

POINT SPREAD FUNCTION FOR MULTI-RECTANGULAR APERTURE

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ABSTRACT

In this paper the point spread function (PSF) for multi-rectangular aperture with no aberration were found numerically with a Q-basic computer program and analytically by deriving suitable equations. And by comparing the two results they were shown a good coincidence which assure the trueness of the computer program.

The values of PSF of the single rectangular aperture was compared with that of single square aperture previously found , and it is found that the PSF for a rectangular aperture has become wider by square root of aspect ratio value in one direction and narrower by the inverse of the square root of the aspect ratio value in the other direction.

The PSF of multi-rectangular aperture were also computed when the focus error were found with a values of 0.25 and 0.5, and it found that the value of PSF of the peak have been decreased by 73.6% of its original value with no aberration for the value of the focus error $w_{20}=0.25$ and by 28% of its original value for $w_{20}=0.5$.

الخلاصة

في هذا البحث تمت دراسة دالة الانتشار النقطية للفتحة المنفردة والمتعددة المستطيلة والخالية من الزيغ عددياً وباستخدام برنامج الحاسوب (Q- basic) وتحليلياً باستخدام المعادلات المناسبة التي تم اشتقاقها ، وبمقارنة القيم المستخرجة بالطريقتين فإن التوافق الشديد للقيم دل على صحة البرنامج المطبق.

أن قيم دالة الانتشار النقطية للفتحة المنفردة المستطيلة قد تمت مقارنتها بالقيم المستخرجة سابقاً للفتحة المنفردة المربعة وقد لوحظ أن دالة الانتشار النقطية تقل بمقدار الجذر التربيعي لنسبة ضلعي المستطيل في أحد الاتجاهين وتزداد بمقدار معكوس الجذر التربيعي لنسبة ضلعي المستطيل بالاتجاه الآخر .

أن دالة الانتشار النقطية للفتحة المنفردة والمركبة المستطيلة وبوجود الخطأ البؤري قد تم حسابها أيضاً ووجد أن قيمة دالة الانتشار النقطية قد هبطت إلى 73.6% من قيمتها الأصلية بدون زيوغ عند وجود الخطأ البؤري $w_{20}=0.25$ وهبطت إلى 28% من قيمتها الأصلية عند $w_{20}=0.5$

1- INTRODUCTION

The image formed by an ideal, perfect lens would be a precise point– by– point reproduction of the object. The failure of an actual lens to achieve such perfection is due to diffraction by the lens aperture, aberrations, and errors in constructions.

The image of a point source object formed by a lens is Known as the Point Spread Function (PSF) of the lens, which describes the luminance blur associated with the image of a point source. It is the graphical or mathematical representation of the image luminance as a function of a lateral distance in the image plane from the location of the ideal image of a point source.

There were many researchers studied the (PSF) for different apertures, the first one who studied the intensity distribution for the image of a point source with no aberration by using a circular aperture was Airy in 1834 [1]. While the study of the influence of aberration on the image of a point source were studied by Strehl [2]. The PSF for apertures other than circular were also studied like square, rectangular, and also annular were studied by Ali H. alhamdani[3]. Also the PSF for a multi- apertures were studied, like the multi circular apertures which studied by Adnan F.[4] and multi square apertures which studied by Sundus Y[5].And in this paper the PSF for multi rectangular apertures were studied.

2-THEORY

The PSF intensity can be expressed as[6,7]:

$$F(u, v) = \left(A \int_y \int_x f(x, y) e^{ik(ux+vy)} dx dy \right)^2 \quad (1)$$

where A is the normalization constant, u and v are the image coordinates, x and y are the coordinates of the interance pupil, and f(x,y) is the exit pupil function and is given by:

$$f(x, y) = \tau(x, y) e^{ikW(x,y)} \quad (2)$$

where the term $\tau(x,y)$ is the transmission function and $W(x,y)$ is the aberration function.

2.1- POINT SPREAD FUNCTION FOR A RECTANGULAR APERTURE

For a rectangular aperture of an area equal to π the limits of integration will be from -1 to 1 for y and from $-\pi/4$ to $\pi/4$ for x, so equation (1) will be[1]:

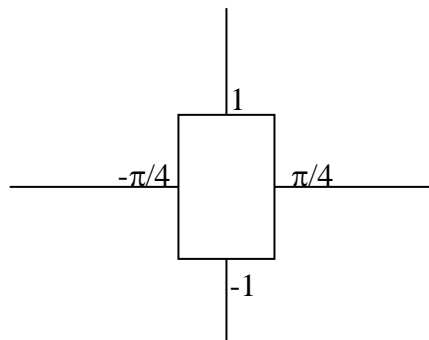


Fig.(1) A rectangular aperture of an area equal to π .

$$F(u, v) = \left(A \int_{-1}^1 \int_{-\pi/4}^{\pi/4} \tau(x, y) e^{ik(W(x, y) + ux + vy)} dx dy \right)^2 \quad (3)$$

and the normalization function can be easily found by putting $F(u, v) = 1$ when u and v equal to zero and with no aberration, hence A will be equal to $1/\pi$, then the normalized intensity point spread function is equal to:

$$F(u, v) = \left(\frac{1}{\pi} \int_{-1}^1 \int_{-\pi/4}^{\pi/4} \tau(x, y) e^{ik(W(x, y) + ux + vy)} dx dy \right)^2 \quad (4)$$

The solution of the above equation when there is no aberration is [8]

$$F(u, v) = \text{sinc}^2(ku\pi/4) \text{sinc}^2(kv) \quad (5)$$

If there is a sub-division of small rectangular apertures each with an area equal to π/N (fig.2 for example), where N is the number of apertures, equation (4) becomes [9]:

$$F(u, v) = \left(\frac{1}{\pi} \sum_n \int_{-\frac{1}{\sqrt{N}}}^{\frac{1}{\sqrt{N}}} \int_{-\frac{\pi}{4\sqrt{N}}}^{\frac{\pi}{4\sqrt{N}}} \tau(x, y) e^{ik(W(x, y) + u(x + X_n) + v(y + Y_n))} dx dy \right)^2 \quad (6)$$

where X_n, Y_n are the coordinates of the origin of the sub-apertures with respect to the origin of the aperture and now x and y refers to the coordinates of a point in the sub-aperture with respect to its origin.

The above equation can be simplified to be:

$$F(u, v) = (c^2 + s^2)(c'^2 + s'^2) \quad (7)$$

where

$$\left. \begin{aligned} c &= \frac{1}{\pi} \int_{-1}^1 \int_{-\pi/4}^{\pi/4} \cos(k(ux + vy)) dx dy \\ s &= \frac{1}{\pi} \int_{-1}^1 \int_{-\pi/4}^{\pi/4} \sin(k(ux + vy)) dx dy \\ c &= \sum_n \cos(k(uX_n + vY_n)) \\ c' &= \sum_n \sin(k(uX_n + vY_n)) \end{aligned} \right\} \quad (8)$$

2.2- POINT SPREAD FUNCTION FOR A SQUARE APERTURE

The PSF for a square aperture of an area equal to π were studied before [5] using equation (6) with the limits of integration from $-\pi^{1/2}/2$ to $\pi^{1/2}/2$ for both x and y coordinates:

$$F(u, v) = \left(\frac{1}{\pi} \sum_n \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \tau(x, y) e^{ik(W(x, y) + u(x + X_n) + v(y + Y_n))} dx dy \right)^2 \quad (9)$$

2.3- ABERRATION

For a perfect lens and monochromatic point source, the wave aberration W measures the optical path difference of each ray compared to that of the principal ray and it depends on the lens aperture radius ρ , azimuth angle ϕ , and the object field size h . The wave aberration polynomial in polar coordinates is [10]:

$$W = W(\sigma^2, \rho^2, \rho \cos \phi) = \sum_l \sum_m \sum_j W_{mj} \sigma^l r^m \cos^j \phi \quad (10)$$

where r and σ are the fractional values: $r = \rho/\rho_{\max}$ and $\sigma = h/h_{\max}$ with ρ_{\max} represents the radius of the exit pupil and h_{\max} represents the off axis image height at the edge of the field of view while l , m , and j represent a power of σ , r , and $\cos \phi$ respectively.

The first five aberrations of the polynomial are called Seidel aberration [11] or primary aberration, which are spherical aberration, Comma, Astigmatism, Field curvature and distortion.

2.3.1- SPHERICAL ABERRATION

This aberration is an on-axis and symmetrical aberration which causes a change in the focal length by changing the height of a non-paraxial ray, this aberration is given by the equation [12]:

$$W = \sum_{m=even} W_{m0} r^m = W_{40} r^4 + W_{60} r^6 + \dots \quad (11)$$

where the term $W_{20} r^2$ is not appeared in the above equation because it represents the focus error aberration which is not considered with the spherical aberration, and when a system has a focus error it is not a reason to have a spherical error, but when it has a spherical aberration, the focus error always found with it. The focus error can be represented by [12]

$$W = W_{20} r^2 = W_{20} (x^2 + y^2) \quad (12)$$

The above equation was used in this research for the aberration function in equations (4) and (6).

3- PRESENT WORK

In this research the point spread function for a single and multi-rectangular aperture of an area equal to π with no aberration and with focus error equal to 0.25 and 0.5 were found as follows:

3.1-PSF for a single rectangular aperture with no aberration

The values of PSF for a single rectangular aperture system with no aberration is found i) numerically by programming Equation (4) with a Q-basic program by using Gaussian method, which depends on the irregular distribution of points which have unequal weights in the integration range, to solve the integrands in this equation and ii) analytically using equation (5). These results were compared and showed a complete coincidence as it is shown in figure(2) which is drawn for $v=0$, i.e. in one-dimension..

To make a comparison between these values and that for a single square aperture with a same area figure(3) is drawn in u-direction ($v=0$), and figure (4) is drawn in v-direction ($u=0$), and it can be noticed that the PSF for a rectangular aperture is wider than that for a square aperture in v-direction and it is thinner in u-direction.

In three dimensions, the diffraction pattern is drawn with a mathcad program as in figure(5) which shows the diffraction pattern for a) a rectangular aperture and b) a square aperture and figure (6) was drawn to show the two patterns together with the dotted lines represent the values of PSF for a square aperture and the solid lines for the rectangular aperture.

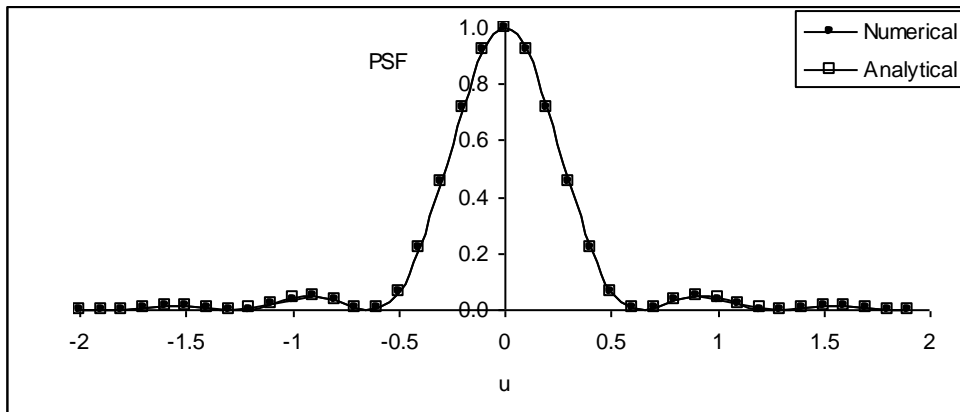
3.2- PSF for multi-rectangular aperture with no aberration

The PSF for different multi aperture systems were studied numerically by programming equation(6) and analytically using equation(7) which consists of two parts, the solution of the first part, which is $(c'^2+s'^2)$, is $\text{sinc}^2(ku\pi/4N^{1/2}) \text{sinc}^2(kv/N^{1/2})$, while the second part which is $(c'^2+s'^2)$ is calculated using these equations:

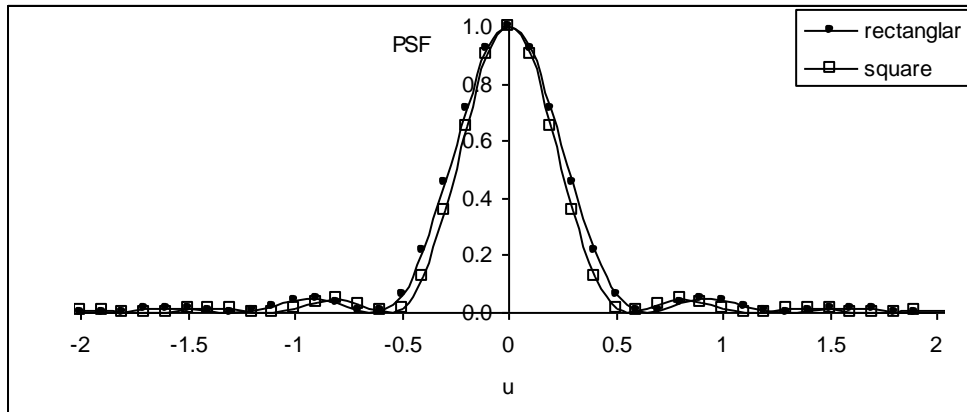
$$c' = \sum_n \cos(2\pi(uX_n + vY_n))$$
$$s' = \sum_n \sin(2\pi(uX_n + vY_n))$$

and $X_n = R \sin(2\pi n/N) \quad Y_n = R \sin(2\pi m/N) \quad (13)$

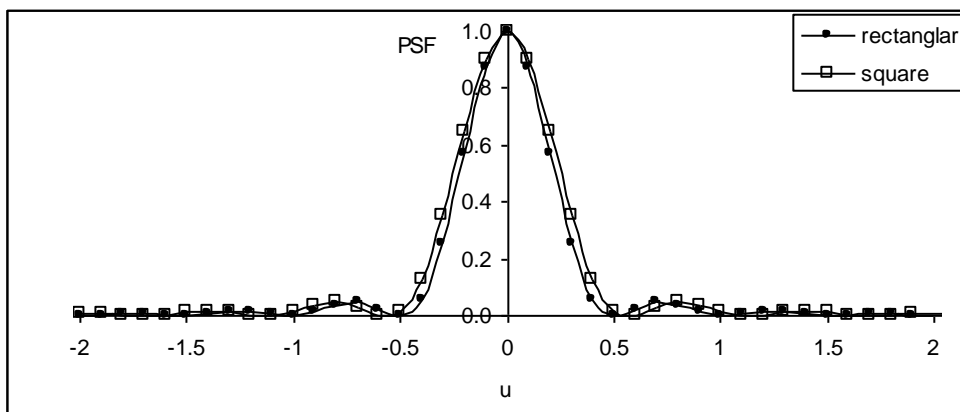
where R is the distance from the centre of the sub-aperture to the centre of the system.



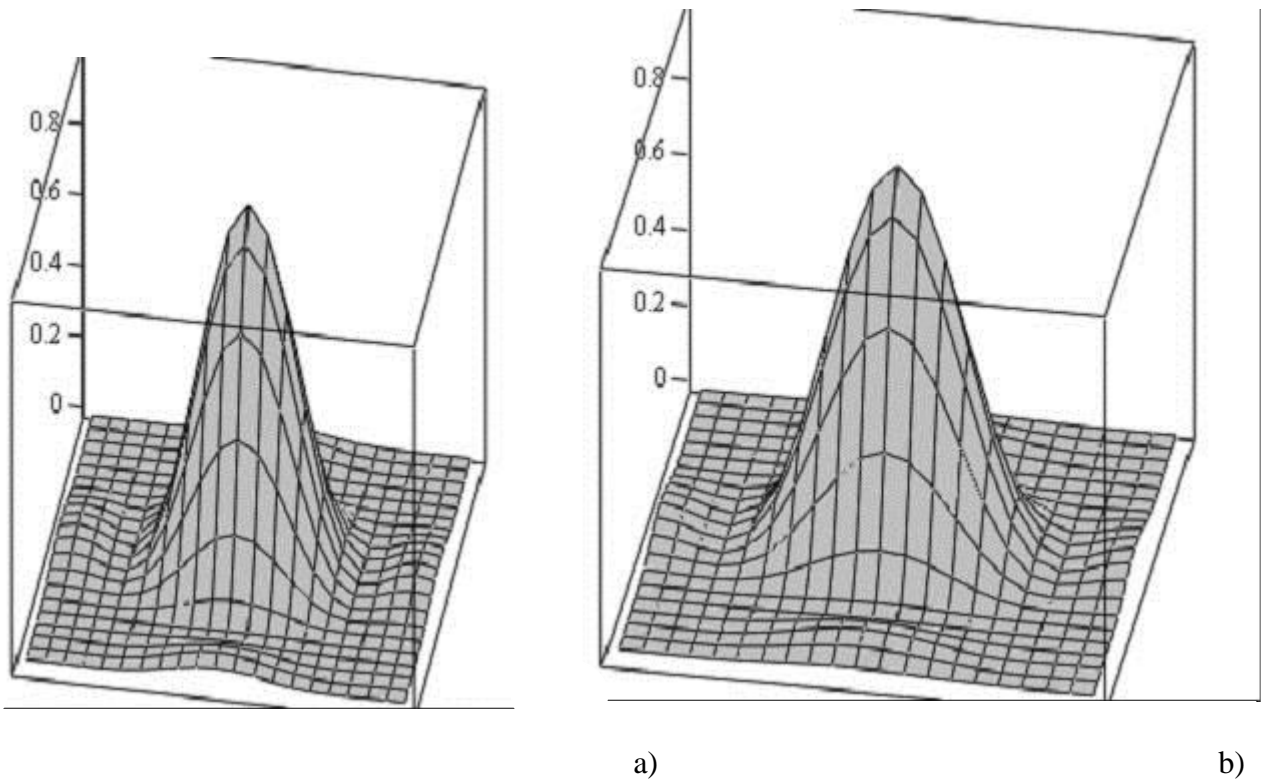
Figure(2) : represents the numerical and analytical PSF values in 1-D for a single rectangular aperture of an area equal to π .



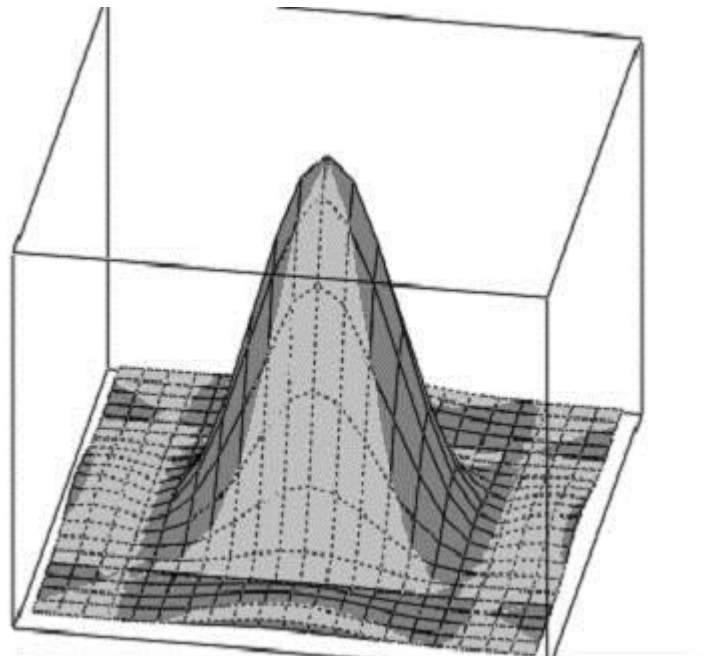
Figure(3) : represents a comparison of the PSF values in u-direction for a single rectangular aperture and a single square aperture of an area equal to π .



Figure(4) : represents a comparison of the PSF values in v-direction for a single rectangular aperture and a single square aperture of an area equal to π .



Figure(5) : represents a comparison of the PSF values in 3-dimension for a) a single rectangular aperture and b) a single square aperture of an area equal to π .



Figure(6) : represents the two previous patterns together with dotted lines represent the values of PSF for a single square aperture and the solid lines represent the values for a single rectangular aperture.

For the number of apertures (N)=4

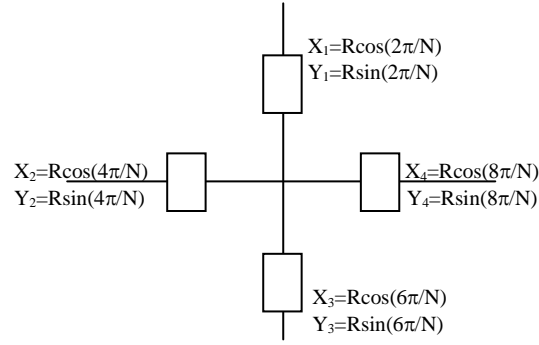


Figure (7): four rectangular aperture

so

$$c' = 2\cos(2\pi Rv/N^{1/2}) + 2\cos(2\pi Ru/N^{1/2}), \quad s' = 0 \quad (14)$$

For N=6

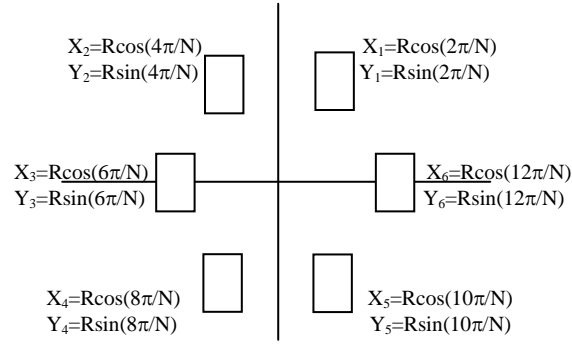


Figure (8): six rectangular aperture

$$c' = 2\cos(2\pi Ru/N^{1/2}) + 4\cos(\pi R(u + 3^{1/2}v)/N^{1/2}) \quad s' = 0 \quad (15)$$

For N=8

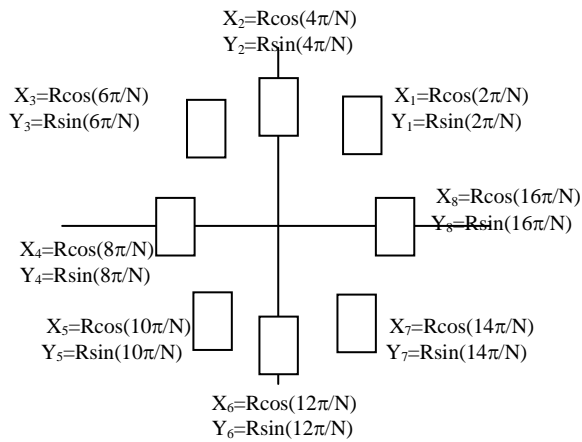


Figure (9): Eight rectangular aperture

$$c' = 2\cos(2\pi Rv/N^{1/2}) + 2\cos(2\pi Ru/N^{1/2}) + 4\cos(2^{1/2}\pi R(u+v)/N^{1/2}) \quad s' = 0$$

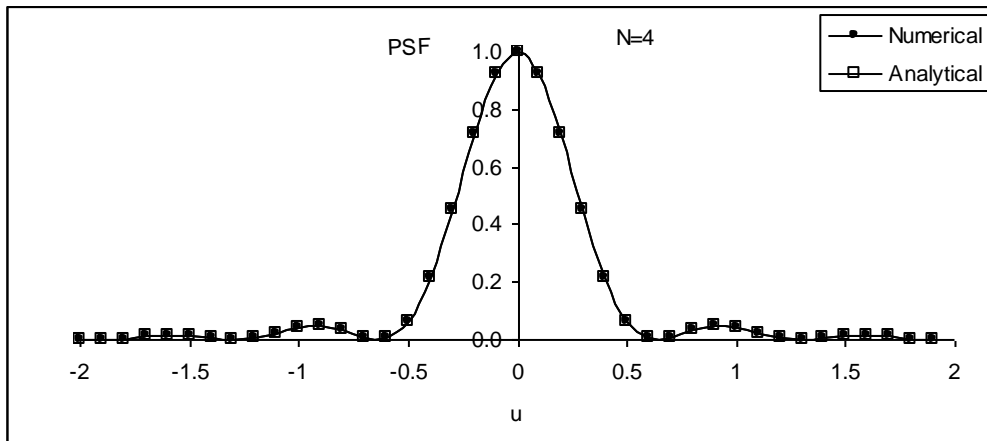
where s' for all the above configuration (figures 7-9) is equal to zero because the function of sine cancels each other due to the symmetrical configuration of the apertures.

Figures (10-12) shows the coincidence of the analytical values of PSF calculated by the above equations and the numerical one for multi-rectangular apertures ($N=4,6,8$), while figure(13) represents the values of PSF for the different sub-apertures together and it is obvious that the full width half maximum is smaller for $N=4$, and this result is similar to that obtained before for a square aperture of the same area[5].

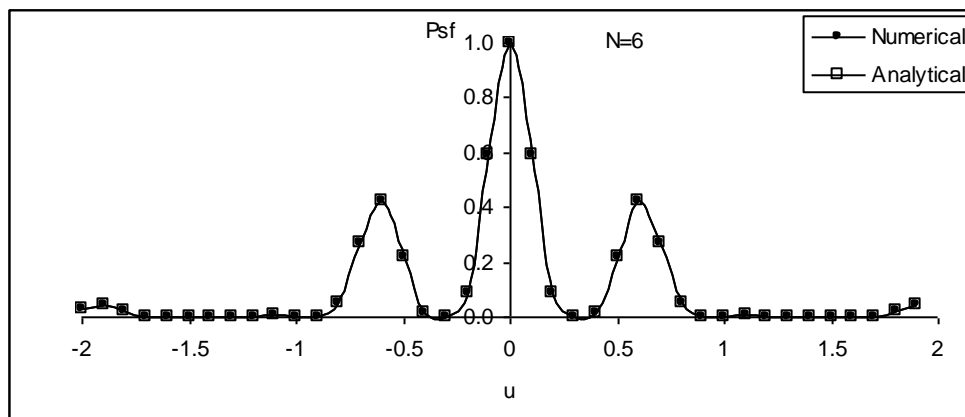
3.3-PSF FOR SINGLE AND MULTI-RECTANGULAR APERTURES WITH PRESENCE OF FOCUS ERROR

When a focus error presents the equations of PSF become difficult to solve analytically, so these values were computed numerically with two values of focus error W_{20} , and figures (14)and(15) represent the PSF for different N with focus error W_{20} equal to 0.25 and 0.5 respectively, and its shown that the value of the peak became less by 0.736 for $W_{20}=0.25$ and by 0.28 for $W_{20}=0.5$.

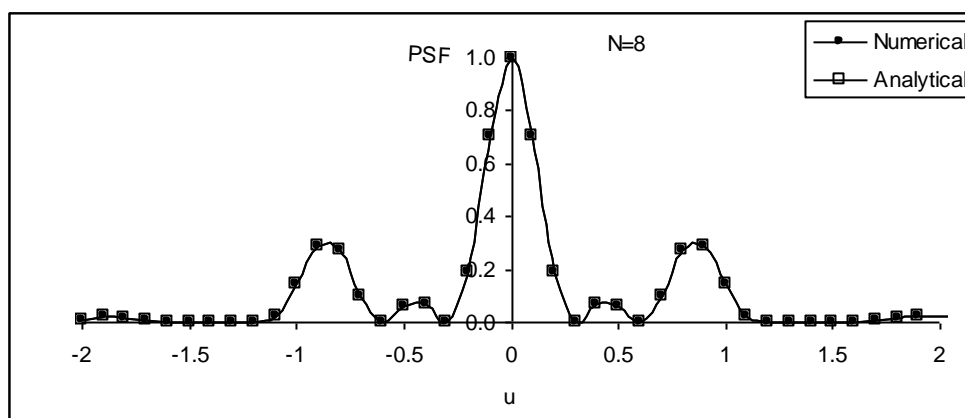
The percent of lowering of the peak of PSF with the two values of W_{20} were compared with those of the square aperture of the same area in figures (16) and (17) and found that the peak for the rectangular aperture is lower than that for the square aperture which means that the effect aberration on the square aperture is less than that on the rectangular aperture.



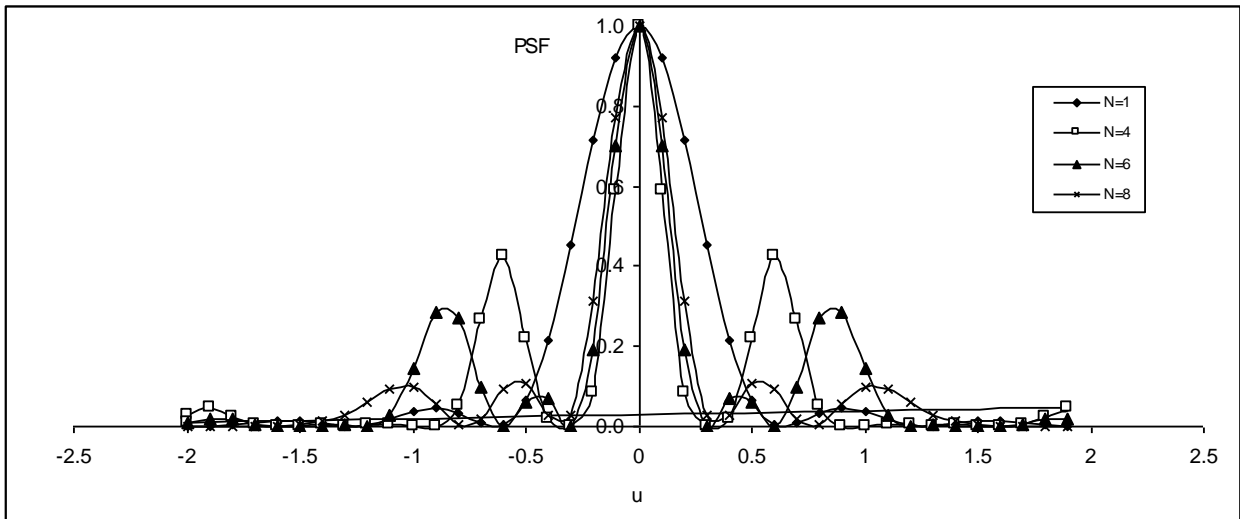
Figure(10) : represents the analytical and numerical values of PSF for $N=4$



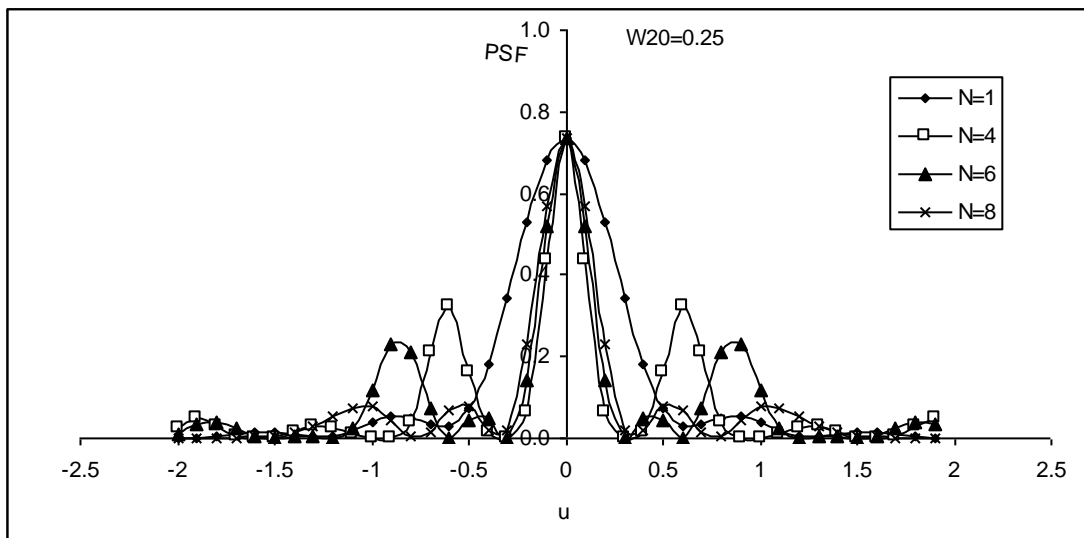
Figure(11) : represents the analytical and numerical values of PSF for $N=6$



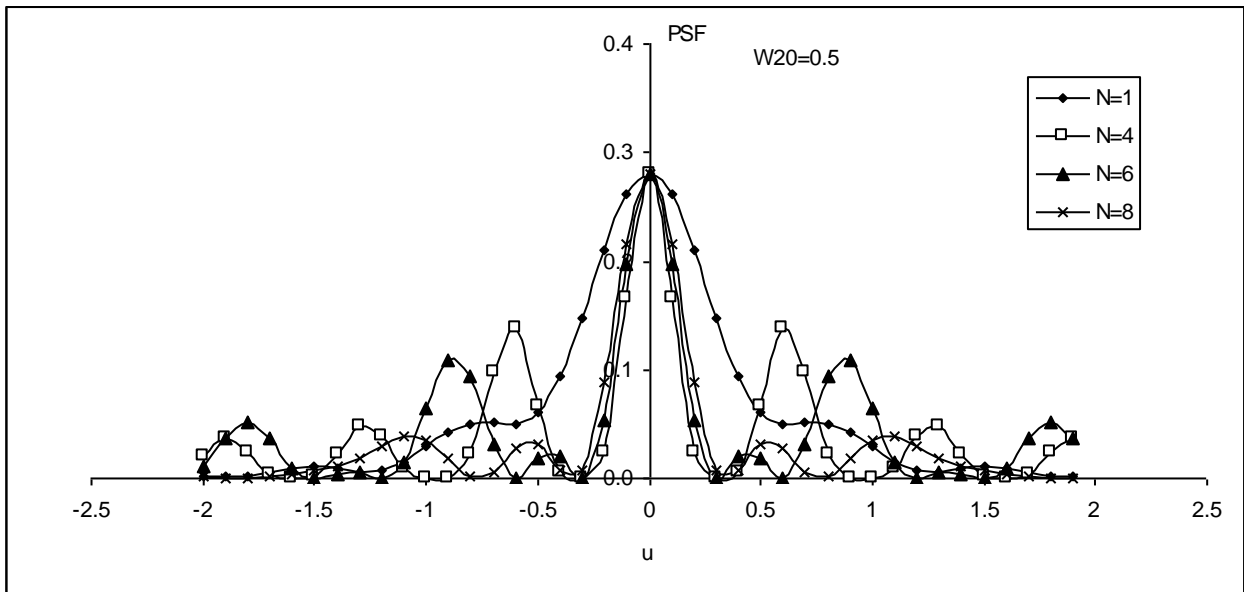
Figure(12) : represents the analytical and numerical values of PSF for $N=8$



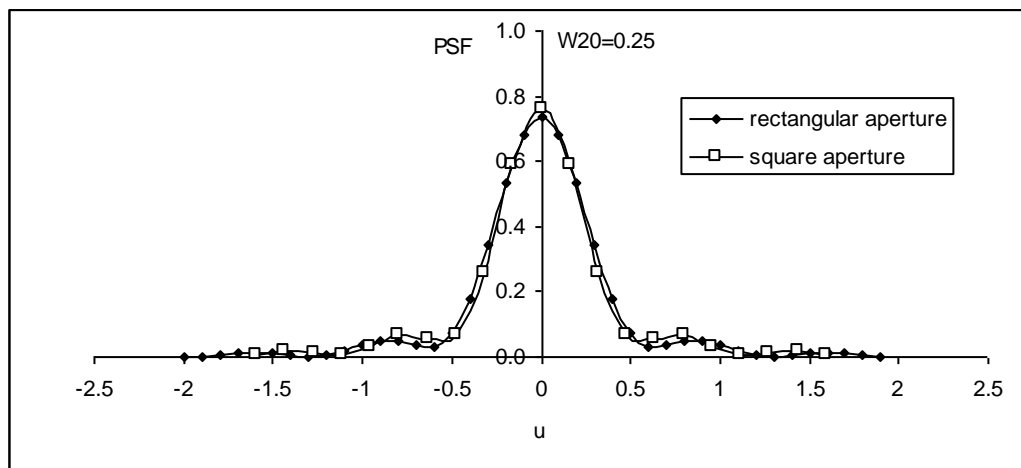
Figure(13) : represents the values of PSF for different rectangular sub-apertures.



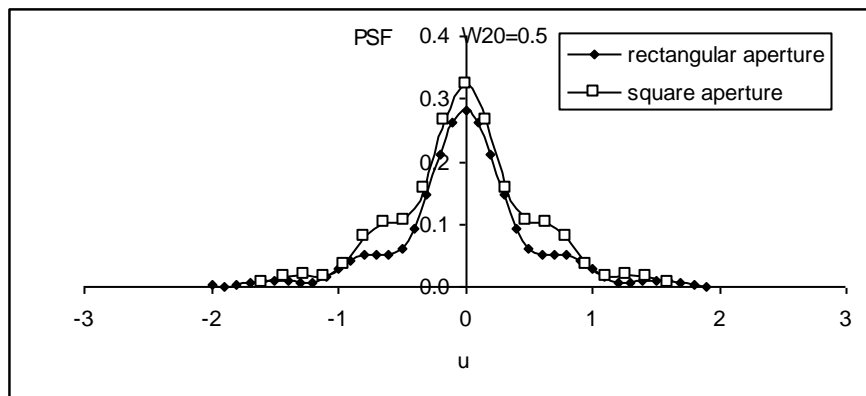
Figure(14) : represents the values of PSF for different rectangular sub-apertures with $W20=0.25$.



Figure(15) : represents the values of PSF for the rectangular aperture with $W20=0.5$



Figure(16) : represents the values of PSF for the rectangular and square apertures with $W20=0.25$



Figure(17) : represents the values of PSF for the rectangular and square apertures with $W20=0.5$

4- DISCUSSION

The PSF of a rectangular aperture with an area equal to π is similar to that of a square aperture but the width of the central peak being wider for a rectangular aperture than that of a square aperture in u-direction and it is thinner in v- direction, and the amount of increase and decrease can be known as follows

For the rectangular aperture

$$F(u,v)=\text{sinc}^2(2\pi ua) \text{sinc}^2(2\pi vb) \quad \text{where} \quad a=\pi/4 \quad \text{and} \quad b=1$$

and for the square aperture

$$F(u,v)=\text{sinc}^2(2\pi ud) \text{sinc}^2(2\pi vd) \quad \text{where} \quad d= \pi^{1/2}/2$$

so for $v=0$

$$F(u,v)=0 \text{ when the angle equal to } \pi, -\pi$$

$$\text{hence } \text{sinc}^2(2\pi ua)=\sin^2(2\pi ua)/2\pi ua=0 \rightarrow (2\pi ua)=\pi \rightarrow u=1/2a=2/\pi$$

i.e. the width of the central peak of PSF for a rectangular aperture with an area equal to π in u-direction = $4/\pi$

$$\text{and } \text{sinc}^2(2\pi ud)=\sin^2(2\pi ud)/2\pi ud=0 \rightarrow (2\pi ud)=\pi \rightarrow u=1/2d=1/\pi^{1/2}$$

i.e. the width of the central peak of PSF for a square aperture with an area equal to π in u-direction = $2/\pi^{1/2}$

$$\text{so the width of the rectangular aperture is wider by the amount } \frac{4/\pi}{2/\sqrt{\pi}} = \frac{2}{\sqrt{\pi}} = \sqrt{r}$$

where r is the aspect ratio a/b

and in the same way in v direction i.e. when $u=0$ then $F(u,v)=0$ when $v=1/2$

i.e. the width of the central peak of PSF for a rectangular aperture with an area equal to π in v-direction = 1

$$\text{so the width of the rectangular aperture is thinner by the amount } \frac{1}{2/\sqrt{\pi}} = \frac{\sqrt{\pi}}{2} = 1/\sqrt{r}$$

From the above discussion it can be concluded that the resolution for the rectangular aperture is better than that for a square aperture with the same area in one-direction and it is better for the square aperture in the other direction.

CONCLUSIONS

- I. The resolution for the rectangular aperture is better than that for a square aperture with the same area in one-direction and it is better for the square aperture in the other direction.
- II. The resolution of the aperture is better when it consists of sub-apertures and the best resolution is obtained when the number of sub-apertures equal to 4, and this is the same result for the square sub- apertures.
- III. The central peak of the PSF becomes lower when a focus error exist and as the value of the focus error increases the peak decreases.

- IV.** The central peak of the PSF when there exist a focus error for a rectangular aperture is lower than that for a square aperture with the same area and the same amount of focus error. Which means that the effect of the aberration on the square aperture is less than that on the rectangular aperture.

5- REFERENCES

- [1] Jenkins A. and White e. "fundamentals of optics " third ed. Mc Graw-Hill Japan (1957).
- [2] Zajac A. "Optics" Addison – Wesley Publishing Company Inc. U.S.A.(1974).
- [3] Al-Hamdani A. H. ,Ph.D. thesis "Numerical Evaluation of Lenses Quality Using Computer Software " Al-Mustansiriyah University (1997)
- [4] Falih A. Ph.D. thesis "Improvement of Resolving Power of Optical System Design by Using Circular Synthetic Aperture " Al-Mustansiriyah University (2004).
- [5] Hasan S.Y." Computational Image Evaluation of Multi-square Aperture Optical systems " Ph.D. thesis, Al- Mustansiriya university (2005).
- [6] Hussein S. A. " Design and Evaluation of Image Quality for IR Scanner" Ph. D. Thesis, Military College of Engineering (2001).
- [7] Jeremy N., Vandebei R., Spergel D., and Littman M. "Extrasolar Planet Finding via Optimal and Shaped Pupil Coronagraphs" AAs Latmacros (2002).
- [8] Born M. and Wolf E."Principle of Optics " 6th ed. Pergamon Press ,Oxford (1970).
- [9] Ditchburn R.W. " Light "Blackie & son Limited, London (1952) .
- [10] Kingsliake R. " Applied Optics and Optical Engineering " volume 8 ,Academic Press, New York (1980).
- [11] Sears F. W. "Optics" 3rd. ed. Eddisin –Wesley Publishing Company,INC(1974).