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Right Derivations on σ-Prime Rings

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Abstract

Let R be a 2-torsion free σ-prime ring in the present paper it is shown that if R is non-commutative, then any Right derivation of R which commutes with σ is zero. Moreover, if d is an a addative mapping of R into it self commuting with σ and satisfying $d(u^2) = 2d(u)u$ for all u in a non zero σ-square closed Lie ideal U of R, Then $U \subseteq Z(R)$ or $d(U) = 0$.

الخلاصة:

لتكن Rحلقة اوليه من نوع σ طليفة الالتواء من النمط 2 في هذا البحث المقدم بينّا بأنه، اذا كانت R حلقة ليست ابداليه فان أي اشتقاق من اليمين على R والذي يتبادل مع σيكون صفري بالأضافه الى ذلك اذا كانت D أي تطبيق جمعي من R الى R يتبادل مع σ ويحقق العلاقة $d(u^2)=2d(u)u$ لكل $u \in U$ و U مثالي لي غير صفري تحقق العلاقه $u^2 \in U$ لكل $u \in U$ فان $d(U) = 0$ او $U \subseteq Z(R)$.

1-Introduction

Throughout this paper, R will represent an associative ring with center Z(R). R is said to be 2-torsion free if whenever 2x=0, with $x \in R$, then x=0. As usual the commutator xy-yx will be denoted by $[x, y]$. We shall use basic commutator identities $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$. If R has an involution σ, we set $Sa_{\sigma}(R) := \{r$ in R such that σ(r)=$\pm$ r}. Recall that R is σ-prime if aRb=σ(a) Rb =0 implies that a=0 or b=0.
One can easily see that every prime ring having an involution σ is a σ-prime ring but the converse is in general not true. A Lie ideal of R is an additive subgroup U of R such that $[U, R] \subset U$. A Lie ideal U of R is called a σ-square closed Lie ideal if $u^2 \in U$ for all $u \in U$ and U invariant under σ.An additive mapping d:R→R is called right derivation if $d(xy) = d(y)x + d(x)y$ and called Jordan right derivation if $d(x^2) = 2d(x)x$, every right derivation is Jordan right derivation but the converse need not be true in general.

Theorem 1.1:

Let R be a 2-torsion free σ-prime ring and d be a right derivation which commutes with σ. Then either $d = 0$ or R is a commutate ring.

Proof:

$d(xyz) = d(z)xy + d(xy)z = d(z)xy + d(y)xz + d(x)yz$.................(1)

on the other hand

$d(xyz) = d(yz)x + d(x)yz = d(z)yx + d(y)zx + d(x)yz$.................(2)

comparing (1) and (2), we conclude

$d(y)[z, x] = d(z)[x, y]$...(3)

Take in (3) $y = z$

$d(z)[z, x] = d(z)[x, z]$

$d(z)[zx - xz - xz + zx]$

$d(z)[2zx - 2xz] = 0 \Rightarrow 2d(Z)[zx - xz] = 0$ *Since Char.* $R \neq 2$ *we find*

$d(z)[z, x] = 0$...(4)

replace x by rx in (4)

$d(z)r[z, x] + d(z)[r, x]z = 0$

$d(z)R[z, x] = 0$...(5)

Hence for $z \in Sa\sigma(R)$ we have either $z \in Z(R)$ or $d(z) = 0$. Let $y \in R$, the fact that $y + \sigma(y) \in Sa\sigma(R)$
assures that $y + \sigma(y) \in Z(R)$ or $d(y + \sigma(y)) = 0$
if $d(y + \sigma(y)) = 0$, then $d(y) \in Sa_{\sigma}(R)$ and in view of (5) this yields $d(y) = 0$ or $y \in Z(R)$. Now suppose that $y + \sigma(y) \in Z(R)$ if $y - \sigma(y) \in Z(R)$, then $2y \in Z(R)$ so that $y \in Z(R)$
if $d(y - \sigma(y)) = 0$, then $d(y) \in Sa_{\sigma}(R)$ and (5) leads to $d(y) = 0$ or $y \in Z(R)$. In conclusion, for all $y \in R$, we have either $y \in Z(R)$ or $d(y) = 0$. Thus R is union of two additive subgroups K and L

where $K=Z(R)$ and $L=\{y\in R/d(y)=0\}$ if $d\neq 0$ then $R=Z(R)$ proving that R is commutative ring.

Lemma 1.2:
If $U\not\subset Z(R)$ is a σ-Lie ideal of a 2-torsion free σ-prime ring and $a,b\in R$ s.t $aUb=\sigma(a)Ub=0$
Then a=0 or b=0
Proof:
$\exists M$ ideal s.t $[M,R]\subset U$ But $[M,R]\not\subset Z$
if $u\in U,\ m\in M\ \ and\ \ y\in R$ then $[mau,y]\in[M,R]\subseteq U$. Then
$a[mau,y]b=a[ma,y]ub+ama[u,y]b=0$
We get $a[ma,y]ub=0$ since $amyub-aymaub=0$
$amayub=0$
thus $aMaRUb=0$ since R is σ-prime ring
$\sigma(a)MaRUb=0$ if $a\neq 0$
we obtain $Ub=0$, so if $x\in R,u\in U$ then $(ux-xu)\in U$. Hence $(ux-xu)b=0$ and so $uxb=0\Rightarrow uRb=0$, if $u\neq 0$ obtain $b=0$.

Lemma 1.3:
Let R be a 2-torsion free ring and let U be a square closed Lie ideal of R if $d:R\rightarrow R$ is an additive mapping satisfying $d(u^2)=2d(u)u$ for all $u\in U$ then
(i) $d(uv+vu)=2d(v)u+2d(u)v$
(ii) $d(uvu)=d(v)u^2+3d(u)vu-d(u)uv$
(iii) $d(uvw+wvu)=d(v)(uw+wu)+3d(w)vu+3d(u)vw-d(w)uv-d(u)wv$
(iv) $d(u)u[v,u]=d(u)[v,u]u$
(v) $[d(vu)-d(v)u-d(u,v)][v,u]=0$ for all $u,v,w\in U$.

Lemma 1.4:
Let R be 2-torsion free ring and let U be asquare closed Lie ideal of R if $d:R\rightarrow R$ is an additive mapping satissfing $d(u^2)=2d(u)u$ for all $u\in U$ .Then
(i)d[u,v][u,v]=0 (ii)$d(v)(u^2-2uvu+vu^2)=0$

Lemma 1.5:
Let $U\neq 0$ be a σ-Lie ideal of 2-torsion free σ-prime ring R. if $[U,U]=0$,
Then $U\subseteq Z(R)$
Proof:

Let $u \in U$, since $[u, rt] \in U \quad \forall r, t \in R$ it follows that $[u,[u, rt]] = 0$. Therefore $u[u, rt] = [u, rt]u$ so that

$ur[u,t] + u[u,r]t = r[u,t]u + [u,r]tu$

Using the fact that $u[u,r] = [u,r]u$ and $[u,t]u = u[u,t]$, we obtain

$ur[u,t] + [u,r]ut = ru[u,t] + [u,r]tu$

In such a way that $2[u,r][u,t] = 0$. As char. $R \neq 2$, this leads to $[u,r][u,t] = 0$ for all $r,t \in R$, replace *r* by *rz* in this equality, where $z \in R$, we find that.

$[u,r]Z[u,t] = 0$ and thus $[u,r]R[u,t] = 0, \forall u \in U, \quad r,t \in R$

for $u \in U \cap Sa\sigma(R)$ we then have $[u,r]R[u,t] = 0 = \sigma([u,r])R[u,t], \forall r,t \in R$ and the σ-prime ness of R forces $u \in Z(R)$ which proves $U \cap Sa\sigma(R) \subseteq Z(R)$

Now, let $u \in U$; since char. $R \neq 2$ and $u + \sigma(u), u - \sigma(u) \in U \cap Sa\sigma(R)$, then $u \in Z(R)$ so that $U \subseteq Z(R)$.

Theorem: (1-6)

Let R be a 2-torsion free σ-prime ring and U a σ-square closed Lie ideal of R. If $d : R \to R$ is an additive mapping commutative with σ and satisfying $d(u^2) = 2d(u)u$ for all $u \in U$, then either $U \subseteq Z(R)$ or $d(U) = 0$.

Proof:

Suppose that $U \not\subset Z(R)$, by lemma 1.4 (ii) we have

$d(v)(u^2 v - 2uvu + vu^2) = 0, \forall u, v \in U$..........(6)

Linear zing (6) in u, by $(u + w)$ and use (6) we get

$d(v)(u^2 v + uwv + wuv + w^2 v - 2uvu - uvw - 2wvu - 2wvw + vu^2 + vw^2 + vuw + vwu)$

$d(v)(uwv + wuv - 2uvw - 2wvu + vuw + vwu) = 0,$ *for all* $u, v, w \in U$................(7)

Taking $w = v = [x, y]$ in (7), where $x, y \in U$ and using lemma 1.4 (i), we conclude that $d[x,y]u[x,y]^2 = 0$ which would force $d([x,y]) \cup [x,y]^2 = 0$ for all $x, y \in U$

Let $x, y \in U \cap S_\sigma(R)$, using lemma 1.2 either $d[x,y] = 0$ or $[x,y]^2 = 0$

Suppose that $[x,y]^2 = 0$, write $[x,y]$ instead of v in (7) by lemma 1.4 (i), we find that

$d([x,y])\{-2u[x,y]w - 2w[x,y]u - [x,y]uw + [x,y]wu\} = 0$ for all $u, w \in U$...............(9)

replace u by $2u[x,y]$ in (9) and once again using lemma 1.4 (i), as $[x,y]^2 = 0$, we obtain $2d([x,y])w[x,y]u[x,y] = 0$, as char $R \neq 2$, this leads to

$d[x,y]U[x,y]u[x,y] = 0$ for all $u \in U$.

The fact that $x, y \in Sa_\sigma(R)$ together with $\sigma(U) = U$ assure that

$d[x,y]U[x,y]u[x,y] = d[x,y]U\{\sigma\}[x,y]u[x,y]\} = 0$ for all $u \in U$

applying lemma 1.2 we get $d([x,y])=0$ or $[x,y]U[x,y]=0$ for all $u\in U$, if $[x,y]U[x,y]=0$, then $[x,y]=0$ and therefore $d([x,y])=0$ which leads us to conclude that

$d([x,y])=0, \textit{ for all } x,y\in U\cap Sa_{\sigma}(R)$

Let $x,y\in U$, since

$d([x+\sigma(x),y+\sigma(y)])=0=d([x+\sigma(x),y-\sigma(y)])$

then $d([x+\sigma(x),y])=0$, on the other hand.

$d([x-\sigma(x),y+\sigma(y)])=0=d([x-\sigma(x),y-\sigma(y)])$,

so that $d([x-\sigma(x),y])=0$, therefore $d([x,y])=0$ for all $x,y\in U$. Hence $d(xy)=d(x)y+d(y)x$ for all $x,y\in U$ by Lemma (1.3) (i) for $x,y\in U$, we have

$d(x^2y)=d(y)x^2+d(x^2)y=d(y)x^2+2d(x)xy$

since $d(yx^2)=d(x)yx+d(yx)x=d(x)yx+d(x)yx+d(y)x^2=2d(x)yx+d(y)x^2$

it then follows that $d(x)[x,y]=0$ for all $x,y\in U$, because char $R\neq 2$.

Replace y by *2vy*, where $u\in U$, and once again using char $R\neq 2$, we find that $d(x)V[x,y]=0$ and therefore $d(x)U[x,y]=0$ for all $x,y\in U$ as $\sigma(U)=U$, by Lemma 1.2 for all $x\in U\cap Sa_{\sigma}(R)$ either $[x,u]=0$ or $d(x)=0$ since $x+\sigma(x), x-\sigma(x)\in U\cap Sa_{\sigma}(R)$ for all $x\in U$, we easily deduce that $[x,U]=0$ or $d(x)=0$ for all $x\in U$, because $do\sigma=\sigma od$. Consequently U is union of two additive subgroups L and k, where $L=\{x\in U/[x,u]=0\}$ and $k=\{x\in U/d(x)=0\}$ since $U\not\subset Z(R)$, by lemma 1.5 $U\neq L$ and therefore $U=k$, proving $d(U)=0$.

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