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Almost Hermitian Manifold of Constant Type and J-Invariant

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Abstract :

we find in this paper the necessary and the sufficient condition for which an almost Hermitian manifold is a constant type and j-invariant.

Keyword:

Almost Hermitian manifold, adjoint G-structure space, constant type, J- invareant curvature tensor.

Introduction :

Almost Hermitian manifold (AH-manifold) one of the most important subjects of the differential geometrical structure. Conclusions behind the study of almost Hermitian manifold were found in many mathematical and theoretical physics aspects, such as Kahler manifold which is highly involved in the teaching of differential geometry, algebraic geometry, theory of Lie groups and topology. For that importance the problem of classification of different kinds of AH-manifold according to its manipulations came to the roof. First attempts were done by Koto [6] in 1960, who found a relationship which was considered as an entry to manifold, and it was almost similar to Kahler manifold. The other studies concentrated on the properties of some classes of almost Hermitian manifold of constant type, which ensured the important facts of these studies, which were the class NK-manifold of zero constant type coincide with the class of Kahler manifold, then the class NK-manifold of non-zero constant type coincided with the 6-dimensional NK-manifold [4]. A. Gray [3] defined three special classes of AH-manifold, which are defined by the following:

1) class R_1 if $<R(X,Y)Z,W>=<R(X,Y)JZ,JW>$

2) class R_2 if $<R(X,Y)Z,W>=<R(JX,JY)Z,JW>+$
$$<R(JX,Y)JZ,W>+<R(JX,Y)Z,JW>$$

3) class R_3 if $<R(X,Y)Z,W>=<R(JX,JY)JZ,JW>$

A. Gray [3] proved that for random AH-manifold, the relation between them is $R_1 \subset R_2 \subset R_3$. The class R_1 is called parakahler manifold [7]. The manifold of class R_3 has been study under the name RK-manifold [9] or J-invariant curvature tensor almost Hermitian manifold.

1- Almost Hermitian manifold

Let M be an $2n$-dimintional smooth manifold, $C^\infty(M)$ be a Lie algebra of C^∞ vector fields on M, $T_p(M)$ be a tangent space on smooth manifold M at $p \in M$. Let ∇ denotes to the operator of the covariant derivative with respect to the Levi- Civita connection (Riemannian connection).

Definition 1.1 [5]

The pair of tensors $\{ J, g = \langle \cdot,\cdot \rangle \}$, where J is an almost complex structure and $g = \langle \cdot,\cdot \rangle$ is a Riemannian metric on M is called almost Hermitian structure (*AH*- structure) such that satisfies the condition: $g(JX, JY) = g(X, Y)$.

Definition 1.2 [5]

Almost Hermitian manifold denoted by $\{ M, J, g = \langle \cdot,\cdot \rangle \}$ is a smooth manifold with (*AH*- structure).

From [1] we have that given an almost Hermitian structure on smooth manifold equivalent to the given Hermitian metric
$\langle\langle X, Y \rangle\rangle = g(X,Y) + \sqrt{-1}\,\Omega(X,Y)$ where $\Omega(X,Y) = g(X, JY)$ is a 2-form which is called fundamental (Kahlerian) form of almost Hermitian manifold.

Definition 1.3 [5]

The Riemannian curvature tensor of M, is covariant tensor field of order 4, whose value at any point $p \in M$ is determined as:
$R(X,Y,Z,W) = \langle R(Z,W)Y, X \rangle$
where $R(Z,W)Y = [\nabla_Z, \nabla_W]Y - \nabla_{[Z,W]}Y$ and $X,Y,Z,W \in T_p(M)$.

The Riemannian curvature tensor $R(X,Y,Z,W)$ is algebraic curvature tensor i.e satisfies the following conditions:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$$
$$R(X,Y,Z,W) = R(Z,W,X,Y)$$
$$R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0$$

Remark:

The specification of an almost Hermitian structure on manifold is equivalent to the setting of a *G*-structure, where *G* is the unitary group $U(n)$. Its elements are the frames adapted to the structure (*A*- frames). The looks as $(p,\varepsilon_1,\cdots,\varepsilon_n,\varepsilon_{\hat{1}},\cdots,\varepsilon_{\hat{n}})$ where $p \in M$, ε_a are eigenvector corresponding to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are eigenvector correspond to the eigenvalue $-i$. Here $a = 1,...,n,\ \hat{a} = n + a$. Therefor, the matrices of the operators of the almost complex structure J , Riemannian metric g and fundamental form written in an *A*-frame looks as follows respectively:

$$(J^i_j) = \begin{pmatrix} iI_n & 0_n \\ 0_n & -iI_n \end{pmatrix} \quad (g_{ij}) = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix} \quad (\Omega_{ij}) = \begin{pmatrix} 0 & \sqrt{-1}\, I_n \\ -\sqrt{-1}\, I_n & 0 \end{pmatrix} \qquad (1.1)$$

where I_n is the identity matrix $i, j = 1,...,2n$ [2].

Definition 1.4 [1]

The Kähler manifold is an almost Hermitian manifold satisfying the condition $\nabla J = 0$.

The condition $\nabla J = 0$ implies

$$R(X,Y,JZ,JW) = R(X,Y,Z,W) \qquad (1.2)$$

The relation (1.2) is the Kahler identity

An almost Hermitian manifold is said to be of J-invariant curvature tensor if

$$< R(X,Y)Z,W > = < R(JX,JY)JZ,JW > \qquad (1.3)$$

such manifolds, called also RK-manifold [9]. The relation

$$< R(X,Y)Z,W > = < R(X,Y)JZ,JW > \qquad (1.4)$$

is said to be of Kähler identity, an almost Hermitian manifold satisfy (1.4) is said to be paraKähler manifold[7].

To define the constant type of almost Hermitian manifold, we first consider the tensor

$$\lambda(X,Y,Z,W) = < R(X,Y)Z,W > - < R(X,Y)JZ,JW >$$

So λ measures the defect from the Kähler identity. Now, we put

$$\lambda(X,Y) = \lambda(X,Y,Y,X)$$

and say that an almost Hermitian manifold $\{M, J, g = <\cdot,\cdot>\}$ is of constant type at $p \in M$ provided that for all $X \in T_p(M)$ ((where $T_p(M)$ is the tangent space at the point $p \in M$)), we have: $\lambda(X,Y) = \lambda(X,Z)$ (1.5)

Whenever the planes $\{X,Y\},\{X,Z\}$ are antiholomorphic and

$g(X,Y)=g(X,Z)=0$, $g(Z,Z)=g(Y,Y)$, $Y,Z\in T_p(M)$. If (1.5) hold for all $p\in M$, then $\{M,J,g=<\cdot,\cdot>\}$ has pointwise constant type. $\{M,J,g=<\cdot,\cdot>\}$ has global constant type if (1.5) is the constant function.

Now, let us put

$$L(X,Y,Z,W)=<R(X,Y)Z,W>-\beta[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)] \qquad (1.6)$$

by theorem [8]. The tensor (1.6) satisfies the Kähler identity.

$L(X,Y,Z,W)=L(X,Y,JZ,JW)$, if and only if the tensor R satisfies the following conditions:

a) $R(JX,JY,JZ,JW)=R(X,Y,Z,W)$.

b) R has constant type.

Thus, if the curvature tensor is J-invariant curvature tensor and has constant type β where $\beta\in C^\infty(M)$, then

$$\begin{aligned} R(X,Y,Z,W)-R(X,Y,JZ,JW)=\beta[&g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\ &-\Omega(X,W)\Omega(Y,Z)+\Omega(X,Z)\Omega(Y,W)]. \end{aligned} \qquad (1.7)$$

With respect to the local coordinates, (1.7) can be expressed as follows:

$$R_{ijkl}-R_{ijJcJd}=\beta[g_{il}g_{jk}-g_{ik}g_{jl}-\Omega_{il}\Omega_{jk}+\Omega_{ik}\Omega_{jl}]$$

$$R_{ijkl}-R_{ijkl}J_k^cJ_l^d=\beta[g_{il}g_{jk}-g_{ik}g_{jl}-\Omega_{il}\Omega_{jk}+\Omega_{ik}\Omega_{jl}]. \qquad (1.8)$$

Theorem

The necessary and sufficient condition for which an almost Hermitian manifold is a constant type and j-invariant are the following:

1) $R_{abcd}=R_{\hat{a}bcd}=R_{\hat{a}b\hat{c}d}=0$

2) $R_{\hat{a}\hat{b}\hat{c}d}=\beta\,\delta_{dc}^{ab}$

Proof:

In the adjoint G-structure space we get the following components:

1) put $i=a$, $j=b$, $k=c$, $l=d$ then (1.8) becomes

$$R_{abcd}-R_{abJcJd}=\beta[g_{ad}g_{bc}-g_{ac}g_{bd}-\Omega_{ad}\Omega_{bc}+\Omega_{ac}\Omega_{bd}]$$

$$R_{abcd}-R_{abcd}J_c^cJ_d^d=\beta[g_{ad}g_{bc}-g_{ac}g_{bd}-\Omega_{ad}\Omega_{bc}+\Omega_{ac}\Omega_{bd}]$$

According to the (1.1) and (1.2) we have:

$$R_{abcd} + R_{abcd} = 0$$
$$\Rightarrow R_{abcd} = 0$$

2) put $i = \hat{a}\,,\ j = b\,, k = c\,, l = d$ then (1.8) becomes

$$R_{\hat{a}bcd} - R_{\hat{a}bcd} J_c^c J_d^d = \beta\,[\, g_{\hat{a}d}\, g_{bc} - g_{\hat{a}c}\, g_{bd} - \Omega_{\hat{a}d}\Omega_{bc} + \Omega_{\hat{a}c}\Omega_{bd}\,]$$

According to the (1.1) and we have:

$$R_{\hat{a}bcd} + R_{\hat{a}bcd} = \beta[\,\delta_d^a(0) - \delta_c^a(0) - (i\delta_d^a(0)) + (i\delta_c^a(0))]$$

$$R_{abcd} + R_{abcd} = 0$$
$$\Rightarrow R_{abcd} = 0$$

3) put $i = \hat{a}\,,\ j = \hat{b}\,, k = c\,, l = d$ then (1.8) becomes

$$R_{\hat{a}\hat{b}cd} - R_{\hat{a}\hat{b}cd} J_c^c J_d^d = \beta\,[\, g_{\hat{a}d}\, g_{\hat{b}c} - g_{\hat{a}c}\, g_{\hat{b}d} - \Omega_{\hat{a}d}\Omega_{\hat{b}c} + \Omega_{\hat{a}c}\Omega_{\hat{b}d}\,]$$

According to the (1.1) and we have

$$R_{\hat{a}\hat{b}cd} + R_{\hat{a}\hat{b}cd} = \beta\,[\,\delta_d^a\delta_c^b - \delta_c^a\delta_d^b - (-i\delta_d^a(-i\delta_c^b)) + (-i\delta_c^a(-i\delta_d^b)]$$

$$2R_{\hat{a}\hat{b}cd} = \beta\,[\,\delta_d^a\delta_c^b - \delta_c^a\delta_d^b + \delta_d^a\delta_c^b - \delta_c^a\delta_d^b\,]$$

$$2R_{\hat{a}\hat{b}cd} = \beta\,[\,2\delta_d^a\delta_c^b - 2\delta_c^a\delta_d^b\,]$$

$$R_{\hat{a}\hat{b}cd} = \beta\ \delta_{dc}^{ab}$$

Where $\delta_{dc}^{ab} = \delta_d^a\delta_c^b - \delta_c^a\delta_d^b$

4) put $i = \hat{a}\,,\ j = b\,, k = \hat{c}\,, l = d$ then (1.8) becomes

$$R_{\hat{a}b\hat{c}d} - R_{\hat{a}b\hat{c}d} J_{\hat{c}}^c J_d^d = \beta\,[\, g_{\hat{a}d}\, g_{b\hat{c}} - g_{\hat{a}\hat{c}}\, g_{bd} - \Omega_{\hat{a}d}\Omega_{b\hat{c}} + \Omega_{\hat{a}\hat{c}}\Omega_{bd}\,]$$

According to the (1.1) and we have

$$R_{\hat{a}b\hat{c}d} + 0 = \beta[\,\delta_d^a\delta_b^c - (-i\delta_d^a(i\delta_b^c))]$$

$$\Rightarrow R_{\hat{a}b\hat{c}d} = 0$$

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