## The Steiner System S(5,6,12) Constructed From Projective Planes PG (2,3)

## إنشاء نظام شتايز S(5,6,12) من مستويات ألإسقاط من المرتبة الثالثة


*Esam Refeeq* — *Technical Inst. Of Kurkuk* — *Computer Center*

*Habeeb Kereem AL- Bederee* — *Dep. Of Mathemation* — *College of Education* — *Kufa University*

*Sttar Hamrrd AL- Janabi* — *Dep. Of Mathematics* — *College of Education* — *AL- Qadisiya University*



## Abstract

The aim of this work is to construct the Steiner system S(5,6, 12) , from the twelve points of four affine planes $\Pi^{\ell_1}, \Pi^{\ell_2}, \Pi^{\ell_3}$ *and* $\Pi^{\ell_4}$ , derived from $\Pi = PG(2,3)$ (the projective plane of order 3 ) by deleting four different lines that have a common point.

## المستخلص

هدف هذا البحث إنشاء نظام شتايز S(5,6,12) ، من النقاط الاثنى عشر للمستويات الأفنية $\Pi^{\ell_1}$ ، $\Pi^{\ell_2}$، $\Pi^{\ell_3}$ ، $\Pi^{\ell_4}$، المأخوذة من $\Pi = PG(2,3)$ (مستوي الإسقاط من المرتبة الثالثة) ، بحذف أربعة خطوط مختلفة ومتقاطعة في نقطة واحدة.


## Introduction

A Steiner system S(t,k,v) is an incidence structure of v point, together with a collection of subsets (blocks) of k points, such that any t-distinct points lies in exactly one of these blocks.

The Steiner system S(5,6,12) can be constructed by extending S(2,3,9) (the affine plane of order 3) three times, [6].

Also the Stiener system S(5,6,12) can be constructed by extending the Stiener system S(4,5,11), which is uniquely extendable, [5].

In section 1 of this work, we derive four different affine planes AG(2,3) each associated with a special deleted line.

The points of these affine planes are associated with the twelve points of the Stiener system S(5,6,12). Also we fined the number of the three non- collinear points contained in each of them.

In section 2, we construct the Stiener system S(5,6,12) and classify the hexads of S(5,6,12) into four different types with respect to the triads. For this purpose, we find it convenient to present results showing by Kadir [4] and then we present ou results.

## 1- Affine Planes Associated With $\Pi = PG(2,3)$

Let $\Pi$ be a projective plane of order 3 , then $\Pi$ will have 13 points, and let us call them {1,2,…,13}, and 13 lines, each point is on lines and each line contains 4 points as shown below (Hughes and Piper, 1973) :
{1 2 3 13}, {1 4 7 10}, {1 5 8 11}, {1 6 9 12}, {4 5 6 13}, {2 5 9 10}, {2 6 7 11}, {2 4 8 12}, {7 8 9 13}, {3 6 8 10}, {3 4 9 11}, {3 5 7 12}, {10 11 12 13}.

Now, take any point, say 13. So 13 is on four lines each of which contains 3 distinct points called triads; and the fourth point 13, as it is shown in the diagram below:

| | | | |
|---|---|---|---|
| 1 | 4 | 7 | 10 |
| 2 | 5 | 8 | 11 |
| 3 | 6 | 9 | 12 |

13

Let us denote these four lines by $\ell_1, \ell_2, \ell_3$ *and* $\ell_4$ respectively, and their triads by $t_1, t_2, t_3$ *and* $t_4$ respectively.
On the deletion of a line $\ell_1$ say, we will be left with 9 points namely {4,5,6,…,12} and 12 lines represented by 3×3 array $A_1 = \Pi^{\ell_1}$ below:

$A_1 = \Pi^{\ell_1}$

| $t_1$ | 10 | 11 | 12 |
|---|---|---|---|
| 4 | 7 | 9 | 8 |
| 5 | 9 | 8 | 7 |
| 6 | 8 | 7 | 9 |

There the rows are labeled by the points of the triad $t_2$ and the columns by the points of the triad $t_4$ and entries from $t_3$, such that i in the (m,n) th position indicates that {m,i,n} is a line in $A_1$. Thus the 12 lines of $A_1$ are the nine entries of the array above with the diagonal line of the array, i.e {7,8,9}, the row {10,11,12} and the column {4,5,6}.

A similar situation occurs when $\ell_2, \ell_3$ *and* $\ell_4$ are deleted to get $A_2 = \Pi^{\ell_2}, A_3 = \Pi^{\ell_3}$ and $A_4 = \Pi^{\ell_4}$ , $A_1, A_2, A_3$ *and* $A_4$ are all together contains 12 points of $\Pi$, except the point 13 which was already omitted.

These 12 points are used as 12 points of the Stiener system S(5,6,12).

**(1.1):** Here we find the number of the three non- collinear points in each of $A_1, A_2, A_3,$ *and* $A_4$. And we constructed the 132 hexads contained in S(5,6,12).

**(1.2): Theorem:** In any of the affine planes $A_1, A_2, A_3,$ *and* $A_4$, there are exactly 18 three non- collinear points.

**Proof**: there are $\binom{3}{1}\binom{3}{1}\binom{3}{1} = 27$ possibilities to choose one point from each of the three remaining triads in each of the affine planes $A_1, A_2, A_3,$ *and* $A_4$. But nine of them are lines of $A_1, A_2, A_3,$ *and* $A_4$, respectively. Hence there are 27-9=18 possibilities which are the three non- collinear points in each of the affine planes $A_1, A_2, A_3,$ *and* $A_4$.

We shall arrange the 18 three non- collinear points of $A_1, A_2, A_3,$ *and* $A_4$ in two arrays as show in (1.2.1), and they will be denoted as follows:

1. For $A_1$ *by* $A_1'$ *and* $A_1''$,
2. For $A_2$ *by* $A_2'$ *and* $A_2''$,
3. For $A_3$ *by* $A_3'$ *and* $A_3''$,
4. For $A_4$ *by* $A_4'$ *and* $A_4''$.

| $t_1$ | 10 | 11 | 12 |
|---|---|---|---|
| 4 | 7 | 9 | 8 |
| 5 | 9 | 8 | 7 |
| 6 | 8 | 7 | 9 |

$A_1$

| | 10 | 11 | 12 |
|---|---|---|---|
| 4 | 8 | 7 | 9 |
| 5 | 7 | 9 | 8 |
| 6 | 9 | 8 | 7 |

$A_1'$

| | 10 | 11 | 12 |
|---|---|---|---|
| 4 | 9 | 8 | 7 |
| 5 | 8 | 7 | 9 |
| 6 | 7 | 9 | 8 |

$A_1''$

| $t_2$ | 10 | 11 | 12 |
|---|---|---|---|
| 1 | 7 | 8 | 9 |
| 2 | 9 | 7 | 8 |
| 3 | 8 | 9 | 7 |

$A_2$

| | 10 | 11 | 12 |
|---|---|---|---|
| 1 | 8 | 9 | 7 |
| 2 | 7 | 8 | 9 |
| 3 | 9 | 7 | 8 |

$A_2'$

| | 10 | 11 | 12 |
|---|---|---|---|
| 1 | 9 | 7 | 8 |
| 2 | 8 | 9 | 7 |
| 3 | 7 | 8 | 9 |

$A_2''$

| $t_3$ | 10 | 11 | 12 |
|---|---|---|---|
| 1 | 4 | 5 | 6 |
| 2 | 5 | 6 | 4 |
| 3 | 6 | 4 | 5 |

$A_3$

| | 10 | 11 | 12 |
|---|---|---|---|
| 1 | 5 | 6 | 4 |
| 2 | 6 | 4 | 5 |
| 3 | 4 | 5 | 6 |

$A_3'$

| | 10 | 11 | 12 |
|---|---|---|---|
| 1 | 6 | 4 | 5 |
| 2 | 4 | 5 | 6 |
| 3 | 5 | 6 | 4 |

$A_3''$

| $t_4$ | 7 | 8 | 9 |
|---|---|---|---|
| 1 | 4 | 5 | 6 |
| 2 | 6 | 4 | 5 |
| 3 | 5 | 6 | 4 |

$A_4$

| | 7 | 8 | 9 |
|---|---|---|---|
| 1 | 5 | 6 | 4 |
| 2 | 4 | 5 | 6 |
| 3 | 6 | 4 | 5 |

$A_4'$

| | 7 | 8 | 9 |
|---|---|---|---|
| 1 | 6 | 4 | 5 |
| 2 | 5 | 6 | 4 |
| 3 | 4 | 5 | 6 |

$A_4''$

(1 -2 – 1)

## 2- Construction of hexads

Here we show how to form the hexads on the 12 points of $A_1, A_2, A_3, \textit{ and } A_4$. There are 132 hexads in the Seiener system S(5,6,12), [4].

(i) There are 9.4 = 36 hexads by joining the nine lines of each the entries of the arrays $A_1, A_2, A_3, \textit{ and } A_4$ with the deleted triads $t_1, t_2, t_3 \textit{ and } t_4$ respectively.

(ii) In the arrays $A_1, A_2, A_3, \textit{ and } A_4$ there are 12 -3 =9 , (i.e except the triads) pair of parallel lines. Each pair of parallel lines form a hexad; so there are 9.4 =36 hexads.

(iii) There are 6 hexads formed from any two complete triads.

(iv) Let $\ell$ be any line in the entries of $A_1, I = 1,2,3,4$. Join $\ell$ to the set of three non- collinear points in $A_j' \textit{ and } A_j''$;. $j = 1,2,3,4$ such that $i \neq j$, with the following conditions, to a void the repeated hexads.

**(2.1) Conditions:**

1) The three non- collinear points must not intersect $\ell$ even in $\Pi$ .

2) The three non- collinear points must not intersect the lines parallel to $\ell$ in $A_2$ in two points.

Following the conditions (2.1), we get 54 different hexads.

Finally, adding the number of the hexads in (i), (ii), (iii) and (iv), we get 36 +36+6+54= 132 hexads.

**(2.2) Types of hexads:**

Considering the triads $t_1, t_2, t_3 \textit{ and } t_4$ the hexads are of four different types and as follows:

**Type (i)**

36 hexads formed from any complete triad, and point from each of the three remaining triads.

**Type (ii)**

36 hexads of the form: two distinct point from three different triads.

**Type (iii)**

6 hexads formed from any two complete triads.

**Type (iv)**

54 hexads of the form: two distinct points of the triads, and one point from each of remaining two triads.

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