مجلة القادسية للعلوم الصرفة المجلد 13 العدد 1 لسنة 2008
المؤتمر العلمي الاول لكلية العلوم المنعقد في 26-27 اذار لسنة 2008

Equivalent Martingale Measures on L^0-space

By

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الخلاصة

نفرض العملية $(X_t)_{t\in I}$ متتابعة ملائمة من المتغيرات العشوائية ذات البعد d المعرفة على الفضاء الاحتمالي الملائم $(\Omega,F,\{F_t\}_{t\in I},P)$. في هذا ثحبنا سوف نحصل على شروط ضرورية وكافية إلى وجود قياس مارتينجل المكافئ Q إلى P (ويسمى عادة قياس-τ المكافئ) بحيث كل مركبة من المتتابعة لها خاصية مار تنجل بالنسبة إلى Q. هذه المشكلة وجدنا حلها في الفضاء -L^0 بالنسبة إلى التبولوجي المتولد من المعياري-Δ.

Abstract

Suppose that $(X_t)_{t\in I}$ is an adapted sequence of d-dimensional random variable defined on some filtered probability space $(\Omega,F,\{F_t\}_{t\in I},P)$.

In this paper we obtain conditions which are necessary and sufficient for the existence of a probability measure Q to P (which we call an equivalent τ-measure) such that each of the d component sequence of $(X_t)_{t\in I}$ has a prescribed martingale property with respect to Q.This problem have been solved in L^0-space with respect to Δ-norm topology

Introduction

In recent years the mathematical theory of stochastic integration (stochastic process) has become of interest because of its several areas of mathematical stochastic integrals with respect to martingales which were first discussed by Winer (1923). The extension of this definition is due of square martingale. The relation between the theorem of price asset and arbitrage were introduced from (Arrow-Debreu)(1959) model, formula of Black and Scholes (1973), linear price model of Cox and Ross (1976).

In their fundamental paper, Harrison and Kreps (1979), discussed the fundamental theorem and introduced the concepts of equivalent martingale measure. Absence of arbitrage alone was not sufficient to obtain an equivalent martingale measure for the stochastic process. . Different solutions have been introduced to relate the topological conditions of arbitrage [see Back and pliska (1991), Dalank, Morton

The triple (Ω,F,P) is called probability space, where Ω be a non –empty set ,*F is a σ-field on* Ω, and P is a probability measure. i.e $P(\Omega)=1$ where P be a measure on (Ω,F) ,the process X, Sometimes denoted $(X_t)_{t\in I}$ is supposed to be $\Re^d$-valued,

i.e The function $X:\Omega\to\Re$ is called **random variable** if X is measurable function. This mean the function $X:\Omega\to\Re$ is random variable if and only if $\{X\le a\}\in F$ for all $a\in\Re$.This paper contains three sections. In section one we introduce some definitions and concepts about martingale and equivalent measure**.** Section two we proved L^0-space is topological space with respect to Δ-norm topology. In section three we proved main result about existence equivalent measure which the process is martingale.

1-Basic definitions and concepts:

In this section we state some definitions which are related to main result.
We start by the following definition:-

Definition (1.1), [6]:

Let X be a random variable defined on probability space (Ω,F,P). We define **the expectation** (or expected value or mean) of X, denoted E(X), as:

$$E(X) = \int_\Omega X\,dP$$

Provided the integral exists. Thus E(X) is the of the Borel measurable

Let (Ω,F,P) be a probability space, I be any subset of $\Re_+$ and (E,β) be a measurable space. We begin with some notation, definitions and theorems from martingales.

Definition (1.2), [4]:

A mapping $X: I\times\Omega\to E$ is called **E-valued stochastic process** if for each $t\in I$, the map $\omega\to X(t,\omega)$ is an E-valued random variable.

i.e.

$$\{\omega: X(t,\omega)\in A\}\in F \text{ for every } A\in\beta.$$

The mapping $\omega\to X(t,\omega)$ from Ω into E is called the random function of the process, this mapping is often denoted by X_t (sometimes is called the state at time t), and the process itself by:

$$X=(X_t)_{t\in I} \text{ or } X=\{X_t: t\in I\}.$$

The probability space (Ω,F,P) is called **the base of the E-valued stochastic process X**, and the space Ω is often called **sample space** of X, the point $\omega\in\Omega$ is called **sample point**. The measurable space (E,β) is called **the state space of the E-valued stochastic process X**. [i.e. the initial value $\lambda\in E$ if $0\in I$ and $X_0(t.\lambda)=X(0,\lambda)=\lambda$, if $E=\Re^d$, $\beta=\beta(\Re^d)\Rightarrow$ X is called **$\Re^d$-valued stochastic process**.

In particular if d=1, (i.e. $E=\Re$, $\beta=\beta(\Re)$), then X is called **$\Re$-valued stochastic process** (or more briefly, X is called stochastic process).

i.e. A stochastic process is a collection of random variables.

$$X=\{X_t: t\in I\}, \text{ on } (\Omega,F).$$

Definition (1.3), [4]:

A filtration $\{F_t: t\in I\}$ is an increasing family of sub σ-fields $F_t\subset F$, increasing means that if $s\le t$, then $F_s\subset F_t$. When F is not specified, it is assumed that F equals the σ-fields generated by $\bigcup_{t\in I} F_t$ which denoted $\bigvee_{t\in I} F_t$, i.e $F = \bigvee_{t\in I} F_t = \sigma(\bigcup_{t\in I} F_t)$

When $I=\Re_+$, we say $F_\infty = \bigvee_{t\ge 0} F_t = \sigma(\bigcup_{t\ge 0} F_t)$

A filtration $\{F_t: t\in I\}$ is said to satisfy the usual conditions if it is

(1) Right continuous, i.e. $F_t = \bigcap_{s>t} F_s$

(2) Complete, i.e. F_0 contains all the null set in F.
[$\forall A\in F$ with $P(A)=0 \Rightarrow A\in F_0$].

Given stochastic process $X=\{X_t: t\in I\}$; the simplest choice of a filtration is that generated by the process itself:

i.e. $F_t^X = \sigma(X_s: 0\le s\le t)$.

The smallest σ-fields with respect to which X_s is measurable for every $0\le s\le t$, (F_t^X is called the natural filtration of X).

Definition (1.4), [6]:

A stochastic process $X=\{X_t : F_t ; t\in I\}$ is said to be **adopted** (to the filtration $\{F_t\}_{t\in I}$) if for each $t\ge 0$; X_t is F_t-measurable random variable;

i.e. $X_t\in F_t$

Every stochastic process X is an adopted to $\{F_t\}_{t\in I}$.

Definition (1.5), [7]:

A stochastic process $X=\{X_t : t\in I\}$ is said to be **martingale with respect to the filtration** $\{F_t\}_{t\in I}$ if,

(1) X is integrable, i.e. $E(|X_t|)<\infty$ for all $t\in I$.
(2) $E(X_t| F_s)=X_s$ a.e. for all s,t with s<t.

Definition (1.6), [7]:

An equivalent martingale measure for stochastic process X is probability measure Q on (Ω,F) which has the following properties:

(1) $P\sim Q$, i.e. $P(A)=0 \Leftrightarrow Q(A)=0, \forall A\in F$.
(2) X is a martingale with respect to the filtration $\{F_t\}_{t\in I}$ and the probability measure Q.

2- L^0-space:

In this section we proved L^0-space is metric space.

We start by the following definition:-

Definition (2.1), [6]:

Let X be a vector space over F. A function $\|.\|: X\to\Re$ is called **a Δ-norm** on X if:

(1) $\|x\|>0 \quad \forall\, x\neq 0,\ x\in X$.

(2) $\|\lambda x\|\le\|x\| \quad \forall\, x\in X,\ 0<|\lambda|\le 1$.

(3) $\lim\limits_{\lambda\to 0}\|\lambda x\|=0 \quad \forall\, x\in X$.

(4) $\|x+y\|\le c.\max\{\|x\|,\|y\|\} \quad \forall\, x,y\in X$ where $c>0$ is some constant independent of x,y.

Lemma (2.2), [3]:

Let X be a vector space. If $\|.\|$ is a Δ-norm on X, then it induces on X a vector topology τ which is metrizable.

Definition (2.3), [3]:

A sequence $\{x_n\}$ in X converges **to** $x\in X$ if and only if $\|x-x_n\|\to 0$ as $n\to\infty$.

Example (2.4), [3]:

Suppose τ is a topology with a countable local basis β of 0 in X, in the form $\beta=\{U_n:n\in N\}$ such that $\bigcap U_n=\{0\}$, each U_n is balanced and $U_{n+1}+U_{n+1}\subset U_n$ for every n. Then we can define a Δ-norm on X by $\|x\|=\mathrm{Sup}\,\{2^{-n}:x\notin U_n\}$ and the Δ-norm induces the origin topology where c=2.

Definition (2.5):

A Δ-norm $\|.\|$ on a vector space X is called **an F-norm** if it satisfies :

$\|x+y\|\le\|x\|+\|y\| \quad \forall x,y\in X$.

Theorem (2.6):

If $\|.\|$ is any F-norm then $d(x,y)=\|x-y\|$ is a (translation) invariant metric on X.

Proof:

Define $d:X\times X\to\Re$ by $d(f,g)=\|x-y\|$ for all $x,y\in X$.

(1) Let $x,y\in X$, if $x\neq y \Rightarrow x-y\neq 0 \Rightarrow \|x-y\|>0$ (by (1) of definition (4.1.1))

$\Rightarrow d(x,y)>0$.

If $x-y=0 \Rightarrow \|x-y\|=\|0\|=\lim\limits_{\lambda\to 0}\|\lambda.0\|=0$ (by (3) of Definition (4.1.1))

$\Rightarrow d(x,y)\ge 0$.

(2) Let $x,y\in X$.

If $x=y \Rightarrow x-y=0 \Rightarrow \|x-y\|=\|0\|=\lim\limits_{\lambda\to 0}\|\lambda.0\|=0$

$\Rightarrow d(x,y)=0 \Rightarrow \|x-y\|=0$.

If $x\neq y \Rightarrow x-y\neq 0 \Rightarrow \|x-y\|>0$ (by (1) of Definition.(4.1.1))

$\Rightarrow 0<\|x-y\|=d(x,y) \Rightarrow d(x,y)>0$ this contradiction.

$\Rightarrow x=y$.

(3) $d(x,y)=\|x-y\|=\|-(y-x)\|\le\|y-x\|=d(y,x)$ ……………………….(1)

$d(y,x)=\|y-x\|=\|-(x-y)\|\le\|x-y\|=d(x,y)$ ……………………….(2)

from (1) and (2), we have $d(x,y)=d(y,x)$ for all $x,y\in X$.

(4) Since X is F-norm

$\Rightarrow d(x,y)=\|x-y\|=\|x-z+z-y\|\le\|x-z\|+\|z-y\|=d(x,z)+d(z,y)$.

$\Rightarrow$ X is metric space.

Corollary (2.7), [3]:

Let X be a Hausdorff topological vector space with a countable local basis of 0, then X is metrizable and the topology may be given by on invariant.

Definition (2.8), [7]:

Let (Ω,F,P) be a probability space. Define $L^0=L^0(\Omega,F,\mu)$ to **the space of all *F*-measurable functions** on Ω with the usual convention about identifying functions equal almost every where.

Define $\|.\|_0:L^0\to\Re$ by:

$$\|f\|_0=\int\frac{|f(x)|}{1+|f(x)|}dP(x) \quad \forall f\in L^0.$$

Proposition (2.10):

$\|.\|_0$ is an F-norm on L^0.

Proof:

In order to prove $\|.\|_0$ is an F-norm on L^0, we proof the following :

(1) Let $f\in L^0$ such that $f\neq 0$

$\Rightarrow f(x)\neq 0 \quad \forall x\in\Omega$.

$\Rightarrow |f(x)|>0$ and $1+|f(x)|>0$

$$\Rightarrow \frac{|f(x)|}{1+|f(x)|}>0$$

$$\Rightarrow \int\frac{|f(x)|}{1+|f(x)|}dP(x)>0$$

$\Rightarrow \|f\|_0>0$

(2) Let $f\in L^0$ and $|\lambda|\le 1$.

Since $|\lambda|\le 1$

$\Rightarrow |\lambda||f(x)|\le|f(x)|$

$\Rightarrow |\lambda f(x)|\le|f(x)|$

$$\Rightarrow \frac{1}{|f(x)|}\le\frac{1}{|\lambda f(x)|}$$

$$\Rightarrow 1+\frac{1}{|f(x)|}\le 1+\frac{1}{|\lambda f(x)|}$$

$$\Rightarrow \frac{|f(x)|+1}{|f(x)|} \le \frac{1+|\lambda f(x)|}{|\lambda f(x)|}$$

$$\Rightarrow \frac{|\lambda f(x)|}{1+|\lambda f(x)|} \le \frac{|f(x)|}{|f(x)|+1}$$

$$\Rightarrow \int \frac{|\lambda f(x)|}{1+|\lambda f(x)|}\, dP(x) \le \int \frac{|f(x)|}{|f(x)|+1}\, dP(x)$$

$\Rightarrow \|\lambda f\|_0 \le \|f\|_0$.

(3) Let $f \in L^0$.

$$\lim_{\lambda\to 0} \|\lambda f\|_0 = \lim_{\lambda\to 0} \int \frac{|\lambda f(x)|}{1+|\lambda f(x)|}\, dP(x) = \int \lim_{\lambda\to 0} \frac{|\lambda f(x)|}{1+|\lambda f(x)|}\, dP(x) = 0$$

(4) Let $f,g \in L^0$.

$|f(x)+g(x)| \le |f(x)|+|g(x)|$

$$\frac{1}{|f(x)+g(x)|} \ge \frac{1}{|f(x)|+|g(x)|}$$

$$1+\frac{1}{|f(x)+g(x)|} \ge 1+\frac{1}{|f(x)|+|g(x)|}$$

$$\frac{1+|f(x)+g(x)|}{|f(x)+g(x)|} \ge \frac{1+|f(x)|+|g(x)|}{|f(x)|+|g(x)|}$$

$$\frac{|f(x)+g(x)|}{1+|f(x)+g(x)|} \le \frac{|f(x)|+|g(x)|}{1+|f(x)|+|g(x)|} = \frac{|f(x)|}{1+|f(x)|+|g(x)|} + \frac{|g(x)|}{1+|f(x)|+|g(x)|}$$

$$\frac{|f(x)+g(x)|}{1+|f(x)+g(x)|} \le \frac{|f(x)|}{1+|f(x)|+|g(x)|} + \frac{|g(x)|}{1+|f(x)|+|g(x)|}$$

$$\int \frac{|f(x)+g(x)|}{1+|f(x)+g(x)|}\, dP(x) \le \int \frac{|f(x)|}{1+|f(x)|+|g(x)|}\, dP(x) + \int \frac{|g(x)|}{1+|f(x)|+|g(x)|}\, dP(x)$$

$\|f+g\|_0 \le \|f\|_0 + \|g\|_0$

(5) Let $f,g \in L^0$.

$$\max\{\|f\|_0,\|g\|_0\} = \begin{cases} \|f\|_0 & \text{if } \|f\|_0 \ge \|g\|_0 \\ \|g\|_0 & \text{if } \|f\|_0 \le \|g\|_0 \end{cases}$$

If $\|f\|_0 \le \|g\|_0$

$\Rightarrow \max\{\|f\|_0,\|g\|_0\} = \|g\|_0$ and $\|f\|_0+\|g\|_0 \le \|g\|_0+\|g\|_0 = 2\|g\|_0 = 2\max\{\|f\|_0,\|g\|_0\}$. Since $\|f+g\|_0 \le \|f\|_0+\|g\|_0$

$\Rightarrow \|f+g\|_0 \le 2\max\{\|f\|_0,\|g\|_0\}$.

Proposition (2.11):
L^0 is metric space.

Proof:
By Theorem (4.3.6).

Proposition (2.12), [5]:
L^0 have trivial dual when P is non-atomic.

3- Main result .

Let (Ω,F,P) be a probability space, and let $L^0_+(\Omega, F,P)$ be **the positive part** of $L^0(\Omega,F,P)$.

i.e. $L^0_+(\Omega,F,P) = \{f \in L^0(\Omega,F,P) : f \ge 0\}$.

Definition (3.1), [6]:
Let $B \subset L^0$, we say that B is **bounded in probability** in L^0 if there exists $k>0$ such that $|f(x)|<k$ a.e. for all $f \in B$.

Theorem (3.2):
Let K be a convex subset of $L^0(\Omega,F,P)$ such that $0 \in K$ then the following statements are equivalent:

(1) $\forall f \in L^0_+(\Omega,F,P)$, $\exists \lambda>0$ such that $\lambda f \notin \overline{K-B}$ where :

$K-B = \{f \in L^0 : \exists g \in K \text{ such that } f \le g\}$

(2) $\forall$ $A\in F$ such that $P(A)>0$, $\exists$ $\lambda>0$ such that $\lambda I_A \notin \overline{K-B}$.

(3) $\exists$ $z\in B$ such that $\sup_{y\in K} E(yz)<\infty$.

(4) There exists probability measure Q equivalent to P such that K is bounded in $L^0_+(\Omega, F, P)$.

Proof:

(1) $\to$(2): Let $A\in F$ be such that $P(A)>0$

$\Rightarrow I_A\neq 0$ Since $I_A\in L^0_+$

$\therefore$ by condition (1) there exists $\lambda>0$ such that $\lambda I_A \notin \overline{K-B}$.

(2) $\to$(3): Suppose $\forall$ $A\in F$, such that $P(A)>0$, $\exists$ $\lambda>0$ such that $\lambda I_A \notin \overline{K-B}$.

Let $H=\overline{K-B}$, $G=\{I_A\}$

Since K is convex and B is convex in $L^0_+(\Omega,F,P)$

$\Rightarrow$ K-B is convex $\Rightarrow$ $\overline{K-B}$ is convex.
i.e. H is convex, and is closed with respect to topology convergence in probability.
Since G has one element.

$\Rightarrow$ G is convex and compact subset of L^0_+, also, since I_A is bounded function.

$\Rightarrow$ by (2) choose $\lambda=1$

$\Rightarrow \lambda I_A \notin \overline{K-B}$ $\Rightarrow I_A \notin \overline{K-B}$.
$\Rightarrow G\cap H=\phi$.

$\Rightarrow$ by a version separation theorem, there exists a linear functional $g\neq 0$ on $(L^0_+)^* = L^0_+$ such that $g|_{\overline{K-B}} \leq \alpha$ and $gI_A>\beta$ when $\alpha<\beta$.

If $\alpha\geq 0 \Rightarrow g$ is zero or negative function.
$\Rightarrow \beta>0$.
$\Rightarrow E[gI_A]>0$.

Let $T=\{g\in L^0_+ : E[gI_A]<\infty\}$.

Let £ be the family of subset of Ω formed by the support of the element $g\in T$. Note that £ is closed under countable union, as for a sequence $(g_n)_{n=1}^{\infty}\in T$, we can fined strictly positive scalar $(\alpha_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}\alpha_n g_n \in T$.

$\Rightarrow$ there exists $g_0\in T$ such that $A_0=\{\omega\in\Omega: g_0(\omega)\neq 0\}$
$\Rightarrow A_0\in F$
$\Rightarrow P(A_0)=\sup\{P(A): A\in £\}$
to prove g_0 is bounded.
i.e. $P(A_0)=1$
Suppose $P(A_0)<1$

$$E[gI_A]=\int g(\omega).I_A(\omega)dP(\omega)$$

$$=\int_A g(\omega)dP(\omega) >0.$$

Take $A_{00}=\Omega|_{A_0}$

$\Rightarrow A_{00}=\{\omega\in\Omega: g_0(\omega)=0\}$.

$I_{A_{00}}\in L^0_+$.

$\Rightarrow E[gI_{A_{00}}]=\int_{\Omega|_{A_0}} g(\omega)dP(\omega)$

Since $I_{\Omega|_{A_0}} \notin \overline{K-B}$ (by (1))

$\Rightarrow E[gI_{A_{00}}]>0$.

$\Rightarrow E[(g+g_0)\, I_{A_{00}}]=E[g\, I_{A_{00}}]+E[g_0\, I_{A_{00}}]$.

$\Rightarrow P\{\omega\in\Omega: (g+g_0)(\omega)\neq 0\}>0$ this contradiction.

$\Rightarrow g\in B$ and $\operatorname{Sup} E[gI_A]<\infty$.

(3) $\to$(4): Let $g\in L^0_+$

$$\Rightarrow \|g\|_0=\int \frac{|g(x)|}{1+|g(x)|}\, dP(x).$$

Let $\varpi(\omega)=\max\{\|g\|_{L^0_+},1\}$.

Let Q be the measure on g with Randon-Nikodym derivative $dQ/dP=c.g(\omega)/\varpi(\omega)$ where the normalize factor $c\in\Re_+$ is chosen such that $Q(\Omega)=1$. Then Q is a probability measure equivalent to P and for each $g\in L^0_+(\Omega,F,P)$, the function $f=E_Q(g(S_t(\omega)-S_s(\omega)I_A)$ where $s<t$.

$\Rightarrow f \geq 0$.

Suppose that $f\neq 0$, since $g\in T$.

$\Rightarrow E_Q(gI_A)<\infty$.

$\Rightarrow 0\leq E_Q(g(S_t(\omega)-S_s(\omega)I_A)<\infty$.

$\Rightarrow f\in T$.

$\Rightarrow E[f\, I_A]<\infty$.

Also, $0\leq -f= E_Q(-g(S_t(\omega)-S_s(\omega)I_A))<\infty$.

$\Rightarrow -f\in T$

Since $-f \geq 0$

$\Rightarrow f \geq 0$ and $f\leq 0 \Rightarrow f=0$.

$\Rightarrow$ there exists equivalent martingale measure $Q\sim P$.

(4) $\to$(1): Suppose $f\in L^0_+(\Omega,F,P)$ and $f\neq 0$.

$\Rightarrow \forall\ \lambda>0,\ \lambda f\in \overline{K-B}$.

$\Rightarrow \exists\ x_n\in K,\ y_n\in B$ such that $\lambda f=x_n-y_n-\delta_n$

where $0\leq\|\delta_n\|_{L^0_+}<1$.

Since by (4), there exists equivalent martingale measure $Q\sim P$. Suppose the density function of Q is $g=dQ/dP\in L^0_+$.

$\lambda gf =gx_n-gy_n-g\delta_n$.

$\Rightarrow E(gx_n)=\lambda E(fg)+E(gy_n)+E(g\delta_n)$.

Since $E[fg]<\infty$.

$\Rightarrow E(gx_n)\geq\lambda E(fg)+1$

$\Rightarrow \operatorname*{Sup}_{y\in K} E(gy)=\infty$, this contradiction.

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