



Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
 ISSN:2521-3504(online) ISSN:2074-0204(print)



Approximately 2-Absorbing Submodule

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ARTICLE INFO

Article history:

Received: dd /mm/year

Revised form: dd /mm/year

Accepted : dd /mm/year

Available online: dd /mm/year

Keywords:

2-absorbing submodule,
 approximately 2-Absorbing
 submodule, prime submodule

ABSTRACT

In this paper present and study the notion of approximately 2-Absorbing submodules as generalization of 2-Absorbing submodules. Many new properties of this class obtain.

MSC..

<https://doi.org/10.29304/jqcm.2023.15.31307>

1. Introduction

Throughout this paper R is a commutative ring with $1 \neq 0$ and M is a unitary R -module. A. Y. Darainm and F. Soheilnia in [1] introduced the concept of 2-Absorbing submodule where a proper submodule N of M is called 2-absorbing submodule of M if whenever $a, b \in R, m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N:M)$.

Our concern in this paper is define and study the concept approximately 2-Absorbing submodule and give many new properties and characterizations.

2. Approximately 2-Absorbing Submodule

Definition 2.1: let $N < M$, N is called Approximately 2-Absorbing Submodule if whenever $a, b \in R, m \in M, abm \in N$, then either $a^2m \in N$ or $b^2m \in N$ or $(ab)^2 \in (N:M)$.

Remarks and Example 2.2:

1. every 2- absorbing submodule \Rightarrow Approximately 2-Absorbing Submodule

proof:

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Communicated by 'sub editor'

let $a, b \in R, abm \in N \rightarrow am \in N$ or $bm \in N$ or $ab \in (N:M)$. Then we get

$a^2m \in N$ or $b^2m \in N$ or $(ab)^2 \in (N:M)$ but the converse is not true for example

$M = \mathbb{Z}_{24}$ as \mathbb{Z} -Module let $N = \langle 8 \rangle$ N is not 2-absorbing by 2.2.2 $\in N$ but $2.2 \notin (N:M) = 8\mathbb{Z}$ and $2.2 \notin M$. Now N is approximately 2-absorbing

$2.2.2 \in N$ we get $2^2.2 = 8 \in N$ or $2.4.1 \in N$ we get $4^2.1 = 16 \in N$ or $2^2.2 \in N$ or $(2.2)^2 = 16 \in 8\mathbb{Z} = (N:M)$ or $4.2.2 \in N$ or $4^2.2 \in N$ or $8.1.1 \in N$ or $8^2.1 \in N$.

2. "The intersection of each pair of distinct prime submodules of R -module M is Approximately 2-Absorbing Submodule "[1].
3. It is clear that every prime submodule is Approximately 2-Absorbing Submodule. However, the converse is not true in general, for example: Consider \mathbb{Z}_6 as \mathbb{Z} -module, $\langle 0 \rangle$ not prime submodule of \mathbb{Z}_6 since $2.3 = 0$ but $3 \notin \langle 0 \rangle$ and $2 \notin \langle 0 \rangle$ but $\langle 0 \rangle = \langle 2 \rangle \cap \langle 3 \rangle$ and is approximately 2-absorbing submodule of \mathbb{Z}_6 as \mathbb{Z} -module by part 1.
4. Let N be Approximately 2-Absorbing Submodule of M , then for each $A \subseteq M$, either $A \subseteq N$ or $A \cap N$ is Approximately 2-Absorbing Submodule of A

Proof: Suppose $A \not\subseteq N$ then to prove $A \cap N$ is T of A

Let $abx \in A \cap N$ with N is approximately 2 absorbing and $x \in M, a, b \in R$

We get $a^2x \in N$ or $b^2x \in N$ or $(ab)^2 \in (N:M)$ then $a^2x \in A \cap N$ or $b^2x \in A \cap N$ or $(ab)^2 \in (A \cap N : A)$

5. Let N, W be two submodules of an R -module M and $N \leq W$. If N is Approximately 2-Absorbing Submodule of M , then N is Approximately 2-Absorbing Submodule of W

proof: If $W = M$ nothing to prove. Let $abx \in N, a, b \in R, x \in W$ then $x \in M$. But N is T of M so either $a^2x \in N$ or $b^2x \in N$ or $(ab)^2 \in (N:M)$ and since $N \leq W$ implies $(N:M) \leq (N:W)$ then either $a^2x \in N$ or $b^2x \in N$ or $(ab)^2 \in (N:W)$ hence N is Approximately 2-Absorbing Submodule of W

6. since every quasi-prime submodule is 2-absorbing, (where a proper submodule N of M is called quasi-prime submodule if whenever $a, b \in R, m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$) [2] hence it is Approximately 2-Absorbing Submodule. However Approximately 2-Absorbing Submodule may not be quasi-prime, as the following example shown: Consider the \mathbb{Z} -module \mathbb{Z} . The submodule $N = 4\mathbb{Z}$ is Approximately 2-Absorbing Submodule of \mathbb{Z} since, if $a, b, c \in \mathbb{Z}$ with $abc \in 4\mathbb{Z} = N$, then at least two of a, b, c are even. Hence either $(ab)^2 \in N$ or $a^2c \in N$ or $b^2c \in N$. But $4\mathbb{Z}$ is not quasi-prime, since $2.2.1 \in 4\mathbb{Z}$ but $2.1 \notin 4\mathbb{Z}$.
7. Let N and W be two submodules of an R -module M such that $N \cong W$

If N is Approximately 2-Absorbing Submodule, it is not necessary that W is as the following example explains this:

Consider the \mathbb{Z} -module \mathbb{Z} , the submodule $2\mathbb{Z}$ is T but $2\mathbb{Z} \cong 30\mathbb{Z}$ and $30\mathbb{Z}$ is not T since $2.3.5 = 30 \in 30\mathbb{Z}$ but $2^2.5 \notin 30\mathbb{Z}$ and $3^2.5 \notin 30\mathbb{Z}$ and $(2.3)^2 = 36 \notin 30\mathbb{Z}$.

8. The intersection of two Approximately 2-Absorbing Submodule need not be approximately 2-Absorbing Submodule for example: $6\mathbb{Z}$ and $5\mathbb{Z}$ are Approximately 2-Absorbing Submodule in the \mathbb{Z} -module \mathbb{Z} , but $6\mathbb{Z} \cap 5\mathbb{Z} = 30\mathbb{Z}$ which is not approximately 2-absorbing.

Proposition 2.3: Let $\varphi: M \rightarrow M'$ be an R-epimorphism. If W is Approximately 2-Absorbing Submodule of M' , then $\varphi^{-1}(W)$ is Approximately 2-Absorbing Submodule of M .

proof: Let $\varphi: M \rightarrow M'$ epimorphism, If w is Approximately 2-Absorbing Submodule then $\varphi^{-1}(W)$ is Approximately 2-Absorbing Submodule of $M < M'$ then $\varphi^{-1}(W) < M$ since φ epimorphism, let $abm \in \varphi^{-1}(W)$ $a, b \in R, m \in M$ $a b \varphi(m) \in W$ and as W is T then either $b^2 \theta(m) \in w$ or $(ab)^2 \in (W: M')$ or $a^2 \varphi(m) \in W$ if $a^2 \varphi(m) \in W$ then $a^2 m \in \varphi^{-1}(W)$ if

$b^2 \varphi(m) \in w$ then $b^2 m \in (\varphi^{-1}W)$ if $(ab)^2 \in (W: M')$ then $(ab)^2 M' \subseteq W$ but

$\varphi(M) \subseteq M'$ so $(ab)^2 \varphi(M) \subseteq W$ that is $(ab)^2 M \subseteq \varphi^{-1}(W)$

$(ab)^2 \in (\varphi^{-1}(W): M)$ $\varphi^{-1}(W)$ is Approximately 2-Absorbing Submodule of M .

Proposition 2.4: $f: M \rightarrow M'$ be an epimorphism, $N < M$ such that $\ker f \subseteq N$ then N is T of M if and only if $f(N)$ is Approximately 2-Absorbing Submodule of M' .

proof: Let $abm' \in f(N)$, $m' \in M'$, $a, b \in R$ $m' = f(m)$ for some $m \in M$

$abf(m) \in f(N)$ so $abf(m) = f(n)$, for some $n \in N$ $f(abm) - f(n) = 0$

we get $abm - n \in \ker f \subseteq N$ implies that $a b m - n \in N$ but $(N$ is T) so either $a^2 m \in N$ or $b^2 m \in N$ or $(ab)^2 \in (N: M)$ if $a^2 m \in N$ Then $f(m) \in f(N)$ that is $a^2 f(m) \in f(N)$ so $a^2 m' \in f(N)$ similarly if $b^2 m \in N$ then $f(b^2 m) \in f(N)$ implies that $b^2 m' \in f(N)$ if $(ab)^2 \in (N: M)$ then $(ab)^2 M \subseteq N$ and so $f((ab)^2 M) \subseteq f(N)$ implies that $(a b)^2 M' \subseteq f(N)$ and we get $(a b)^2 \in (f(N): M')$ Then $f(N)$ is Approximately 2-Absorbing Submodule of M' .

\Leftarrow let $a b m \in N$ then $f(abm) \in f(N)$ so $abf(m) \in f(N)$ since $f(N)$ is Approximately 2-Absorbing Submodule of M' we get

1. $a^2 f(m) \in f(N)$ then $f(a^2 m) = f(n)$, for some $n \in N$ hence $a^2 m - n \in \ker f \subseteq N$ so $a^2 m \in N$
2. $b^2 f(m) \in f(N)$ then similarly $b^2 m \in N$
3. if $(ab)^2 \in (f(N): M')$ then $(ab)^2 M' \subseteq f(N)$ so $(ab)^2 f(x) \in f(N), \forall x \in M$ so that $f((ab)^2 x) = f(n)$, for some $n \in N$ and hence $(ab)^2 x \in N$ for each $x \in M$ thus $(ab)^2 \in (N: R M)$, N is Approximately 2-Absorbing Submodule of M .

Corollary 2.5: R be a ring Let, M an R- module and N, K submodules of M with $K \leq N$. Then N is Approximately 2-Absorbing Submodule of M if and only if $\frac{N}{K}$ is Approximately 2-Absorbing Submodule of $\frac{M}{K}$.

Proof: Let N is Approximately 2-Absorbing Submodule of M

Let $ab(m+K) \in \frac{N}{K}$ then $abm \in N$ with N is Approximately 2-Absorbing Submodule of M then $a^2 m \in N$ or $b^2 m \in N$ or $(ab)^2 \in (N: M)$ we get $a^2(m+K) \in \frac{N}{K}$ or $b^2(m+K) \in \frac{N}{K}$ or $(ab)^2 \in (\frac{N}{K}: \frac{M}{K})$

\Leftarrow let $abm \in N \Rightarrow abm + K \in \frac{N}{K}$ With $\frac{N}{K}$ is Approximately 2-Absorbing Submodule of $\frac{M}{K}$ then $a^2 m + K \in \frac{N}{K}$ or $b^2(m+K) \in \frac{N}{K}$ or $(ab)^2 \in (\frac{N}{K}: \frac{M}{K})$

We get $a^2 m \in N$ or $b^2 m \in N$ or $(ab)^2 \in (N: M)$

Proposition 2.6: Let N a proper submodule of an R -module M . Then N is Approximately 2-Absorbing Submodule of M if and only if $abK \subseteq N$ for some $a, b \in R, K \subseteq M$ implies $(ab)^2 \in (N:M)$, or $a^2 K \subseteq N$ or $b^2 K \subseteq N$. Then N is Approximately 2-Absorbing Submodule of M .

proof: Let N be Approximately 2-Absorbing Submodule and $abK \subseteq N$ suppose that $(ab)^2 \notin (N:M)$, $a^2 K \not\subseteq N$ and $b^2 K \not\subseteq N$. Then there exists m_1, m_2 in K such that $a^2 m_1 \notin N$ and $b^2 m_2 \notin N$. Since $abm_1 \in N$ and $(ab)^2 \notin (N:M)$, $a^2 m_1 \notin N$ then $b^2 m_1 \in N$ also $abm_2 \in N$ and $(ab)^2 \notin (N:M)$, $b^2 m_2 \notin N$ then $a^2 m_2 \in N$. Now $(ab)^2 (m_1 + m_2) \in N$ and $(ab)^2 \notin (N:M)$. We have $a^2 (m_1 + m_2) \in N$, i.e. $a^2 m_1 + a^2 m_2 \in N$ we get $a^2 m_1 \in N$ contradiction! If $b^2 (m_1 + m_2) \in N$, i.e. $b^2 m_1 + b^2 m_2 \in N$ so $b^2 m_2 \in N$ contradiction! then either $(ab)^2 \in (N:M)$ or $a^2 K \subseteq N$ or $b^2 K \subseteq N$.

Theorem 2.7: Let N a proper submodule of an R -module M , then the following statement are equivalent :

1. N is Approximately 2-Absorbing Submodule of M
2. If $IJK \subseteq N$, for some ideal I and J of R and some submodule K of M then either $I^2 K \subseteq N$ or $J^2 K \subseteq N$ or $(IJ)^2 \subseteq (N:M)$.

proof: (1) \Rightarrow (2)

Suppose N is Approximately 2-Absorbing Submodule of M and $IJK \subseteq N$ for some ideals I and J of R and some submodule K of M and $(IJ)^2 \not\subseteq (N:M)$.

let $(IJ)^2 \not\subseteq (N:M)$ to show that $I^2 K \subseteq N$ or $J^2 K \subseteq N$ suppose $I^2 K \not\subseteq N$ and $J^2 K \not\subseteq N$ there exist a_1, a_2 show that $a_1 \in I$ and $a_2 \in J$ and $a_1^2 K \not\subseteq N$ and $a_2^2 K \not\subseteq N$ have $a_1^2 a_2^2 K \subseteq N$ since N is T so we have $(a_1 a_2)^2 \in (N:M)$ Since $(IJ)^2 \not\subseteq (N:M)$, $\exists b_1 \in I, b_2 \in J$ such that $(b_1 b_2)^2 \notin (N:M)$, But $b_1 b_2 K \subseteq N$ We have $b_1^2 K \subseteq N$ or $b_2^2 K \subseteq N$

Case (1) $b_1^2 K \subseteq N$ and $b_2^2 K \not\subseteq N$ Since $a_1^2 b_2^2 K \subseteq N$ and $b_2^2 K \not\subseteq N$ and $a_1^2 K \not\subseteq N$ so $(a_1 b_2)^2 \in (N:M)$ by Proposition 2.6 Since $b_1^2 K \subseteq N$ and $a_1^2 K \not\subseteq N$ we get $(a_1^2 + b_1^2) K \not\subseteq N \Rightarrow (a_1^2 + b_1^2) b_2^2 K \subseteq N$ neither $(a_1^2 + b_1^2) K \subseteq N$ nor $b_2^2 K \subseteq N$ We get $((a_1^2 + b_1^2) b_2^2) \in (N:M)$ by Proposition 2.1.6 but $(a_1^2 + b_1^2) b_2^2 = a_1^2 b_2^2 + b_1^2 b_2^2 a_1^2 b_2^2 \in (N:M) \Rightarrow (b_1 b_2)^2 \in (N:M)$ contradiction

Case 2) $b_2^2 K \subseteq N$ and $b_1^2 K \not\subseteq N$. By a similar argument of case(1),

we reach to a contradiction

Case 3) $b_1^2 K \subseteq N$ and $b_2^2 K \subseteq N$ Since $b_2^2 K \subseteq N$ and $a_2^2 K \not\subseteq N$ we conclude $(a_2^2 + b_2^2) K \not\subseteq N$ but $a_1^2 (a_2^2 + b_2^2) K \subseteq N$ and Neither $a_1^2 K \subseteq N$ nor $(a_2^2 + b_2^2) K \subseteq N$ hence $a_1^2 (a_2^2 + b_2^2) \in (N:M)$ by Proposition 2.6 since $(a_1 a_2)^2 \in (N:M)$ and $a_1^2 a_2^2 + a_1^2 b_2^2 \in (N:M)$ we have $a_1^2 b_2^2 \in (N:M)$ $(a_1^2 + b_1^2) a_2^2 K \subseteq N \Rightarrow a_2^2 K \subseteq N$ or $(a_1^2 + b_1^2) K \subseteq N$ or $(a_1^2 + b_1^2) a_2^2 \in (N:M)$ by Proposition 2.6. But $(a_1^2 + b_1^2) a_2^2 = a_1^2 a_2^2 + b_1^2 a_2^2$ So $= a_1^2 a_2^2 + b_1^2 a_2^2 \in (N:M)$ we get $b_1^2 a_2^2 \in (N:M)$ Now, $(a_1^2 + b_1^2) (a_2^2 + b_2^2) K \subseteq N$ we Have $a_1^2 a_2^2 + a_1^2 b_2^2 + b_1^2 a_2^2 + b_1^2 b_2^2 \in (N:M)$ by Proposition 2.6 so $b_1^2 b_2^2 \in (N:M)$ is contradiction!

Theorem 2.8: If N is Approximately 2-Absorbing Submodule of M , then $(N:_R M)$ is approximately 2-absorbing ideal of R .

proof: Let $abc \in (N:{}_R M)$, $abc \in R$, $a^2c \notin (N:M)$, $b^2c \notin (N:M)$, show that $(ab)^2 \in (N:M)$ there exist $m_1, m_2 \in M$ such that $a^2cm_1 \notin N$, $b^2cm_2 \notin N$. But $ab(cm_1 + cm_2) \in N$ since N is approximately 2-absorbing so $a^2(cm_1 + cm_2) \in N$ or $b^2(cm_1 + cm_2) \in N$ or $(a^2b^2) \in (N:{}_R M)$ we get $a^2cm_1 + a^2cm_2 \in N$ then $a^2cm_1 \in N$ which is contradiction! if $b^2cm_1 + b^2cm_2 \in N$ then $b^2cm_1 \in N$ which is contradiction then $(ab)^2 \in (N:M)$.

Theorem 2.9: Let N be a submodule of a multiplication R -module M such that $(N:{}_R M)$ is approximately 2-absorbing ideal of R . Then N is Approximately 2-Absorbing Submodule of M .

Proof: As $(N:{}_R M) \neq R$, $N \neq M$. Let $a, b \in R$, $m \in M$, and $abm \in N$. Since M is a multiplication R -module, there exists an ideal I of R such that $Rm = IM$. Thus $abIM \subseteq N$. Hence, $abI \subseteq (N:{}_R M)$. Now by assumption, $(ab)^2 \in (N:{}_R M)$ or $a^2I \subseteq (N:{}_R M)$ or $b^2I \subseteq (N:{}_R M)$. Therefore, $(ab)^2 \in (N:{}_R M)$ or $a^2IM \subseteq N$ or $b^2IM \subseteq N$. Thus $(ab)^2 \in (N:{}_R M)$ or $a^2m \in N$ or $b^2m \in N$. by [3]

Proposition 2.10: Let R be a commutative ring, M is a cyclic R -module and N is a submodule of M . Then N is Approximately 2-Absorbing Submodule of M if and only if $(N:{}_R M)$ is approximately 2-absorbing ideal of R .

Proof: \Rightarrow Let N is T of M . Assume $M = Rm$, $m \in M$. Let $abc \in (N:{}_R M)$ when $a, b, c \in R$ but $(ab)^2 \notin (N:M)$ and $b^2c \notin (N:M)$,

prove $a^2c \in (N:{}_R M)$ it following from $abcm \in N$, and $(ab)^2m \notin N$, $b^2cm \notin N$. That, $a^2cm \in N$ then $a^2c \in (N:M)$

\Leftarrow let $(N:{}_R M)$ is approximately 2-absorbing ideal of R , let $abx \in N$, $x \in M$, $x = cm$, $c \in R$ so $abcm \in N \Rightarrow abc \in (N:{}_R m) = (N:M)$ by $(N:{}_R M)$ is approximately 2-absorbing ideal of R $a^2c \in (N:M)$ or $b^2c \in (N:M)$ or $(ab)^2 \in (N:M)$ $a^2cm \in N$ or $b^2cm \in N$ or $(ab)^2 \in (N:M)$ $a^2x \in N$ or $b^2x \in N$ or $(ab)^2 \in (N:M)$ N is Approximately 2-Absorbing Submodule of M .

Proposition 2.11: Suppose M is a finitely generated multiplication R -module. If I approximately is 2-absorbing ideal of R such that $\text{ann}M \subseteq I$, then IM is Approximately 2-Absorbing Submodule of M .

Proof: Let $abm \in IM$ where $a, b \in R$, $m \in M$, hence $ab(m) \subseteq IM$. Since M is multiplication then $(m) = JM$ for some ideal J of R . Thus $abJM \subseteq IM$

$abJ \subseteq I + \text{ann}M = I$ by [3]. But I is approximately 2-absorbing ideal of R , so either $(ab)^2 \in I$ or $a^2J \subseteq I$ or $b^2J \subseteq I$, it follows that $(ab)^2 \in (IM:{}_R M)$

or $a^2JM \subseteq IM$ or $b^2JM \subseteq IM$; that is either $(ab)^2 \in (IM:{}_R M)$ or $a^2(m) \subseteq IM$ or $b^2(m) \subseteq IM$. Thus $(ab)^2 \in (IM:{}_R M)$ or $a^2m \in IM$ or $b^2m \in IM$ and so IM is Approximately 2-Absorbing Submodule of M .

Corollary 2.12: Suppose M is a faithful finitely generated multiplication R -module. If I is approximately 2-absorbing ideal of R , then IM is Approximately 2-Absorbing Submodule of M .

Proof: It follows directly by Proposition 2.11.

Corollary 2.13: Let M be a faithful finitely generated multiplication R -module. Then every proper submodule of M is Approximately 2-Absorbing Submodule if and only if every proper ideal of R is approximately 2-absorbing.

Proof: \Leftarrow It follows directly by Corollary (2.12) .

\Rightarrow Let I be a proper ideal of R . Then $N = IM$ is a proper submodule of M

So it is Approximately 2-Absorbing Submodule and hence by Theorem (2.8) , $(N:M)$ is approximately 2-absorbing ideal . But M is faithful finitely generated multiplication R -module , so $(N:M)=I$ by [3,Theorem.3.1].

Proposition 2.14: Let M be an R -module , N a proper submodule of M ,if N is approximately 2-absorbing then $(N :_R \langle m \rangle)$ is approximately 2-absorbing ideal for each $m \in M - N$.

Proof: Let $abc \in (N :_R \langle m \rangle)$ then $abcm \in N$, but N is approximately 2-absorbing submodule then $a^2(cm) \in N$ or $b^2(cm) \in N$ or $(ab)^2 \in (N :_R M)$, so that $a^2 cm \in N$ or $b^2 cm \in N$ or $(ab)^2 M \subseteq N$ that is $a^2 c \in (N : (m))$ or $b^2 c \in (N :_R (m))$ or $(ab)^2 \in (N :_R (m))$ hence $(N :_R (m))$ is approximately 2-absorbing ideal.

Proposition 2.15: Let N be a proper submodule of an R -module M . The following statements are equivalent:

1. N is Approximately 2-Absorbing Submodule of M
2. $(N :_M I)$ is approximately 2-absorbing , for each ideal I of R with $IM \not\subseteq N$
3. $(N :_M (r))$ is approximately 2-absorbing submodule for each $r \in R$ with $rM \not\subseteq N$

Proof: (1) \Rightarrow (2) Let I be an ideal of R with $IM \not\subseteq N$ is a proper submodule of M

Let $abm \in (N :_M I)$ where $a, b \in R, m \in M$, then $ab(Im) \subseteq N$ But N is approximately 2-absorbing submodule of M , so by Proposition(2.6) , either $a^2(Im) \subseteq N$ or $b^2(Im) \subseteq N$ or $(ab)^2 \in (N :_R M) \subseteq ((N : I) : M)$. Hence either $a^2 m \in (N : I)$ or $b^2 m \in (N : I)$ or $(ab)^2 \in ((N :_M I) : M)$. Thus $(N :_M I)$ is approximately 2-absorbing submodule .

(2) \Rightarrow (3) It is clear .

(3) \Rightarrow (1) Take $r=1$ then $(N : (1))= N$, so N is approximately 2-absorbing.

Proposition 2.16: Let N is Approximately 2-Absorbing Submodule of an R -module M . For any $a, b, c \in R, m \in M$, $abcm \in N$ implies either $(ab)^2 m \in N$ or $a^2 c m \in N$ or $b^2 c m \in N$

Proof: Let $abm \in N$, then $ab(cm) \in N$ Since N is T then $a^2(cm) \in N$ or $b^2(cm) \in N$ or $(ab)^2 \in (N:M)$.So $a^2 c m \in N$ or $b^2 c m \in N$ or $(ab)^2 m \in N$.

Theorem 2.1.17: Let N be a proper submodule of an R -module M . Consider the following statements:

1. N is Approximately 2-Absorbing Submodule of M
2. For each $a, b \in R, m \in M$. If $a^2 b^2 m \notin N$, then $(N : abm) \subseteq (N : a^2 m) \cup (N : b^2 m)$.
3. For each $a, b \in R, m \in M$ if $a^2 b^2 m \notin N$ then $(N : abm) \subseteq (N : a^2 m)$ or

$(N : abm) \subseteq (N : b^2 m)$ Then (1) \Rightarrow (2) \Rightarrow (3) and if M is cyclic , then (3) \Rightarrow (1)

Proof: (1) \Rightarrow (2) Let $c \in (N : abm)$ then $cabm \in N$ so $abcm \in N$. Since N is T , we get either $a^2cm \in N$ or $b^2cm \in N$ or $a^2b^2m \subseteq N$. but $a^2b^2m \notin N$ (by hypotheses), Then only $c \in (N : a^2m)$ or $c \in (N : b^2m)$

Thus $(N : abm) \subseteq (N : a^2m) \cup (N : b^2m)$.

1. **(3)** It is clear.

2. **(1)** Now suppose that $M = (m_1)$ for some $m_1 \in M$. Let $abm \in N$. But $m = m_1r$ for some $r \in R$ then $abrm_1 \in N$ and suppose that $a^2b^2 \notin (N : M)$ so, $a^2b^2m_1 \notin N$ and $r \in (N : abm_1)$. Hence either $r \in (N : a^2m_1)$ or $r \in (N : b^2m_1)$ by condition (3). And so $a^2rm_1 \in N$ or $b^2rm_1 \in N$, that is either $a^2m \in N$ or $b^2m \in N$. Hence N is Approximately 2-Absorbing Submodule.

N of M is called a pure submodule of an R -module M if $IM \cap N = IN$ or any ideal I of R [4].

Proposition 2.18 : Let N be a proper pure submodule of an R -module M , If (0) is Approximately 2-Absorbing Submodule of M , then N is Approximately 2-Absorbing Submodule.

Proof: Let $abm \in N$ where $a, b \in R, m \in M$ Put $I = (ab)$ then $abm \in IM \cap N$, but $IM \cap N = IN$, so $abm = abn$ for some $n \in N$, then $ab(m - n) = 0$, but (0) is approximately 2-absorbing then $a^2(m - n) = 0$ or $b^2(m - n) = 0$ or $(ab)^2 \in \text{ann}M \subseteq (N : M)$. So we get $a^2m = a^2n \in N$ or $b^2m = b^2n \in N$ or $(ab)^2 \in (N : M)$, Thus N is Approximately 2-Absorbing Submodule.

proposition 2.19 : Let N be a proper submodule of an R -module M and S multiplicative subset of R , then $S^{-1}N$ is T of $S^{-1}R$ -submodule of $S^{-1}M$ if N is Approximately 2-Absorbing Submodule.

proof: Let $\bar{a}, \bar{b} \in S^{-1}R, \bar{x} \in S^{-1}M$, then $\bar{a} = \frac{a_1}{s_1}, \bar{b} = \frac{b_1}{s_2}, \bar{x} = \frac{x_1}{s_3}$ for some

$a_1, b_1, \in R, x_1 \in M, s_1, s_2, s_3 \in S, \frac{a_1}{s_1} \frac{b_1}{s_2} \frac{x_1}{s_3} \in S^{-1}N$ then $\frac{a_1 b_1 x_1}{s_1 s_2 s_3} \in S^{-1}N$ where $s_1 s_2 s_3 = t \in S$. Then there exists $t_1 \in S$ such that $t_1 a_1 b_1 x_1 \in N$ since N is T we get $t_1 a_1^2 x_1 \in N$ or $b_1^2 x_1 \in N$ or $t_1 (a_1 b_1)^2 \in (N : M)$. If $t_1 a_1^2 x_1 \in N$, then $\frac{t_1 a_1^2 x_1}{t_1 s_1 s_3} \in S^{-1}N$, so $\frac{a_1^2}{s_1} \frac{x_1}{s_3} \in S^{-1}N$. If $b_1^2 x_1 \in N$, then $\frac{b_1^2 x_1}{s_2 s_3} \in S^{-1}N$ so $\frac{b_1^2}{s_2} \frac{x_1}{s_3} \in S^{-1}N$. If $t_1 (a_1 b_1)^2 \in (N : M)$ then $\frac{t_1 (a_1 b_1)^2}{t_1 s_1 s_2} \in S^{-1}N$ so $\frac{a_1^2}{s_1} \frac{b_1^2}{s_2} \in S^{-1}(N : M)$. But $S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$ then $\frac{a_1^2}{s_1} \frac{b_1^2}{s_2} \in (S^{-1}N : S^{-1}M)$. Hence $S^{-1}N$ is Approximately 2-Absorbing Submodule.

Let $\varphi: M \rightarrow S^{-1}M$ defined by $\varphi(m) = \frac{m}{1}$ for each $m \in M$. It is clear that φ is an R -homomorphism

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