 \ge \sum_{i=0}^{8}\alpha_i$$

If $u \ge 0$ on S_h and $-L_h[u] \ge 0$ on R_h, then $u \ge 0$ in R_h.

Lemma 2 [6]: Let $L_h[u]$ satisfy the hypothesis of lemma 1. If $|u| \le v$ on S_h and $|L_h[u]| \le -L_h[v]$ in R_h, then $|u| \le v$ in $R_h + S_h$

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Lemma 3[6]: Let $L_h[u]$ satisfy the hypothesis of lemma 1. Let w(x,y) be some function such that $-L_h[u]>0$ in R_h and $w\geq 0$ on S_h. For any function e(x,y), we have

$$|e(x,y)|\leq W\max_{R_h}\left\{\frac{|L_h[e]|}{-L_h[w]}\right\}+\max_{S_h}|e(x,y)|$$

where

$$W=\max_{R_h+S_h}|w(x,y)|.$$

Lemma 4[6]: Let $\bar{u}$ satisfy $L_h[\bar{u}]=G$, and the boundary condition $\bar{u}=g$. Let u satisfy $L_h[u]=G$ in R_h, If $\bar{u}$ has partial derivatives up to fourth order and continuous and b in R+S, then

$$|L_h[u-\bar{u}]|\leq\frac{h^2M_4}{6},$$

where

$$M_4=\max\left[\max_{R+S}\left|\frac{\partial^4 u}{\partial x^4}\right|,\max_{R+S}\left|\frac{\partial^4 u}{\partial y^4}\right|\right].$$

Main Result we prove the following theorem.

Theorem [6]: Under the hypothesis of lemma4 , for all $(x,y)\in R_h+S_h$ we have

$$|u(x,y)-\bar{u}(x,y)|\leq\frac{h^2r^2}{24}\max_{R+S}[M_4]+\max_{S_h}|u(x,y)-\bar{u}(x,y)|\text{......(1)}$$

where r is the radius of the circle which contains R+S.

Proof: for the proof of the first term in the right hand side of the equation (1) see [9]. To find the second term, we use in R_h' the equation:

$$L_h[u]=\sum_{i=0}^{8}\alpha_i u_i=G\text{......(2)}.$$

Solving for u(x, y) we have

$$u(x,y)=\frac{8s_2s_4s_5s_6s_7s_8}{Q}\left[\frac{s_3}{s_1+s_3}u_1+\frac{s_1}{s_1+s_3}u_3\right]+$$
$$\frac{8s_1s_3s_5s_6s_7s_8}{Q}\left[\frac{s_4}{s_2+s_4}u_2+\frac{s_2}{s_2+s_4}u_4\right]+$$
$$\frac{2s_1s_2s_3s_4s_6s_8}{Q}\left[\frac{s_7}{s_5+s_7}u_5+\frac{s_5}{s_5+s_7}u_7\right]+$$
$$\frac{2s_1s_2s_3s_4s_5s_7}{Q}\left[\frac{s_8}{s_6+s_8}u_6+\frac{s_6}{s_6+s_8}u_8\right]\text{............(3)}$$

where $Q=8s_2s_4s_5s_7s_6s_8+8s_1s_3s_5s_7s_6s_8+2s_1s_2s_3s_4s_5s_8+2s_1s_2s_3s_4s_5s_6$.

The first expression in brackets corresponds to the linear interpolation in the points $u(x+h_1,y)$ and $u(x-h_3,y)$ the second expression in brackets corresponds to linear interpolation in points $u(x,y+h_2)$ and $u(x,y-h_4)$, the third expression in brackets corresponds to linear interpolation in the points $u(x+h_5,y+h_5)$ and $u(x-h_7,y-h_7)$, the fourth expression in brackets corresponds to the linear interpolation in the points $u(x-h_6,y+h_6)$ and $u(x+h_8,y-h_8)$. The overall expression represents linear interpolation in two interpolated values. By the properties of linear interpolation, we have:

$$\left|\bar{u}(x,y)-\left\{\frac{s_3}{s_1+s_3}u_1+\frac{s_3}{s_1+s_3}u_3\right\}\right|\leq\frac{h^2(s_1+s_3)^2}{8}M_2$$

$$\left|\bar{u}(x,y)-\left\{\frac{s_4}{s_2+s_4}u_2+\frac{s_4}{s_2+s_4}u_2\right\}\right|\leq\frac{h^2(s_2+s_4)^2}{8}M_2$$

$$\left|\bar{u}(x,y)-\left\{\frac{s_7}{s_5+s_7}u_5+\frac{s_5}{s_5+s_7}u_7\right\}\right|\leq\frac{h^2(s_5+s_7)^2}{8}M_2$$

$$\left|\bar{u}(x,y)-\left\{\frac{s_8}{s_6+s_8}u_6+\frac{s_6}{s_6+s_8}u_8\right\}\right|\leq\frac{h^2(s_6+s_8)^2}{8}M_2\text{......(4)}$$

Where

$$\bar{u}_1=\bar{u}(x+h_1,y),\bar{u}_2=\bar{u}(x,y+h_2),$$
$$\bar{u}_3=\bar{u}(x-h_3,y),\bar{u}_4=\bar{u}(x,y-h_4)$$
$$\bar{u}_5=\bar{u}(x+h_5,y+h_5),\bar{u}_6=\bar{u}(x-h_6,y+h_6),$$
$$\bar{u}_7=\bar{u}(x-h_7,y-h_7),\bar{u}_8=\bar{u}(x+h_8,y-h_8)$$

and

$$M_2=\max\left[\left|\frac{\partial^2\bar{u}}{\partial x^2}\right|,\left|\frac{\partial^2\bar{u}}{\partial y^2}\right|\right].$$

Therefore

$$\left| \begin{array}{l} \bar{u}(x,y) - \\ \frac{8s_2s_4s_5s_6s_7s_8}{Q}\left[\frac{s_3}{s_1+s_3}\bar{u}_1 + \frac{s_1}{s_1+s_3}\bar{u}_3\right] \\ -\frac{8s_1s_3s_5s_6s_7s_8}{Q}\left[\frac{s_4}{s_2+s_4}\bar{u}_2 + \frac{s_2}{s_2+s_4}\bar{u}_4\right] \\ -\frac{2s_1s_2s_3s_4s_6s_8}{Q}\left[\frac{s_7}{s_5+s_7}\bar{u}_5 + \frac{s_5}{s_5+s_7}\bar{u}_7\right] \\ \frac{2s_1s_2s_3s_4s_5s_7}{Q}\left[\frac{s_8}{s_6+s_8}\bar{u}_6 + \frac{s_6}{s_6+s_8}\bar{u}_8\right] \end{array} \right| \le \frac{h^2}{2}M_2 \ldots(5)$$

since $si \le 1, i = 1,2,3,\ldots,8.$ Thus we have

$$|u(x,y) - \bar{u}(x,y)| \le \gamma_1|e(x+h_1,y)| + \gamma_2|e(x,y+h_2)| + \gamma_3|e(x-h_3,y)| + \gamma_4|e(x,y-h_4)| + \gamma_5|e(x+h_5,y+h_5)| + \gamma_6|e(x-h_6,y+h_6)| + \gamma_7|e(x-h_7,y-h_7)| + \gamma_8|e(x+h_8,y-h_8)| + \frac{h_2}{2}M_2 \ldots\ldots\ldots(6)$$

$$\gamma_1 = \frac{8s_2s_4s_5s_6s_7s_8}{Q}\left(\frac{s_3}{s_1+s_3}\right), \gamma_3 = \frac{8s_2s_4s_5s_6s_7s_8}{Q}\left(\frac{s_1}{s_1+s_3}\right)$$
$$\gamma_2 = \frac{8s_1s_3s_5s_6s_7s_8}{Q}\left(\frac{s_4}{s_2+s_4}\right), \gamma_4 = \frac{8s_1s_3s_5s_6s_7s_8}{Q}\left(\frac{s_2}{s_2+s_4}\right).$$
$$\gamma_5 = \frac{2s_1s_2s_3s_4s_6s_8}{Q}\left(\frac{s_7}{s_5+s_7}\right), \gamma_7 = \frac{2s_1s_2s_3s_4s_6s_8}{Q}\left(\frac{s_5}{s_5+s_7}\right)$$
$$\gamma_6 = \frac{2s_1s_2s_3s_4s_5s_7}{Q}\left(\frac{s_8}{s_6+s_8}\right), \gamma_8 = \frac{2s_1s_2s_3s_4s_5s_7}{Q}\left(\frac{s_6}{s_6+s_8}\right)$$
...........(7)

We now seek to show that for $u(x,y) \in R_h'$

$$|u(x,y) - \bar{u}(x,y)| \le \frac{4}{5}\max_{R_h}|e(x,y)| + \frac{h^2}{2}M_2 \ldots(8)$$

To do this, first consider the case one of the point, say (x+h1, y) is in S and the other points are not Hence s2=s3=…=s8=1, in this case e(x+h1, y)=0 since $\sum_{i=1}^{8}\gamma_i = 1$, and since

$$\gamma_2 + \gamma_4 = \frac{8s_1s_3s_5s_6s_7s_8}{Q} \le \frac{8s_2s_4s_5s_6s_7s_8}{Q}$$
$$= \gamma_1 + \gamma_3 \ldots(9)$$

we have $\gamma_1 + \gamma_3 \ge \frac{2}{3}$. Moreover, since

$$\frac{s_3}{s_1+s_3} \ge \frac{s_3}{s_1+s_3}, \ldots\ldots(10)$$

we have $\gamma_1 \ge \gamma_3$, and $\gamma_1 \ge -\frac{1}{5}$. Thus $\sum_{i=2}^{8}\gamma_i \le \frac{4}{5}$, then (8) holds.

Let us consider the case where two of the points are on S. There are essentially two different cases. In the first case (x+h₁,y) and (x-h₃,y) are in S and the other six points are not. We have $\gamma_1 + \gamma_3 \ge \gamma_2 + \gamma_4$. Since $\sum_{i=1}^{8}\gamma_i = 1$, and $\gamma_1 + \sum_{i=4}^{8}\gamma_i \le -\frac{2}{5}$. Thus we have

$$|u(x,y) - \bar{u}(x,y)| \le \frac{2}{5}\max_{R_h}|e(x,y)| + \frac{h_2}{2}M_2$$

.

A similar arrangement holds if (x,y+h₂) and (x,y-h₄) are in S, or (x+h₅,y+h₅) and (x-h₇, y-h₇), or (x-h₆, y+h₆) and (x+h₈, y-h₈) are in S. If (x+h₁, y) and (x-h₃, y) are in S we have $\gamma_1 \ge \gamma_3$, $\gamma_2 \ge \gamma_4$ and $\gamma_1 + \gamma_2 \ge \gamma_3 + \gamma_4$ so that $\gamma_1 + \gamma_2 \ge \frac{2}{5}$, and

$$\sum_{i=3}^{8}\gamma_i \le \frac{3}{5}\max_{R_h}|e(x,y)| + \frac{h_2}{2}M_2 \ldots\ldots(12).$$

Similarly in the other cases it holds if any two different points lies on S.

Next, let us consider the case where only one of the points, say (x+h₁, y) is not in S. Evidently

$$\gamma_1 = \frac{8s_2s_4s_5s_6s_7s_8}{Q}\left(\frac{s_3}{s_1+s_3}\right) \le \frac{2}{5}\left(\frac{s_1}{s_1+s_3}\right) \le \frac{1}{5} \ldots(13)$$

Since $8s_2s_4s_5s_6s_7s_8Q^{-1}$ is an increasing function of $s_2s_4s_5s_6s_7s_8$ and $s_2s_4s_5s_6s_7s_8 \le 1$, and since $s_3(1+s_3)^{-1}$ is an increasing function of s₃ and $s_3 \le 1$. Thus we have

$$|u(x,y) - \bar{u}(x,y)| \le \frac{1}{5}\max_{R_h}|e(x,y)| + \frac{h_2}{2}M_2 \ldots(14)$$

In similar discussion for three or more up to seven points do not lies on S.

Finally, if all eight points (x+h₁, y), …, etc are on S we have $|u(x,y) - \bar{u}(x,y)| \le \frac{h_2}{2}M_2$

Therefore, (14) holds in all cases.

We now let

$$v = \max_{R_h}|e(x,y)|, \mu = \max_{R_h'}|e(x,y)|$$

. Evidently, from (2.4.14) and the first part of the theorem we have

$$\nu \le \frac{h^2 r^2}{24} M_4 + \mu,\ \ \mu \le \frac{4}{5}\max(\nu,\mu) + \frac{h^2}{2} M_2 ....(16)$$

If $\mu \le \nu$, then $\mu \le \frac{4}{5}\nu + \frac{h^2}{2} M_2$, and

$$\nu \le \frac{h^2 r^2}{24} M_4 + \frac{4}{5}\nu + \frac{h^2}{2} M_2 ....(17)$$

or

$$\nu \le \frac{5h^2 r^2}{24} + \frac{5}{2} h^2 M_2 ...(18).$$

On the other hand, if $\mu \ge \nu$, then

$$\mu \le \frac{4}{5}\mu + \frac{h^2}{2} M_2 ...(19)$$

and

$$\mu \le \frac{5}{4} h^2 M_2 ...(20)$$

therefore, since $\mu \le \nu$ or else $\mu \le \frac{5}{2} h^2 M_2$,

we have

$$\max_{R_h} \left| u(x,y) - \bar{u}(x,y) \right| \le \frac{5h^2 r^2}{24} M_4 + \frac{5}{2} h^2 M_2 .$$

$$\le \left(\frac{5r^2}{24} M_4 + \frac{5}{2} M_2 \right) h^2 \le O(h^2) ......(21)$$

Implementation

In the following different examples that have curved boundary boundaries and the results are presented as shown in the following tables.

Example 1

The Poisson equation $\Delta u = f$ in the first quadrant bounded by the circle $X^2 + y^2 = 1$, where the exact solution and boundary conditions are given by $u(x, y) = \sin(\pi x)\sin(\pi y)$.

Step Length h	Maximum error in $\overline{H}$	Maximum error in $\overline{K}$
0.5	**1.0390e-01**	**3.24366e-02**
0.25	**3.1358e-02**	**2.36209e-02**
0.125	**8.9441e-03**	**3.72839e-03**
0.0625	**2.3840e-03**	**5.45875e-04**
003125	**6.0972e-04**	**7.02413e-05**
0.015625	**1.5111e-04**	**9.75254e-06**

Example 2

Quarter moon with the exact solution and the boundary conditions given by $u(x, y) = \sin(\pi x)\sin(\pi y)$.

Method of Solution	Maximum absolute error
$\overline{H}$	1.20457e-03
$\overline{K}$	2.787202e-04
PLTMG package (Using Finite Element Method)	7.54e-03

Example 3

Step Length h	Maximum error in $\overline{H}$	Maximum error in $\overline{K}$
0.25	**2.309123e-03**	**8.961037e-04**
0.125	**6.51036e-04**	**6.16542e-05**
0.0625	**1.34793e-04**	**6.107614e-06**
003125	**4.413789e-05**	**7.0241315e-06**
0.015625	**1.015908e-05**	**2.2342435e-07**

Laplace equation $\Delta u = 0$ in the first quadrant bounded by the circle $X^2 + y^2 = 1$, where the exact solution and the boundary conditions are given by the equation $u(x, y) = \exp(-2x)\cos(2y)$.

Conclusion

In this work the finite difference of the operator $\overline{K}$ be proved of order $O(h^2)$. This operator used in Laplace and Poisson equations with curved boundary condition, the results are very satisfactory as shown above tables.

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مؤثر لابلاس في نطاق غير منتظم

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الخلاصة

هدف هذا البحث هو برهان ان مؤثر لابلاس في نطاق غير منتظم يستعمل تسعة نقاط بأنه من الرتبة الثانية، قدمت بعض الأمثلة كتوضيح لهذه الطريقة.