On direct product of Semi – neat subgroup of Abelian Group G

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Abstract:

In [8]. M.A.H. Abdullah gave some new results of pure 1-2 and 3 subgroups of Abelian Group G, and in [9], he gave new definition of subgroups which more general of pure subgroups, called semi-pure.

In This paper we shall gave the general cose of the results by M.A.H Abdullah [9].

We shall define new subgroups which are a family of semi – pure and called semi – neat which more general of semi – neat subgroups. More ever we are studying some general properties of semi – pure, and use these properties to obtain conditions for direct product of semi- neat.

Introduction:

We shall use definitions and some important results of paper [2], [7], [8] and [10]. In this paper we shall give and study some important concept and new results, which is use in the theory of Abelian Groups.

It is well known that a subgroups S of G is said to be pure in G, $(\forall n)$, $x \in S$ and $(\forall n), n \in Z^+$ if n|x in G then n|x in S (See[4]60).

And S is neat in G if $\forall p$ (p is prime number if p|x in G then p|x in S.

In more general case of neat subgroups, we shall define new subgroups as the following:

Definition A

Let S be a subgroup of G, we said S is semi neat in G if S satisfy the following condition:
$(\forall a)$, $a \in S$ and for some P (prime number), if p|a in G then p|a is S …… (1)
Clearly every neat subgroup of G is semi – neat in G.
Example: Suppose that G = Z_{12} and S = {0, 6}.
Take p=3, so we have 3|6 in G, because 3(6)=6, and the solution element (6) belong to S, which means that 3|6 in S.
Thus, S is a semi –neat in G.
By the above definition of the semi – neat subgroup, we consequently, that the pure subgroups are is well known that the following diagram of implications.

$$\begin{array}{ccccc} \text{Divisible} & \rightarrow & \text{pure} & \rightarrow & \text{Neat} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Semi - divisible} & \rightarrow & \text{semi – pure} & \rightarrow & \text{Semi – neat} \end{array}$$

Remark: I (a) = < a > $\cup$ f (a), where <a> is a cyclic - semi group generated by a, and we denoted f (a) by the set of the solution of eq (1).
Clearly f(a) $\supseteq$ f(a) fqr all n $\in Z^+$, we need the following lemma s to get some results.
Lemma A: For all $x \in G$, $x \notin f(a)$ and for all $m \in Z^+$ then $I(a^m) \not\subset I(a)$.

Proof:
Evidently $\langle a^m \rangle \subseteq \langle a \rangle$ and $f(a^m) \subseteq f(a)$.
Hence $I(a^m) \subseteq I(a)$. But a is the only generator of the semi - group <a>. Hence $a \notin \langle a^m \rangle$ what together with a $\notin$ f (a). Implies a $\notin$ f (a^m).
This means that $a \notin \langle a^m \rangle \cup f(a^m) = I(a^m)$.
Therefore $I(a^m) \subset I(a)$.

Lemma B:
IF a $\notin$ f (a), then a is the only generator of I (a) and is also the only generator of a $\cup$ f(a). .
Proof:
Let I (b)=I (a) hold. Ifb $\in \{a^2, a^3, ...\}$ Then I (b) $\subset$ I (a), if $b \in f(a)$ then

$I(b) \subseteq f(a) \subseteq I(a)$, hence $I(b) \subseteq I(a)$ hold too. The prooffor $a \cup f(a)$ is similar.

Lemma C:

IF I (a) = I (b) the either $a \in f(a)$ and $b \in f(b)$ or $a \notin f(a)$ and $b \notin f(b)$.

Proof:

If I (a) =I (b), $a \in f(a)$ and $b \notin f(b)$ then a=b . Since I (b) has the only generator by b. But this is a contradiction with the fact that $a \in f(a)$, $b \notin f(b)$

Lemma D:

Let $a \in JI\{si \mid i \in I\}$, where si are semi- neat subgroups f G and let:

$r(a) = \text{Max}\ \{n \in N | a^n \notin f(a)\}$ and

$r(a) = \text{Max}\ \{n \in N \mid a \notin f(ai)\}$ Then

$r(a) = \text{Max}\ \{r(ai)\ I\ i \in I\}$

Remark

1- Max $\{r(ai), i \in I\}$ always exists, since r (ai) <= r (a)

2- the elements $a, b \in S$, where S is semi neat in G,

Proof

The equality holds if $a \in f(a)$ i.e. if $ai \in f(ai)$, for all $i \in I$. In this case

r (a)=0, r (ai)=0 for all $i \in I$ (see[l] .p121)

Hence Max $\{r(ai)/ i \in I\}=0$ *too.*

If $a \notin f(a)$, then

$R(a) = \text{Max}\ \{n \in N / a^n \notin f(a)\} >= l$

But $a^{r(a)} \notin f(a)$, $a^{r(a)+l} \in f(a)$, therefore there exists $a_k \in I$ such that $a^{r(a)} \notin f(a_k)$, $a_k^{r(a)+1} \in f(a_k)$, hence

$r(a_k) = r(a)$

For all $I \in I$. moreover

$r(a_k) = r(a)$. Thus are have

Max $\{r(ai)/i \in I\} = r(a)$.

Now, we are ready to show the following result.

Theorem 1:

Let S be a semi - neat in G, and let a, b $\in$ S then I (a) $\not\subset$ I (b) if and only if either a $\cup$ f (a) $\not\subset$ b $\cup$ f (b) or there exists an n >=2 such that $a=b^n$.

Proof

Let I (a) $\subset$ I (b), then either a $\in$ f(b) or a= b^n , n $\in$ {2,3 ,...}.
Let a $\in$ f(b). If I (a) =f(b) then b $\in$ f(b), hence b $\cup$ f(b) =f(b).
But in this case a$\cup$ f(a) $\subseteq$I (a) $\subset$ I (b) = f(b)=b $\cup$ f(b) holds i.e
a $\cup$ f(a) $\subset$ b $\cup$ f(b).
If a $\in$ f (b) and I (b) $\supset$ f (b), then b $\in$ f (b), hence
b $\cup$ f(b) $\supset$ f(b) . But in this case a $\cup$ f(a) $\subseteq$ f(b) $\subset$ b $\cup$ f(b) ,hence
a $\cup$ f(a) $\subset$ b $\cup$ f(b) again.
Conversely, let a $\cup$ f(a) $\subset$ b $\cup$ f(a) hold, then b $\neq$ a ,hence
a $\in$ f(a) . This I (a) $\subseteq$f(b) $\subseteq$I(b) . If I (a) = I (b), then
(a $\neq$ b), a $\in$ f(a) and b $\in$ f(a), hence a $\cup$ f(a) $\subseteq$b $\cup$ f(b) and
b $\cup$ f(b) $\subseteq$a $\cup$ f(a) ,therefore b $\cup$ f(b) = a $\cup$ f(a) . But this is a contradiction with
a $\cup$ f(a) $\not\subset$b $\cup$ f(b) . thus I (a) $\neq$ I (b) and since that a= b^n, then I (a) $\not\subset$ I(b)
Now, we shall give the characterization of I(a) $\not\subset$I(b) by the direct product of semi- neat subgroups .

Theorem 2:

Let s = JI {si | i $\in$ I} , where si are semi -neat subgroups, a, b $\in$ S then I(a) $\not\subset$ I (b) $\leftrightarrow$ ai $\cup$ f(ai) $\not\subset$ bi $\cup$ f(bi) for all i $\in$ I.

Proof:

Let I (a) $\not\subset$t I (b). Since a, b $\in$ Sand s= JI { si | i $\in$ I} so
I (ai) $\not\subset$t I (bi) for all i $\in$ I. By using above theorem we get
ai $\cup$ f(ai) $\not\subset$ t bi$\cup$ f(bi) .
Conversely, let ai $\cup$ f (ai) $\not\subset$t bi $\cup$ f (bi) for all i $\in$ I, then bi $\neq$ ai, hence
ai $\in$ f(bi) . This implies I (ai) $\subseteq$f (bi) $\subseteq$I (bi). If I (ai) = I (bi) then (ai $\neq$ bi), ai $\in$f(bi)and bi $\in$ f(ai) ,hence
ai $\cup$ f(ai) $\subseteq$bi $\cup$ f(bi) therefore ,we get ai $\cup$f(ai)=bi $\cup$f(bi).but this is a
contradiction to the fact ai $\cup$ f(ai) $\not\subset$ bi $\cup$ f(bi) we obtain I(ai)$\neq$I(bi)

for all $i \in I$. Thus $I(a) \not\subset I(b)$.

<u>Theorem 3:</u>

Let s= JI {si| $i \in I$ } when si are semi- neat subgroups of G, a, b $\in$ s then

$I(ai) \not\subset I(bi) \leftrightarrow ai \cup f(ai) \not\subset bi \cup f(bi)$ for all $i \in I$.

<u>Proof:</u>

Let $I(ai) \not\subset I(bi)$ for all $i \in I$.

Therefore we have $I(a) \not\subset I(b)$. By using theorem 2 we get

$ai \cup f(ai) \not\subset bi \cup f(bi)$ for all $i \in I$.

Conversely, if we have $ai \cup f(ai) \not\subset bi \cup f(bi)$ for all $i \in I$, so we get

$ai \neq bi$ for all $i \in I$. Therefore $ai \in f(bi)$ and this implies

$I(ai) \subseteq f(bi) \subseteq I(bi)$,

Again by theorem2 we get $I(a) \not\subset I(b)$, but $ai \neq bi$ so $ai \in f(bi)$ and $bi \in f(ai)$, thus $ai \cup f(ai) = bi \cup f(bi)$,but this is contradiction with $ai \cup f(ai) \not\subset bi \cup f(bi)$ we obtain $I(ai) \neq I(bi)$ and so $I(ai) \not\subset I(bi)$

From theorems 1,2 and 3 we get the following characterization of direct product of semi – neat subgroups of G by the last theorem.

<u>Theorem 4:</u>

Let s = JI { si | $I \in I$, si are semi – pure subgroups of G} a, b $\in$ S then the following conditions are equivalent:

(i) $I(a) \not\subset I(b)$

(ii) $ai \cup f(ai) \not\subset bi \cup f(bi)$ for all $i \in I$

(iii) $I(ai) \not\subset I)bi)$ for all $i \in I$

<u>References</u>

[1] ABRHAN, and Stake, 1. R-systems of unary alyebras . J Math (1993) slovaca p.120-123
[2] CATINO,F on complete ideals in semigroups Czech .Math J .40(115) 1990 155-158.
[3] FABBICI, lone-sided direct product of two semigroups (ACTA. Math .1994. Hungary p.160-166).
[4] Fuchs. LASZlO, Abelian Groups. U.S.A (1980).
[5] Fuchs. LASZlO, Infinite Abelian Groups. U.S.A (1980)
[6] Lagos's.Generalized ideals in semi-groups Acta, SCI, Math 22 (1961) 217-222. (Hungary).
[7] MICCOLI, M.M on-ideals in semi-groups Czech .Mathj 39 (114)

1989 . (522-527)

[8] M.A.H.Abdulla on pure-1-2 and 3 subgroups PU .M.A ser A vol. 3 no 3-4 p 135-139 Italy (1992).

[9] M.A.H. Abdulla. Direct product of semi – pure. Abhath Zeka V.4 P: 119-226 (2008).

[10] STEINFElD .0 RIngs and semI-groups. AcadeI'rlla kado Buaapest
(1979).

[11] SULKA .R on Maximal principle ideals of semigroups. PU.M.A ser D vol. 2 NO 1-2 p (111-113) ITALY (1990).