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Pre Open Sets In Topological Spaces

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Abstract

This work consists of three sections. In section one we will study the properties of P-connected spaces [3] and we will show that the product of two P-connected spaces is P-connected. In section two we recall the definition of I- space [2] and we will introduce similar definition PI-space using pre open sets and also we will study some properties of this definition. In [8],[2] T_D-space and MI- space are studied respectively. In section three we introduce similar definition T_P- space using pre open sets. In particular we will prove that in MI-space the T_P- space and T_D-space are equivalent.

Introduction

The concept of pre open set in topological spaces was introduced in 1982[7]. This set was also considered in [6] and [4]. We recall the definition of connected spaces [5]. In section one we study similar definition using pre open sets which is called P-connected space [3] and we give several properties of this definition. Recall that a space is an I-space if each open subset is connected [2]. In section two of this paper we study similar definition PI-space using pre open sets and we study some properties of this definition. Also we recall that a space is T_D-space iff the set of limit points of any singleton is closed [8]. In section three we study similar definition T_P- space using pre open sets and we show that T_D-space and T_P- space are equivalent if the space is MI- space.

Section 1 : P-connected Spaces

Definition 1.1 [7]

Let X be a topological space and $A \subseteq X$. A is called pre open (P-open) in X iff $A \subseteq \overset{o}{\overline{A}}$. A is called p – closed iff A^c is P-open and it is easy to see that A is P-closed set iff $\overline{A^o} \subseteq A$. It is clear that every open set is P-open and every closed set is P-closed , but the converse is not true in general. The intersection of two P-open sets is not in general P-open , but the intersection of P-open set with open set is P-open. Also the union of any P-open sets is P-open.

Definition 1.2 [3, Definition 1.1.21]

Let X be a topological space and $A \subseteq X$. The P-closure of A is defined as the intersection of all P-closed sets in X containing A, and is denoted by $\overline{A}^P$. It is clear that $\overline{A}^P$ is P-closed set for any subset A of X.

Proposition 1.3 [3, Proposition 1.1.23]

Let X be a topological space and $A \subseteq B \subseteq X$.
Then:

(i) $\overline{A}^P \subseteq \overline{B}^P$

(ii) $A \subseteq \overline{A}^P$

(iii) A is P-closed iff $A = \overline{A}^P$

Definition 1.4

Let X be a topological space and $x \in X$, $A \subseteq X$. The Point x is called a P-limit point of A if each P-open set containing x, contains a point of A distinct from x. We shall call the set of all P-limit points of A the P-derived set of A and denoted it by A'^P. Therefore $x \in A'^P$ iff for every P-open set G in X such that $x \in G$ implies that $(G \cap A) - \{x\} \neq \phi$.

Proposition 1.5

Let X be a topological space and $A \subseteq B \subseteq X$.
Then:

(i) $\overline{A}^P = A \cup A'^P$

(ii) A is P-closed set iff $A'^P \subseteq A$

(iii) $A'^p \subseteq B'^P$

(iv) $A'^p \subseteq A'$

(v) $\overline{A}^P \subseteq \overline{A}$

Proof

(i) If $x \notin \overline{A}^P$, then there exists a P-closed set F in X such that $A \subseteq F$ and $x \notin F$. Hence $G = X - F$ is a P-open set such that $x \in G$ and $G \cap A = \phi$. Therefore $x \notin A$ and $x \notin A'^P$, then $x \notin A \cup A'^P$. Thus $A \cup A'^p \subseteq \overline{A}^P$. On the other hand, $x \notin A \cup A'^P$ implies that there exists a P-open set G in X such that $x \in G$ and $G \cap A = \phi$. Hence $F = X - G$ is a P-closed set in X

such that $A \subseteq F$ and $x \notin F$. Hence $x \notin \overline{A}^P$. Thus $\overline{A}^P \subseteq A \cup A'^P$.

Therefore $\overline{A}^P \subseteq A \cup A'^P$.

For (ii), (iii), (iv) and (v) the proof is easy.

Definition 1.6

Let X be a topological space. Two non- empty subsets A and B of X are called P-separated iff $\overline{A}^P \cap B = A \cap \overline{B}^P = \phi$.

Definition 1.7 **[3]**

Let X be a topological space. Then X is called P-connected space iff X can not be expressed as the union of two disjoint, P-open, non- empty subsets of X.

Remark 1.8 **[3]**

A set A is called P-clopen iff it is P-open and P-closed.

Theorem 1.9 **[3]**

Let X be a topological space, then the following statements are equivalent :

(i) X is a P-connected space.

(ii) X can not be expressed as the union of two disjoint, non- empty and P-closed sets.

(iii) The only P-clopen sets in the space are X and ϕ.

Theorem 1.10

Let X be a topological space, then the following statements are equivalent :

(i) X is a P-connected space.

(ii) X is not the union of any two P-separated sets.

Proof

$(\longrightarrow)$

Let A and B are two P-separated sets such that $X = A \cup B$. Therefore $\overline{A}^P \cap B = A \cap \overline{B}^P = \phi$. Since $A \subseteq \overline{A}^P$ and $B \subseteq \overline{B}^P$, then $A \cap B = \phi$. Now $\overline{A}^P \subseteq X - B = A$. Hence $\overline{A}^P = A$. Then A is P-closed set. By the same way we can show that B is P-closed set which is a contradicts with Theorem 1.9 (ii).

$(\longleftarrow)$

Let A and B are two disjoint non-empty and P-closed sets such that $X = A \cup B$. Then $\overline{A}^P \cap B = A \cap \overline{B}^P = A \cap B = \phi$ which is a contradicts with the hypothesis.

Remark 1.11 **[3]**

Every P-connected space is a connected space. But the converse is not true in general. For the example see[3].

Remark 1.12

If $\phi \neq A \subseteq (X,T)$, we call A a P-connected set in X whenever (A,T_A) is a P-connected space.

Example 1.13

In this example we show that P-connectivity is not a hereditary property. Let $X=\{a,b,c,d\}$ and $T_x=\{\{a\},\{a,b\},\{a,c\},\{a,b,c\},\phi,X\}$ be a topology on X. The P-open sets are: $\{a\},\{a,b\},\{a,c\},\{a,b,c\},\{a,d\},\{a,b,d\},\{a,c,d\},\phi,X$. It is clear that X is P-connected space since the only P-clopen sets are X and ϕ. Let $Y=\{b,c\}$, then $T_y=\{\{b\},\{c\},Y,\phi\}$. It is clear that Y is not P-connected space since $\{b\}\neq\phi$, $\{b\}\neq Y$ and $\{b\}$ is P-clopen set in Y. Thus a P-connectivity is not a hereditary property.

Proposition 1.14

Let A be a P-connected set and H,K are P-separated sets. If $A\subseteq H\cup K$ then either $A\subseteq H$ or $A\subseteq K$.

Proof

Suppose A is P-connected set and H,K are P-separated sets such that $A\subseteq H\cup K$. Let $A\not\subseteq H$ and $A\not\subseteq K$. Suppose $A_1=H\cap A\neq\phi$ and $A_2=K\cap A\neq\phi$. Then $A=A_1\cup A_2$. Since $A_1\subseteq H$, hence $\overline{A_1}^P\subseteq\overline{H}^P$. Since $\overline{H}^P\cap K=\phi$, then $\overline{A_1}^P\cap A_2=\phi$. Since $A_2\subseteq K$, hence $\overline{A_2}^P\subseteq\overline{K}^P$. Since $\overline{K}^P\cap H=\phi$, then $\overline{A_2}^P\cap A_1=\phi$. But $A=A_1\cup A_2$, therefore A is not P-connected space which is a contradiction. Then either $A\subseteq H$ or $A\subseteq K$.

Proposition 1.15

If H is P-connected set and $H\subseteq E\subseteq\overline{H}^P$ then E is P-connected.

Proof

If E is not P-connected, then there exists two sets A,B such that $\overline{A}^P\cap B=A\cap\overline{B}^P=\phi$ and $E=A\cup B$. Since $H\subseteq E$, thus either $H\subseteq A$ or $H\subseteq B$. Suppose $H\subseteq A$, then $\overline{H}^P\subseteq\overline{A}^P$, thus $\overline{H}^P\cap B=\overline{A}^P\cap B=\phi$. But $B\subseteq E\subseteq\overline{H}^P$,thus $\overline{H}^P\cap B=B$. Therefore $B=\phi$ which is a contradiction. Thus E is P-connected set. If $H\subseteq E$, then by the same way we can prove that $A=\phi$ which is a contradiction. Then E is P-connected.

Corollary 1.16

If a space X contains a P-connected subspace A such that $\overline{A}^P = X$, then X is P-connected.

Proof

Suppose A is a P-connected subspace of X such that $\overline{A}^P = X$. Since $A \subseteq X = \overline{A}^P$, then by proposition 1.15, X is P-connected.

Proposition 1.17

If A is P-connected set then $\overline{A}^P$ is P-connected.

Proof

Suppose A is P-connected set and $\overline{A}^P$ is not. Then there exist two P-separated sets H, K such that $\overline{A}^P = H \cup K$. But $A \subseteq \overline{A}^P$, then $A \subseteq H \cup K$ and since A is P-connected set then either $A \subseteq H$ or $A \subseteq K$ (by proposition 1.14)

(1) If $A \subseteq H$, then $\overline{A}^P \subseteq \overline{H}^P$. But $\overline{H}^P \cap K = \phi$, hence $\overline{A}^P \cap K = \phi$. Since $K \subseteq \overline{A}^P$, then $K = \phi$ which is a contradiction.

(2) If $A \subseteq K$, then the same way we can prove that $H = \phi$ which is a contradiction.

Therefore $\overline{A}^P$ is P-connected set.

Proposition 1.18

Let X be a topological space such that any two elements a and b of X are contained in some P-connected subspace of X. Then X is P-connected.

Proof

Suppose X is not P-connected space. Then X is the union of two P-separated sets A, B. Since A, B are non-empty sets, thus there exist a, b such that $a \in A$, $b \in B$. Let H be a P-connected subspace of X which contains a and b. Therefore by proposition 1.14 either $H \subseteq A$ or $H \subseteq B$ which is a contradiction since $A \cap B = \phi$. Then X is P-connected space.

Proposition 1.19

If A and B are P-connected subspace of a space X such that $A \cap B \neq \phi$, then $A \cup B$ is P-connected subspace.

Proof

Suppose that $A \cup B$ is not P-connected. Then there exist two P-separated sets H and K such that $A \cup B = H \cup K$. Since $A \subseteq A \cup B = H \cup K$ and A is P-connected, then either $A \subseteq H$ or $A \subseteq K$. Since $B \subseteq A \cup B = H \cup K$ and B is P-connected, then either $B \subseteq H$ or $B \subseteq K$.

(1) If $A \subseteq H$ and $B \subseteq H$, then $A \cup B \subseteq H$. Hence $K = \phi$ which is a contradiction.

(2) If $A \subseteq H$ and $B \subseteq K$, then $A \cap B \subseteq H \cap K = \phi$. Therefore $A \cap B = \phi$ which is a contradiction.

By the same way we can get a contradiction if $A \subseteq K$ and $B \subseteq H$ or if $A \subseteq K$ and $B \subseteq K$. Therefore $A \cup B$ is P-connected subspace of a space X.

Theorem 1.20

If X and Y are P-connected spaces, then $X \times Y$ is P-connected space.

Proof

For any points (x_1, y_1) and (x_2, y_2) of the space $X \times Y$, the subspace $X \times \{y_1\} \cup \{x_2\} \times Y$ contains the two points and this subspace is P-connected since it is the union of two P-connected subspaces with a point in common (by proposition 1.19). Thus $X \times Y$ is P-connected (by proposition 1.18).

Section 2 : PI- spaces

Definition 2.1 [2]

A topological space (X,T) is called I- space iff each open subset of X is connected.

Proposition 2. 2 [2]

The following statements are equivalent in any topological space (X,T):

(i) (X,T) is an I- space.

(ii) Every pair of non- empty open subsets of X has non- empty intersection.

Definition 2.3

A topological space (X,T) is called PI- space iff each P-open subset of X is P-
connected.

It is clear that every PI- space is I- space, but the converse is not true in general. In example 2.7 we will show that not every I-space is PI-space.

Theorem 2.4 [1, Theorem 1.8]

Let (Y,γ) be a subspace of a space (X,T). If $A \subset Y$ and $A \in PO(X,T)$, then $A \in PO(Y,\gamma)$.

Theorem 2.5 [1, Theorem 1.9]

Let (Y,T_y) is a subspace of (X,T), $y \in PO(X,T)$ and $A \in PO(y,T_y)$, then $A \in PO(X,T)$.

Theorem 2.6

The following statements are equivalent in any topological space (X,T):

(i) (X,T) is PI- space.

(ii) Every pair of non-empty P-open subsets of X has a non-empty intersection.

Proof

(i) $\longrightarrow$(ii)

Let A,B be a non-empty P-open subsets of X such that $A \cap B = \phi$. It is clear that $A \cup B$ is P-open subset in X and A,B are P-open subset in $A \cup B$ (by Theorem 2.4). Then $A \cup B$ is not P-connected set . Therefore $A \cap B \neq \phi$.

(ii) $\longrightarrow$(i)

Let A be a P-open subset of X such that $A = U \cup V$ where U,V are non-empty P-open subsets in A and $U \cap V = \phi$. Then U,V are P-open subset in X (by Theorem 2.5) which contradicts with the hypothesis. Thus A is P-connected set.

Example 2.7

Let $X = \{1,2,3\}$, $T = \{\phi, X, \{1,2\}\}$. The P-open sets are :
$\phi, X, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}$ It is clear that (X,T) is I- space, but it is not PI-space, since $\{1\},\{2\}$ are non-empty P-open subsets of X with an empty intersection.

Proposition 2. 8

Let (X,T) be a PI- space. If a non-empty subset A of X is P-open , then $\overline{A}^P = X$.

Proof

Let A be a P-open set such that $\overline{A}^P \neq X$. Then there exist a point $x \notin \overline{A}^P = A \cup A'^P$. Hence there exist a P-open set G contain x such that $G \cap A = \phi$ which is a contradicts with Theorem 2.6.

Theorem 2.9

Let (X,T) be a PI- space . Then the P-closure of the intersection of all non-empty P-open subsets of X is equal to X.

Proof

Let $C = \cap\{U : U$ is P-open in X and $U \neq \phi\}$. If C is non-empty, then C has a non-empty intersection with each non-empty P-open set in X. Therefore $\overline{C}^P = X$.

Section 3 : T_P - spaces

Definition 3.1 [8]

A topological space (X,T) is called T_D-space iff the set of limit points of any singleton is closed.
In a similar way we introduce the following:

Definition 3.2

A topological space (X,T) is called T_P - space iff the set of P-limit points of any singleton is P-closed.
We shall prove later that T_P-space and T_D-space are equivalent if the space is MI-space. The following example shows that the T_D-space the T_P-space are not equivalent in general.

Example 3.3

Let $X=\{a,b,c,d,e\}$ and $T=\{\{a\},\{c,d\},\{a,c,d\},\{a,b,d,e\},\{d\},\{a,d\},\phi,X\}$ be a topology on X. The P-open sets are: $\{a\},\{c,d\},\{a,c,d\},\{a,b,d,e\},\{d\},\{a,d\}$, $\{a,b,d\},\{a,d,e\},\{a,b,c,d\},\{a,c,d,e\},\phi,X$. It is clear that X is T_P- space. But X is not T_D- space since $\{b\}'=\{e\}$ and $\{e\}$ is not closed.

Definition 3.4

Let (X,T) be a topological space, then (X,T) is called $PT_o(resp.,PT_1)$ iff for every $x,y\in X$ such that $x\neq y$, there exists a P-open set containing x but not y or ($resp.$, and) a P-open set containing y but not x.

Proposition 3. 5

Let (X,T) be a topological space. If for every $x\in X$, $\{x\}'^P$ is P-closed set, then (X,T) is a PT_o - space.

Proof

Let $x,y\in X$ such that $x\neq y$. Then either $y\notin\overline{\{x\}}^P$, in which case $N_y=\overline{\{x\}}^{P^c}$ is a P-open set contain y which does not contain x. Or $y\in\overline{\{x\}}^P$, then $y\in\{x\}'^P$. Hence $N_x=\{x\}'^{P^c}$ is a P-open set which does not contain y. If $x\notin N_x$, then $x\in\{x\}'^P$. Hence for each G_x P-open set contain x, $(G_x\cap\{x\})-\{x\}\neq\phi$ which is a contradiction. Then N_x contain x. Therefore (X,T) is PT_0 −space.

Proposition 3. 6

Every T_P - space is PT_o - space.

Proof

This follows immediately from proposition 3.5.

Proposition 3. 7

A topological space (X,T) is PT_1- space iff $\{x\}=\overline{\{x\}}^P$ for each $x\in X$.

Proof

Let (X,T) be a PT_1- space and $x\in X$. If $y\in X-\{x\}$, then there exist P-open set V such that $y\in V$ and $x\in X-V$. Hence $y\notin\overline{\{x\}}^P$ and $\overline{\{x\}}^P=\{x\}$. Conversely, suppose that $\overline{\{x\}}^P=\{x\}$, for each $x\in X$. Let $y,z\in X$ with $y\neq z$. Then $\overline{\{y\}}^P=\{y\}$ implies that $\overline{\{y\}}^{P\,c}$ is P-open set contain z but not y. Also , $\overline{\{z\}}^P=\{z\}$ implies that $\overline{\{z\}}^{P\,c}$ is P-open set contain y but not z. Thus (X,T) is PT_1-space.

Proposition 3. 8

Every PT_1-space is a T_P-space.

Proof

In a PT_1-space X , $\overline{\{x\}}^P=\{x\}$ for all $x\in X$; hence $\{x\}'^P\subseteq\{x\}$. Therefore $\{x\}'^{P'^P}\subseteq\{x\}'^P$. Then $\{x\}'^P$ is P-closed. Hence the space is T_P-space.

Definition 3.9 [2]

An I- space (X,T) is called a maximal I- space if for any topology U on X such that $T\subset U$, then (X,U) is not an I- space. We will denote a maximal I- space briefly by MI- Space.

Proposition 3. 10 [2]

Let (X,T) be a MI-space. If (X,T) is not T_1-space then :

$(X,T)=\{A: x_0\in A$, for some $x_0\in X\}\cup\{\phi\}=\langle X,x_0\rangle$ for some $x_0\in X$.

Proposition 3. 11

Let (X,T) be a T_1- space. Then (X,T) is T_P- space iff it is T_D-space.

Proof

Let (X,T) be a T_1- space. Then $\{x\}' = \{x\}'^P = \phi$ for each $x \in X$. Then for each $x \in X$, $\{x\}'$ and $\{x\}'^P$ are closed and hence they are P-closed sets. Therefore (X,T) is T_P- space iff it is T_D-space.

Proposition 3. 12

Let (X,T) be a MI- space, and is not T_1- space. Then (X,T) is T_P- space iff it is T_D-space.

Proof

Let (X,T) be a MI- space which is not T_1- space. Then $(X,T) = \langle X, x_0 \rangle$ for some $x_0 \in X$ (by Proposition 3.10). Thus $\{x\}' = \phi$ for each $x \neq x_0$ and $\{x_0\}' = X - \{x_0\}$. Therefore $\{x\}'$ is closed for each $x \in X$. Since in this space the P-open sets are the same as the open sets, then $\{x\}'$ is closed iff $\{x\}'^P$ is P- closed for each $x \in X$. Therefore (X,T) is T_P-space iff it is T_D-space.

Theorem 3. 13

Let (X,T) be a MI- space. Then (X,T) is T_P- space iff it is T_D-space.

Proof

The Theorem follows immediately from Propositions 3.11 and 3.12.

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المجموعات الـ Pre – مفتوحة في الفضاءات التبولوجية

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الخلاصة

ان هذا البحث يتكون من ثلاثة فصول . في الفصل الاول سوف ندرس خواص الفضاءات الـ P – متصلة وسنبين ان حاصل الضرب الديكارتي لفضائين P- متصلين هو فضاء P - متصل . في الفصل الثاني نتذكر تعريف الفضاء الغير قابل للتجزئة (فضاء – I) وسوف ندرس تعريف مشابه لهذا التعريف باستعمال المجموعات الـ P- مفتوحة والذي نسميه بالفضاء- PI وكذلك سوف ندرس بعض الخواص لهذا التعريف . في الفصل الثالث نتذكر التعريفيين الفضاء T_D- والفضاء غير قابل للتجزئة اعظم (فضاء – MI) وسوف ندرس تعريف مشابه لتعريف فضاء - T_D باستعمال المجموعات الـ P - مفتوحة والذي نسميه بالفضاء - T_P.

بصورة خاصة سوف نبرهن انه اذا كان الفضاء – MI فان الفضاءين - T_P و - T_D يكونان متكافئين.