مجلة القادسية للعلوم الصرفة المجلد 13 العدد 1 لسنة 2008
المؤتمر العلمي الاول لكلية العلوم المنعقد في 26-27 اذار لسنة 2008

Almost kahler manifold of class *W*-parakahler

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Abstract. The author studies the concept of the almost Kahler manifold of class *W*-parakahler. It is found the necessary and sufficient condition in which that an almost Kahler manifold of class *W*-parakahler can be an Einstein manifold.

Key words. Almost Hermitian manifold, almost Kahler manifold, adjoint *G*-structure space, constant type, Einstein manifold.

Introduction. The class of almost Kahler manifold was one of the sixteen classes of almost Hermitian manifold which were found by Gray and Hervella [9]. This class is a generalization of the class Kahler manifold in which the fundamental form is closed. In 1958 Sasaki [16] proved that the Riemannian metric which was given on a smooth manifold *M* generates a Riemannian metric on the tangent space *T*(*M*). At tha beginning this Riemannian metric was called as a Sasaki metric, and then Tachibana and Okumura [17] defiend an almost complex structure on a tangent fiber Riemannian space with the Sasaki metric. Barros and Naveira[2] are defined the concept of almost Kahler manifold such that its Riemannian curvature tensor satisfies the condition of the special Gray's class R_2. Kasabov [10] found the classification for conformal flat of almost Kahler manifold of class R_2 with dimension grater than 4. Blair[3] proved that for each case where $n \geq 8$, there is no non zero almost Kahler manifold with a constant curvature tensor and this tensor is equal to zero when this manifold is Kahler. Oguro and Sekigava [13] are proved that an 4-dimentional almost Kahler manifold was a Kahler manifold.

One of the important subjects of almost Kahler manifold is a constant type. The first one who studied this concept was Gray [6], [7]. He studied some kinds of nearly Kahler manifolds of a constant type which they have some properties , for example holomorphic sectional curvature tensor . Kirichenko studied the property of a constant type on general kinds of almost Hermitian manifold. He found [11] new method for this study which depends on adjoint G-structure space, with structure group is the unitary group.

The concept of almost Hermitian manifold of class *W*-parkahler found by Habeeb M. Abood [4] as a generalization of the concept of almost Hermitian manifold of class R_1 (parakahler manifold) . He found the sufficient and necessary condition in which that an almost Hermitian manifold is *W*-parakahler manifold.

1-Almost Kahler manifold. Let M be an $2n$ - dimensional smooth manifold, $C^{\infty}(M)$ be an algebra of smooth functions on M, $X(M)$ be a Lie algebra of vector fields on M. Denote by ∇ to the Riemannian connection of metric g. Let d be the exterior differentiation.

Definition 1.1 [11]. An almost Hermitian structure (*AH* – structure) on *M* is a pair of tensors $\{\ J, g=<.,.>\ \}$, where *J* is an almost complex structure, $g=<.,.>$ is a Riemannian metric, such that $<JX, JY>=<X,Y>$, $X,Y \in X(M)$. A smooth manifold *M* with *AH*-structure is called an almost Hermitian manifold (*AH*-manifold).

In $T_p(M)$ there exist a basis of the form $\{\varepsilon_1, \ldots\ldots, \varepsilon_n, \bar{\varepsilon}_1, \ldots\ldots, \bar{\varepsilon}_n\}$. Its corresponding frame is $\{p, \varepsilon_1, \ldots\ldots, \varepsilon_n, \bar{\varepsilon}_1, \ldots\ldots, \bar{\varepsilon}_n\}$.

Suppose that the indices *i, j, k, l* in the range *1, 2, …….,2n* and the indices *a, b, c, d, e, f, g, h* in the range *1,2,… ,n*. Denote $\hat{a} = a + n$, then its corresponding its frame can be written as the form $\{p, \varepsilon_1, \ldots\ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots\ldots, \varepsilon_{\hat{n}}\}$.

It is known [12] that the setting an *AH*-structure on *M* is equivalent to the setting of a *G*- structure in the principle fiber bundle of all complex frames of manifold *M* which contains *G*- structure that is the unitary group *U(n)*, and this *U(n)* is called an adjoint *G*- structure. In the space of the adjoint *G*- structure, the following forms define matrices which gives components of tensor fields *g* and *J*:

$$\left(g_{ij}\right)=\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad , \quad \left(J^i_j\right)=\begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \qquad (1.1)$$

Where I_n is the unit matrix of order n.

Definition 1.2 [9]. Let *M* be an *A*H-manifold with *AH*-structure $\{\ J, g=<.,.>\ \}$. *AH*-structure is called an almost Kähler structure (*AK*-structure) if the fundamental form $\Omega(X,Y)=<X, JY>$ is closed, i.e. $d\Omega=0$. A smooth manifold *M* with *AK*-structure is called an almost Kahler manifold(*AK*-manifold)

Remark. By the Banaru's classification of *AH*-manifold [1], the *AK*-manifold satisfies the following properties:

$$B^{ab}_c = B^c_{ab} = 0\,,\ B^{[abc]} = B_{[abc]} = 0$$

where B^{ab}_c and B^{abc} are called Kirichenko's virtual and structure tensors rspectively.

Proposition 1.1 [5]. In adjoint *G*- structure space, the total group of structure equation of *AK*-manifold is the following forms:

1. $d\omega^a = \omega_b^a \wedge \omega^b + B^{abc}\,\omega_b \wedge \omega_c$

2. $d\omega_a = -\omega_a^b \wedge \omega_b + B_{abc}\,\omega^b \wedge \omega^c$

3. $d\omega_b^a = \omega_c^a \wedge \omega_b^c + B_b^{adc}\,\omega_c \wedge \omega_d + B_{bcd}^a\,\omega^c \wedge \omega^d + (A_{bd}^{ac} + 2B^{ach}\,B_{hbd})\,\omega^d \wedge \omega_c$

4. $dB^{abc} = B_d^{abc}\,\omega^d + B^{abcd}\,\omega_d + B^{dbc}\,\omega_d^a + B^{adc}\,\omega_d^b + B^{abd}\,\omega_d^c$

where $\{A_{bd}^{ac}\}$ are a system of functions in the adjoint *G*-structure space which are symmetric by the lower and upper indices.

Proposition 1.2 [15]. The second group of the structure equation of connection in Riemannian manifold is given by the form:

$$d\omega_j^i = \omega_k^i \wedge \omega_j^k + \frac{1}{2} R_{jkl}^i\,\omega^k \wedge \omega^l,$$

Where $\{R_{jkl}^i\}$ is a system of functions which are the components of Riemannian curvature tensor.

Proposition 1.3 [5] . By using the propositions 1.1 and 1.2, the components of Riemannian curvature tensor of *AK*-manifold in the adjoint *G*- structure space are:

1. $R_{bcd}^a = 2B_{bcd}^a$ 2. $R_{b\hat{c}d}^a = 4B^{cah}\,B_{dbh} - A_{bd}^{ac} - 2B^{ach}\,B_{hbd}$

3. $R_{bc\hat{d}}^a = A_{bc}^{ad} + 2B^{adh}\,B_{hbc} - 4B^{dah}\,B_{cbh}$ 4. $R_{b\hat{c}\hat{d}}^a = 2B_b^{adc}$

5. $R_{\hat{b}cd}^{\hat{a}} = -2B_{acd}^b$ 6. $R_{\hat{b}\hat{c}d}^{\hat{a}} = A_{ad}^{bc} + 2B^{bch}\,B_{had} - 4B_{dah}\,B^{cbh}$

7. $R_{\hat{b}c\hat{d}}^{\hat{a}} = 4B^{dbh}\,B_{cah} - A_{ac}^{bd} - 2B^{bdh}\,B_{hac}$ 8. $R_{\hat{b}\hat{c}\hat{d}}^{\hat{a}} = -B_a^{bcd}$

9. $R_{\hat{b}cd}^a = 4B^{hab}\,B_{hcd}$ 10. $R_{\hat{b}\hat{c}d}^a = -2B_d^{cab}$

11. $R_{\hat{b}c\hat{d}}^a = 2B_c^{dab}$ 12. $R_{\hat{b}\hat{c}\hat{d}}^a = -4B^{[c|ab|d]}$

13. $R_{bcd}^{\hat{a}} = -4B_{[c|ab|d]}$ 14. $R_{b\hat{c}d}^{\hat{a}} = 2B_{dab}^{\;\;\;\;c}$

15 $R^{\hat{a}}_{bc\hat{d}} = -2B_{cab}{}^{d}$ 16. $R^{\hat{a}}_{b\hat{c}\hat{d}} = 4B^{hcd}\,B_{hab}$

and the others are conjugate of them.

Proposition 1.4 [11] . An *AH*-manifold is a manifold of a constant type c if and only if in adjoint *G*-structure space satisfies:

$B^{had}B_{hbc} = 2c\delta^{ad}_{bc}$, where $\delta^{ad}_{bc} = \delta^{a}_{b}\delta^{d}_{c} - \delta^{a}_{c}\delta^{d}_{b}$.

Definition 1.3 [15]. A Ricci tensor r is a tensor of type (2,0) which is defined by: $r_{ij} = R^{k}_{ijk} = g^{kl}R_{kijl}$.

Definition 1.4 [15]. A scalar curvature tensor denoted by k, which is defined by:

$k = g^{ij}r_{ij}$.

Proposition 1.5 . An *AH*-manifold has an *J*-invariant Ricci tensor if and only if in adjoint G-structure space satisfies $r^{\hat{a}}_{b} = 0$.

Proof. An *AH*-manifold has an *J*-invariant Ricci tensor if an only if $J \circ r = r \circ J$.

Then directly by computing this relation in adjoint *G*-structure space, we get:

$(J \circ r)^{i}_{j} = (r \circ J)^{i}_{j}$ which means that $J^{i}_{k}r^{k}_{j} = r^{i}_{k}J^{k}_{j}$

Therefore by (1.1) we get that $r^{\hat{a}}_{b} = 0$.

2- Almost Kahler manifold of class *W*-parakahler. As for as the Riemannian space is concerned, the conformal curvature tensor or Wely's tensor is defined by the form [15]:

$$W_{ijkl} = R_{ijkl} + \frac{1}{m-2}(r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{ik} - r_{jk}g_{il}) + \frac{K}{(m-2)(m-1)}(g_{il}g_{jk} - g_{ik}g_{jl}),$$

where R_{ijkl} are the components of the Riemannian tensor, r_{ij} are the components of Ricci tensor, g_{ij} are components of the Riemannian metric g and K is the scalar curvature tensor. According to our case, the *AH*-manifold which we have, $m = 2n$, then the Weyl's tensor is defined by the following:

$$W_{ijkl} = R_{ijkl} + \frac{1}{2(n-1)}(r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{ik} - r_{jk}g_{il}) + \frac{K(g_{il}g_{jk} - g_{ik}g_{jl})}{2(n-1)(2n-1)}$$

This tensor has similar properties to those of the Riemannian curvature tensor.

Definition 2.1 [4] . The *AK*-manifold is called a conformal parakahler manifold (or of class *W*-parakahler), if the conformal curvature tensor satisfies the following equality:

$\langle W(X,Y)Z,T\rangle = \langle W(JX,JY)Z,T\rangle \quad X, Y, Z \in X(M)$

Note that, this equality is similar to the equality of R_1-manifold(parakahler manifold)[8], so we study this manifold by the name of *W*- parakahler manifold.

Proposition 2.1 [4] . The *AH*-manifold is *W*- parakahler manifold if and only if, in the ajoint *G*-structure space, the following condition satisfies:

$W_{abcd} = W_{\hat{a}bcd} = W_{\hat{a}\hat{b}cd} = 0$.

The main result of this paper is the following theorem.

Theorem 2.1 . Suppose that *M* is an *AK*-manifold of class *W*-parakahler with *J*-invariant Ricci tensor, then *M* is an Einstein manifold if and only if *M* is a manifold of a constant type.

Proof. Suppose that *M* is an *AK*-manifold of class *W*-parakahler with *J*-invariant Ricci tensor.

By the proposition(2.1) we get:

$W_{abcd} = W_{\hat{a}bcd} = W_{\hat{a}\hat{b}cd} = 0$.

The condition $W_{\hat{a}\hat{b}cd} = 0$ means that the Weyl s tensor is given by the form:

$$R_{\hat{a}\hat{b}cd} = \frac{-1}{2(n-1)}(r_{\hat{a}c}g_{\hat{b}d} + r_{\hat{b}d}g_{\hat{a}c} - r_{\hat{a}d}g_{\hat{b}c} - r_{\hat{b}c}g_{\hat{a}d}) - \frac{k(g_{\hat{b}c}g_{\hat{a}d} - g_{\hat{b}d}g_{\hat{a}c})}{2(2n-1)(n-1)} \qquad (2.1)$$

Acording to the components of the metric g in (1.1), the equation (2.1) can be as the form:

$$R_{\hat{a}\hat{b}cd} = \frac{-1}{2(n-1)}(r_c^a\delta_d^b + r_d^b\delta_c^a - r_d^a\delta_c^b - r_c^b\delta_d^a) - \frac{k(\delta_c^b\delta_d^a - \delta_d^b\delta_c^a)}{2(2n-1)(n-1)} \qquad (2.2)$$

Suppose that *M* is Einstein manifold

Since Ricci tensor is *J*-invariant

Then we get $r_b^a = e\delta_b^a$ and (2.2) can be written as the form

$$R_{\hat{a}\hat{b}cd} = \frac{-e}{2(n-1)}(\delta_c^a\delta_d^b + \delta_d^b\delta_c^a - \delta_d^a\delta_c^b - \delta_c^b\delta_d^a) - \frac{k(\delta_c^b\delta_d^a - \delta_d^b\delta_c^a)}{2(2n-1)(n-1)}$$

$$R_{\hat{a}\hat{b}cd} = (\frac{e}{2(1-n)} + \frac{k}{2(2n-1)(n-1)})\,\delta_{cd}^{ab}$$

Since M is AK-manifold.

Then by proposition 1.3 we obtain:

$$B^{hab}B_{hcd} = \frac{1}{4}(\frac{e}{2(1-n)} + \frac{k}{2(2n-1)(n-1)})\,\delta_{cd}^{ab}$$

Therefore, by the proposition 1.4, M is a manifold of the constant type with

$$c = \frac{1}{8}(\frac{e}{2(1-n)} + \frac{k}{2(2n-1)(n-1)})$$

Conversely, suppose that M is an AK-manifold of class W-parakahler such that M of a constant type.

Then by the proposition 2.1 we have $W_{abcd} = W_{\hat{a}bcd} = W_{\hat{a}\hat{b}cd} = 0$.

And by the proposition 1.4 we have $B^{hab}B_{hcd} = 2c\delta_{cd}^{ab}$.

Therefore we obtain :

$$4B^{hab}B_{hcd} = \frac{-1}{2(n-1)}(r_c^a\delta_d^b + r_d^b\delta_c^a - r_d^a\delta_c^b - r_c^b\delta_d^a) - \frac{k(\delta_c^b\delta_d^a - \delta_d^b\delta_c^a)}{2(2n-1)(n-1)}$$

Then we get that: $(8c - \frac{k}{2(n-1)(2n-1)})\delta_{cd}^{ab} = \frac{-1}{2(n-1)}(r_c^a\delta_d^b + r_d^b\delta_c^a - r_d^a\delta_c^b - r_c^b\delta_d^a)$

(2.3)

By folding the equation (2.3) by the indices b and d (d=b), then we obtain:

$$\frac{16c(n-1)(2n-1)-k}{2n-1}\delta_c^a = -(nr_c^a - r_c^a - r_c^a) = (2-n)r_c^a$$

Then $r_c^a = \dfrac{16c(n-1)(2n-1)-k}{(2n-1)(2-n)}\delta_c^a$

Since Ricci tensor is J-invariant.

Then by the proposition 1.5 we have $r_b^{\hat{a}} = 0$.

Then $r_j^i = \dfrac{16c(n-1)(2n-1)-k}{(2n-1)(2-n)}\delta_j^i$

Therefore *M* is the Einstein manifold with Einstein constant $e = \dfrac{16c(n-1)(2n-1)-k}{(2n-1)(2-n)}$.

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