On Uniqueness of Equivalent Martingale Measure

By

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Abstract

Let S= $\{S_t: t \geq 0\}$ is an adapted sequence of random variables defined on filtered probability space $(\Omega, F, \{F_t\}, P)$.

In this paper provides a detailed proof of that result as well as the following:-

No free arbitrage is necessary and sufficient condition to the existence of new measure Q equivalent to measure P such that S is martingale with respect to Q, but this measure not unique, also we give example to explain this condition not sufficient to get unique measure, but no free arbitrage as well as extension property we have uniqueness

1. Introduction

A stochastic process $S=(S_t)_{t\geq 0}$ is a family of random variables defined on probability space (Ω, F, P) .

We say that S is adapted if $S(t)\in F_t$ (i.e. S(t) is F_t – measurable for each t).

A stochastic process $S=(S(t))_{0\leq t<\infty}$ is martingale to (F,P) if :-

1- S is adapted and $E|S(t)|<\infty$, $\forall 0\leq t<\infty$.

2- $E[S(t)| F_s]=S(s)$ $(0\leq s<t)$.

A probability measure Q on (Ω,F_t) equivalent to P is called a martingale measure for S if the process S follows a Q-martingale with respect to the filtration F.

We denoted by $M^e(S)$ the class of equivalent martingale measures.

2. Definitions and notations

We denote by Θ the set of all element H ,where $H=(H(t))_{t\in I}$ is R^{d+1}-valued stochastic process with company $H_i(t)$ for i=1,2,3 ..,d and satisfies:-

1-H_t is an adapted to $\{F_t\}_{t\in I}$ $\forall i=1,2,3 ..,d$

2-$E(H_i(t)\, S_i(t))^2<\infty$ $\forall 0\leq t<\infty$ $\forall i=1,2,3 ..,d$

3-There exists a finite integral k and sequence $0=t_0<\ldots<t_k=T$ such that $t_n\in I$ and H(t,w) is constant over the interval $[t_{n-1},t_n]$for every w, (n=1,2,…,k)

4-$\forall$ n =1,2, ..,k ,we have

$$<H(t_{n-1}),S(t_n)>=<H(t_n),S(t_n)>$$

where < , > denoted the Euclidean inner product.

Remark (2.1):

It is clear that Θ is a subspace of the space of valued stochastic process.

We associate with H.

$$V^H=(V^H(t))_{0<t<\infty} \text{ by } V^H(t)=V^H(0)+\int_0^t H(u)\, dS(u).$$

Now, if we take X the space of all random variable defined on (Ω, F, p) equipped with norm topology.

Let M_a be subset of X defined by:

$M_a=\{x\in X: a + V^H(T) \leq x$, for some $H\in\Theta, a\in R\}$.

i.e. $M_0=\{x\in X: V^H(0)+V^H(T) \leq x$, for some $H\in\Theta\}$

where $a=V^H(0)$

Define $\pi: M_0 \rightarrow R$ by $\pi(x)=V^H(0)$

Proposition (2.2) :

1-M_a is convex set .

2-π is linear function .

proof :

1-let x, y $\in M_a$ and $0\leq\lambda\leq 1$

x $\in M_a$ then $\exists\ H_1\in H \ni a_1+ v^H{}_1(t)\leq x$

y $\in M_a$ then $\exists\ H_2\in H \ni a_2+ v^H{}_2(t)\leq y$

$\lambda(a_1+ v^H(t))= \lambda a_1+ v^{\lambda H}(t)\leq \lambda x$

$(1-\lambda)(a_2+ v^H(t))=((1-\lambda)a_2+v^{(1-\lambda)H}{}_2(t))\leq(1-\lambda)y$

Since $\lambda a_1+ (1-\lambda) a_2 \in R$

$$\therefore v^{\lambda H}{}_1(t) + v^{(1-\lambda)H}{}_2(t)=\int_0^t \lambda H_1(t)dS(u) + \int_0^t (1-\lambda) H_2(t)dS(u)$$

$$=\int_0^t \lambda(H_1(t)+(1-\lambda)H_2)dS(u) = v^{\lambda H1+(1-\lambda)H2}(t)$$

$\therefore\ \lambda x+(1-\lambda)y\geq(\lambda a_1+(1-\lambda)a_2+v^{\lambda H1+(1-\lambda)H2}(t)$

$\therefore\ \lambda x+(1-\lambda)y\in M_a$

2-let x, y $\in M_0$, let $\lambda, \beta\in R$

x $\in M_0$ then $\exists H_1 \in\Theta \ni x \geq V^H{}_1(0)+ V^H{}_1(t)$.

$y \in M_0$ then $\exists H_2 \in \Theta \ni y \geq V^{H}{}_2(0) + V^{H}{}_2(t)$.

$\lambda x+\beta y \geq V^{\lambda H_1 + \beta H_2}(0) + V^{\lambda H_1 + \beta H_2}(t)$

$\pi(\lambda x+\beta y) = V^{\lambda H_1 + \beta H_2}(0) = V^{\lambda H_1}(0) + V^{\beta H_2}(0)$

$= \lambda V^{H}{}_1(0) + \beta V^{H}{}_2(0)$

$= \lambda\pi(x) + \beta\pi(y)$.

$\therefore \pi$ is linear functional.

3.Free – arbitrage .

We started this section by the following definition:-

Definition (3.1) :

Let $X_+=\{x \in X: P\{V^H(t) \geq 0\}=1$ and $P\{V^H(t) > 0\}>0\}$.

A free –arbitrage is an element $H \in \Theta$ such that

$P(V^H(0)=0)=1$ and $V^H(t) \in X_+$.

We denote $M^a(S)$ the set of all (not necessarily equivalent) martingale probability measures. The letter a stands for "absolutely continuous with respect to P .

And let $C_0=\{x \in X: \exists H \in \Theta \ni V^H(t) \geq x\}$.

Lemma (3-2) :

For a probability Q on (Ω, F) the following are equivalent:-

1-$Q \in M^a(s)$.

2-$E_Q(x) = 0 \ \forall x \in M_0$.

3- $E_Q(y) \leq 0 \ \forall y \in C_0$.

Proof: $(1) \rightarrow (2)$

let Q be absolutely continuous of P such that S is martingale with respect to Q .

let $x \in M_0$ then $\exists H \in \Theta \ni x=V^H(t)$

$\therefore$ $E(\Delta S_t | F_t)=0 \Rightarrow (d(\Delta S_t) | F_t)=0$

$\therefore$ $\int_0^t Hu\ E(d(\Delta S_u) | F_u)=0$

$\therefore$ $E \int_0^t Hu\ (d(\Delta S_u) | F_u)=0$

$\therefore$ $E \int_0^t Hu\ (d(\tilde{S}_u) | F_u)=0$

$\therefore$ $E_Q(V^H(t)=0$

$\therefore$ $E_Q (V^H(t))=0$ for some $H \in \Theta$

$\therefore$ $E_Q (x)=0$.

Proof: $(2) \rightarrow (3)$

let $y \in C_0$ then $\exists\ H \in \Theta \ni V^H(\in) \geq y$

$E_Q(V^H(t)) \geq E_Q (y)$

We can choose $H \in \Theta$ such that $V^H(t) \in M_0$

$\therefore$ $E_Q(y) \leq 0$.

Proof: $(3) \rightarrow (1)$

Let Q be a probability on (Ω, F) .

Let $A \in F$ and $p(A)=0$ to prove $Q(A)=0$ and S is martingale with respect to Q

Let $y \in C_0$ then $E_p(y)=\int y\ dP(A)$

Let $x=V^H(t) \in M_0$ for some $H \in \Theta$ then $E_Q(V^H(t)) \leq 0$.

Since $V^H(t)$ is positive random variable. Define $Q: F \rightarrow R$ by $Q(A)= \int y\ dp$, since $y \in C_0$

then y is Borel measurable, $Q(A)=\int_A y(w)\ dp(w)=0$ then Q Equivalent to p.

On the other hand, $E(V^H(t))=0$ for some $H \in \Theta$ and $V^H(t) \geq V^H(t)=y$

$E_Q(\Delta V^H(t) | F_t)=E(\int_0^t Hu\ (d(\Delta S_u) | F_u)=\int H_u dE(\Delta S_u | F_t)=0$.

Assumption (3.3):

The set $M^e(S)$ is not-empty.

Theorem (3.4),[3]:

If H is bounded non-negative process and S is martingale, then $V^H(t)$ is martingale.

Theorem (3.5):

Let (Ω,F,P) be a probability space. The following statements are equivalent:

(1) If there is no Free-arbitrage.

(2) $M^c(S)\neq\phi$.

Proof: (1)$\rightarrow$(2)

Suppose $G=M_0$ and $H=X_+/\{0\}$, G and H are convex sets.

Let $x\in M_0$ then $V^H(t)\leq x$ for some $H\in\Theta$.

Since (1) is true then $E(V^H(T))=0$ for some $H\in\Theta$.

$P(V^H(T)=0)=0$ then $P(V^H(T)>0)=0$

$\therefore$ $V^H(T)\notin X_+$ and $V^H(T)\neq 0 \Rightarrow x\notin H$.

$G\cap H=\phi$

Let $P=\{\sum_{n=1}^{N}\mu_n\delta_n : \mu_n>0 \wedge \sum_{n=1}^{N}\mu_n=1\}$ where $\{\delta_\lambda\}_{\lambda\in(0,1)}$ is the indicater function.

To prove P is convex

Let $\mu_n>0$, $\beta_n>0$, $\sum_{n=1}^{N}\mu_n=1$, $\sum_{n=1}^{N}\beta_n=1$. Let $0\leq\beta<1$, then:

$\beta\sum_{n=1}^{N}\mu_n\delta_\lambda+(1-\beta)\sum_{n=1}^{N}\beta_n\delta_\lambda=\sum_{n=1}^{N}(\beta\mu_n+(1-\beta)\beta_n)\delta_\lambda \Rightarrow \beta\mu_n+(1-\beta)\beta_n>0$ and

$\sum_{n=1}^{N}\beta\mu_n+(1-\beta)\beta_n=\beta.1+(1-\beta).1=1.$

Thus P is convex compact subset of X_+.

Hence we may by strictly separate [1] we can separate the sets P and M_0 by linear functional f such that:

$$E_f(x) \le \alpha \quad , \qquad \forall\, x \in M_0.$$

$$E_f(y) \le \beta \quad , \qquad \forall\, y \in P.$$

Find $\alpha<\beta$, replace α by 0 then $\beta>0$

$\therefore\ E_f(x) \le 0$, $E_f(y)>\beta>0$.

Since $E_f(\delta_\lambda)>0$, we may normalize f such that $E_f(\delta_\lambda)=1$. Since f is strictly positive, we therefore found a probability measure Q on (Ω,F) equivalent to P such that the lemma (3.2) hold: On other words, we founds an equivalent martingale measure Q for the process S.

Proof: (2) $\to$(1)

If $M^c \neq \phi$

let $H\in\Theta$ such that $P(V^H(0)=0)=1$ and $P\{V^H(T)>0\}=0$

by lemma (3.2), $E_Q(y)=0$, $\forall\, y \in C_0$

$\therefore\ \exists\, H\in\Theta$ such that $V^H(t)=y$

$\therefore\ E(V^H(t))\le 0$.

Let $G^H(t)= V^H(t)+ \int_{T=1}^{t} H_u\, ds_u$.

Since S is Q-martingale

$\therefore$ G is Q-martingale (by use Theorem (3.4))

$\therefore\ E(G^H(T))= E(G^H(0))$

Since $P(E(V^H(0)=0)=0$

$\therefore\ E(G^H(T))=0$.

Since $P(V^H(T)\ge 0)=0$ and $Q\sim P$

$\therefore\ Q(G^H(T) \ge 0)=0$ and $Q(G^H(0)=0)=0$

$\therefore\ Q(G^H(T) >0)=0$, this contradiction

$\therefore\ G^H(T)\notin X_+ \quad \forall\, H\in\Theta$.

4. Main result:

We start this section by the following Proposition.

Proposition (4.1):

Let $(\Omega, \mathcal{F}, P)$ be a probability space and Q be an equivalent martingale measure and let $x \in X$ then:

(1) $V^H(0)$, $V^H(t)$ are uniquely determined by $x= V^H(0)+ V^H(t)$.

(2) $V^H(t)$ is Q-martingale with respect to filtrations $\mathcal{F}$.

(3) $V^H(0)=E_Q(x)$.

Proof:

(1) Suppose $x= V^{H_1}(0)+V^{H_1}(T)$, $x= V^{H_2}(0)+V^{H_2}(T)$

then $V^{H_1}(0)-V^{H2}(0)=V^{H_1}(T)-V^{H_2}(T)$ for some $H_1,H_2 \in \Theta$.

We assume $V^{H_1}(0) \neq V^{H_2}(0)$ and $V^{H_1-H_2}(0)>0$

then $V^{H_1-H_2}(T)>0$

Since Q is equivalent martingale measure to P, then S is non-free arbitrage .

If $P\{w: V^{H_1-H_2}(0)=0\}=1$

$\therefore$ $P\{w: V^{H_1(w)-H_2(w)}(0)=0\}=1$ and $P\{w: V^{H_1(w)-H_2(w)}(T)>0\}>0$.

Hence $V^{H_1-H_2}(T) \in X_+$

$\therefore$ S is free-arbitrage . This is contradiction.

To prove $V^H(t)$ is unique.

Let $G^{H_1}(t)=V^{H_1}(0)+V^{H_1}(t)$ and $G^{H_2}(t)=V^{H_2}(0)+V^{H_2}(t)$.

If $H_1 \neq H_2$ then $V^{H_1}(T) \neq V^{H_2}(T)$, $\exists$ $t<T$ such that $V^{H_1}(t) \neq V^{H_2}(t)$, take:

$A=\{w \in \Omega: V^{H_1(w)}(t)>V^{H_2(w)}(t)\}$ then $A \in F_t$ and $P(A)>0$.

Define $Y = V^{H_1}(T) - V^{H_2}(T)$,

$$\text{let } m(u) = \begin{cases} H_1(u)-H_2(u) & ; \quad u \le t, \\ I_{A^c} - (H_1(u)-H_2(u)) + I_A Y B(u) & ; \quad t < u \le T. \end{cases}$$

where $B: X \to R$, $\int_0^t B(u)\,ds_u = 1$.

$$\int_0^t m(u)\,ds(u) = \int_0^t (H_1(u) - H_2(u))\,ds_u = V^{H_1}(t) - V^{H_2}(t).$$

If $t=u$, then:

$$\int_0^t m(u+1)\,ds(u) = \int_0^t I_{A^c}(H_1(u+1) - H_2(u+1))\,ds_u + \int_0^t I_A Y B(u)\,ds_u$$

$$= I_{A^c}\int_0^t (H_1(u+1) - H_2(u+1))\,ds_u + I_A \int_0^t YB\,ds_u \quad , \quad t<u<T.$$

$$= I_{A^c}(V^{H_1}(t) - V^{H_2}(t)) + I_A Y B(T) B^{-1}(T) \quad \text{where } S(T)=B^{-1}(T).$$

$$= Y = V^{H_1}(T) - V^{H_2}(T)$$

$m(0)=H_1(0)-H_2(0)=0$.

Hence m is self financing with zero initial.

$$V^m(T) = \int_0^T I_{A^c}(H_1(u) - H_2(u))\,ds_u + I_A Y B(T)\,ds_u$$

$$= I_A Y B(T) B^{-1}(T) \ge 0$$

and

$$P\{V^m(T) > 0\} = P\{A\}$$

Thus $V^m(T) \in X_t$. This contradiction.

(2) Suppose $Q \in M^e(S)$,

Suppose $G^H(t)=V^H(0)+V^H(t)$.

$G^H(t+1) - G^H(t) = V^H(t+1) - V^H(t)$

$$= \int_0^{t+1} H(u)\,ds_u - \int_0^t H(u)\,ds_u$$

$=(H-S)_{t+1}-(H-S)_t$

$G^H(t)$ is Q-martingale by theorem (3.4).

(3) Since $G^H(t)$ is Q-martingale.

If $E_Q(x)=E_Q(V^H(0)+V^H(T))=E_Q(G^H(t))$

$\therefore$ $E_Q(G^H(t))= E_Q(G^H(0))= V^H(0)$

$\therefore$ $E_Q(x)= V^H(0)$.

Example (4.2):

Let $\Omega=\{w_1,w_2\}$ and F=P{w} and $P\{w_1\}=\frac{1}{2}$, $P\{w_2\}=\frac{1}{2}$.

Let $S(0)=\begin{bmatrix} S_0(0) \\ S_2(0) \end{bmatrix}=\begin{bmatrix} a \\ 90b \end{bmatrix}$, $S_0(T)=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $S_1(T)=\begin{bmatrix} 180 \\ 90 \end{bmatrix}$

$$\bar{S}=\begin{bmatrix} S_0(T) & S_0(T) \\ S_1(T) & S_1(T) \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 180 & 90 \end{bmatrix}$$

We try to find vector $\psi=(\psi_1,\psi_2)$, $\psi_i>0$, i=1,2, such that:

$$\begin{bmatrix} a \\ 90b \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 180 & 90 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad ,$$

we find $\psi_1+\psi_2=a$

$2\psi_1+\psi_2=b$

By solving this equations, we find $\psi_1=b-a$, $\psi_2=2a-b$.

Define Q $\{w_1\}=\psi_1=b-a$

Q $\{w_2\}=\psi_2=2a-b$.

Case one: If a=1 and $b=\frac{1}{3} \Rightarrow Q=\{\frac{2}{3},\frac{1}{3}\}$.

Case two: If a=1, $b=\frac{1}{2} \Rightarrow Q=\{\frac{1}{2},\frac{1}{2}\}$.

Hence we have some of cases of Q.

$M^e(S)$ have at least one element.

Theorem (4.3):

Let (Ω,F,P) be a probability space. If there is no-free arbitrage .The following statements are equivalent:

(1) $M^e(S)$ has a single element.

(2) There is a linear functional ψ such that $\psi|_{M_0}=\pi$.

Proof:(1)→(2)

Suppose $M^e(S)$ has a single element, then there is unique measure Q equivalent to P.

Define $\psi:X\to R$ by $\psi(x)=E_Q(x)$, for all $x\in X$.

$x\in X$ then $\exists$ a positive element $H\in\Theta$, such that $x= V^H(0)+V^H(T)$

Since $V^H(0)$ is unique, then $E_\psi(x)= V^H(0)$

then $\psi(x)= V^H(0)$.

Hence ψ is well define.

Also, ψ is positive.

Since $V^H(t)=\int_0^t H(u)ds_u$ is continuous $\forall$ $0\le t<\infty$. Thus ψ is continuous.

Now, to prove ψ is linear.

Let $x,y\in X$, $\lambda,\beta\in R$.

$x\in X$ then $\exists\ H_1\in\Theta$ such that $x= V^{H_1}(0)+V^{H_1}(T)$.

$y\in X$ then $\exists\ H_2\in\Theta$ such that $y= V^{H_2}(0)+V^{H_2}(T)$.

$\lambda x= V^{\lambda H_1}(0)+V^{\lambda H_1}(T)$

$\beta y= V^{\beta H_2}(0)+V^{\beta H_2}(T)$

$\lambda x+\beta y= V^{\lambda H_1+\beta H_2}(0)+V^{\lambda H_1+\beta H_2}(T)$.

$\psi(\lambda x+\beta y)= V^{\lambda H_1+\beta H_2}(0)= V^{\lambda H_1}(0)+V^{\beta H_2}(0)=\lambda V^{H_1}(0)+\beta V^{H_2}(0)=\lambda\psi(x)+\beta\psi(x)$.

It means, that ψ is linear.

Finally, to prove $\psi|_{M_0}=\pi$.

Let $x\in M_0$ then $\exists\ H\in\Theta$ such that $V^H(0)+V^H(T)\leq x$

$\psi(x)= E_\psi(x)$

since $M^e(S)=\{\psi\}$ and $V^H(0)$, $V^H(T)$ are unique.

Thus $\psi(x)= V^H(0)$.

Since $\pi(x)= V^H(0)$, $\forall\ x\in M_0$, then $\psi(x)=\pi(x)$, $\forall\ x\in M_0$.

Proof :(2)$\rightarrow$(1)

Suppose $M^e(S)$ has at least one element, we assume $Q_1,Q_2\in M^c(S)$ and $Q_1\neq Q_2$.

$\therefore\ E_{Q_1}(x) \neq E_{Q_2}(x) \quad \forall\ x \in X.$

Since there is no free arbitrage

$\therefore$ the measures Q_1, Q_2 are equivalent to P

$\therefore \therefore\ \exists\ H \in \Theta$ (which is positive) such that $x = V^H(0) + V^H(T)$.

Define $\psi : X \to R$ by $\psi(x) = V^H(0) \quad \forall\ x \in X$, then ψ is positive, linear functional (integral are linear) and if we take $m \in M_0$, $\psi(m) = V^H(0) = \pi(m) \quad \forall\ m \in M_0$.

Since $V^H(0)$ is unique and $V^H(0) = E_\psi(x)$ and $\psi(x) = E_{\psi_1}(x)$ and $\psi(x) = E_{\psi_2}(x)$.

Thus $E_{\psi_1}(x) = E_{\psi_2}(x)$ this contradiction.

Hence $M^e(S)$ has at most one element.

حول وحدانية قياس مارتنجل المكافئ

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المستخلص

لتكن $\{ S_t: t \geq 0 \}$ =S متتابعة ملائمة من المتغيرات العشوائية المعرفة على الفضاء الاحتمال.$(\Omega, F, \{ F_t \}, P)$ في هذا البحث برهنا انعدام المخاطرة الحرة هو شرط ضروري وكاف على وجود قياس جديد Q مكافئ للقياس P بحيث S مار تنجل بالنسبة إلى القياس Q ,ولكن هذا الشرط غير كاف للحصول على الوحدانية , وأعطينا مثال لتوضيح , ولكن شرط انعدام المخاطرة الحرة مع خاصية التوسع نحصل على وحدانية القياس المكافئ.

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