On b*- COMPACTNESS and b- COMPACTNESS

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ABSRACT

In this paper we use the notions of b- open sets and b*-open sets to introduce and study new generalizations of compactness, which are termed by b-compactness and b*- compactness. We show that b- compactness is strictly stronger than b*- compactness, and this is stronger than compactness. Several characterizations and properties of these spaces are given. Further we introduce and study the notion of b-irresolute functions.

1. INTRODUCTION.

The notion of b-open sets was introduced in 1996,by D. Anderjivic[1].The class of all b- open (resp. b-closed) sets are denoted by BO(X) (resp. BC(X)). The arbitrary union of b-open sets is a b-open set, while the intersection of a finite number of b-open sets is not necessarily a b-open [1]. It was shown in [1] that, PO(X)$\subseteq$BO(X) $\subseteq \beta$O(X) where PO(X) is the class of all pre-open sets[4], and βO(X) is the class of all β-open sets[2]. The topology generated by BO(X) was introduced in [1] and denoted by τ_b so that, $\tau_b = \{V \subset X : V \cap S \in BO(X)$,whenever $S \in BO(X)\}$. We denote by b*- open the set S$\in \tau_b$, and its complement by b*-closed.

Throughout this paper X and Y will denote topological spaces on which no separation axioms are assumed unless otherwise stated.

2. PRELIMINARIES.

Definition 2.1.

A subset S of a space X is called a b –open set [1] if $S \subset \text{int}(cl(S)) \cup cl(in(S))$ and a b-closed if $cl(in(S)) \cap in(cl(S)) \subset S$, where “int” and “cl” stand for interior and closure operators respectively.

Remark 2.2.

For any topological space (X,τ), we have.
$\tau \subseteq \tau_b \subseteq$ BO(X) and the converse are not true in general as the following example shows.

Example 2.3.

Let X = {a, b, c}, τ = {X, ϕ, {a }, {b}, {a, b}}. Then BO(X) = {X, ϕ, {a}, {b}, {a, b},{a, c},{b, c}} and τ_b = {X, ϕ, {a}, {b}, {a,b}}.If A = { a, c}, then A is b-open set but not b*-open.

Now if τ = { X, ϕ, {a }}, then BO(X) = {X, ϕ, {a},{b}, {a, b},{a, c}}, and τ_b = {X, ϕ, {a}, {a, b}, {a, c}}, then {a, b} is b*-open but not open.

Definition 2.4.

A subset U of a space X is called b*-neighborhood (resp. b- neighborhood) of a point x of X if there exist $S\in\tau_b$ (resp. $S\in$ BO(X)) such that $x\in S\subseteq U$. We denote by $\mathcal{U}^*_x$ (resp. $\mathcal{U}_x$) the family of all b*-neighborhood (resp. b-neighborhood) of x.

Remark 2.5.

Every neighborhood is b*-neighborhood and every b*-neighborhood is a b-neighborhood but the converse is not true as the fallowing example shows:

Example 2.6.

Let X = {a, b, c}, τ = {X, Ø, {a, b}}.
Then BO(X) = {X, Ø, {a}, {a, c}, {c, b}, {a, b}, {b}} and
τ_b = {X, Ø, {a}, {a, b}, {b}}. Therefore {b} is b*-neighborhood of b hence b-neighborhood, but not a neighborhood of b, and {c, b} is a b-neighborhood of c but not b*- neighborhood of c.

Definition 2.7.

Let (X,τ) be a topological space, then the smallest b*-closed (resp. b-closed) set containing a set A is called the b*-closure (resp. b-closure) of A and denoted by *b*cl(A)* (resp. *bcl(A)).*

The following remark is an immediate consequence of Definition 2.7.

Remark 2.8.

For any subset A of a space X, we have.

(1) $A\subseteq bcl(A)\subseteq b^*cl\,(A)\subseteq cl(A)$.

(2) *A* is b- closed (resp. b*- closed)if and only if *bcl(A)=A* (resp. b*cl(A)=A) , and *A* is b*-closed if and only if *b*cl(A)=A.*

Lemma 2.9.

Let (X,τ) be a topological space, Let $A\subseteq X$, then $x\in b^*cl(A)$ (resp. $x\in bcl(A)$) if and only if for each b*-open (resp. b-open) set U containing x, $A\cap U\neq Ø$.

Proof.

Since τ_b is a topology on X, so the proof of the first case is obvious. We prove the second case:

Suppose that $x \notin bcl(A)$.Since *bcl(A)* is a b- closed set, then *U=X-bcl(A)* is b- open set containing *x* and $U \cap A = \phi$.

Conversely, Let $x \in X$, Suppose that there exist a b-open set *U* containing *x* such that $U \cap A = \phi$. Then *X-U* is b-closed and $A \subset X - U$. Hence $bcl(A) \subset X - U$. Therefore $x \notin bcl(A)$.

Now we introduce the following definition.

Definition 2.10.

Let £ be a filter on the space X, then the b*-closure of £ is denoted by b*cl(£) and defined by b*cl(£) = ∩ {b*cl(F): F ∈£} and its elements are called the b*-cluster points of £. It is clear that b*cl(£) is a b*-closed set.

Recall that a filter £ on a space X is convergent to a point x of X if and only if every neighborhood of x is in £.

Definition 2.11.

A filter £ on a space X is said to be b*-convergent to a point x of X if and only if ư$^*_x$ ⊂ £.

Remark 2.12.

By Remark 2.5, it follows that if a filter £ on a space X is b*-convergent to x, then it is convergent to x. but the converse may be false as the following example shows.

Example 2.13.

Let (X,τ) be as in example 2.6, let £= {X, {a,b}}, then £ is a filter on X and £ converges to b, and it does not b*-converges to b.

Remark 2.14.

Since τ_b is a topology on X, then ư$^*_x$ is a filter on X.

Theorem 2.15.

Let £ be a filter on a space X, and x ∈ X, then x is a b*-cluster point of £ if and only if there exists a filter finer than £ which b*-converges to x.

Proof .

Let x be a b*-cluster point of £, then x ∈ b*cl(F) for each F∈£. Hence by Lemma 2.9, for each U∈ ư$^*_x$ and for each F ∈ £, U∩F $\neq \phi$, Therefore the family £ ∪ư$^*_x$ of generates some filter £* on X. Since £∪ư$^*_x$ ⊂ £*, then £* is finer than £ and it b*-converges to x.

Conversely, if there is a filter £* finer than £ and ư$^*_x$ which is b- convergent to a point x, then every b*-open set containing x intersects every F∈£, and hence $x \in b^*cl(F)$ for each $F \in$ £. Thus x is a b*-cluster point of £.

3. b*- COMPACTNESS and b-COMPACTNESS.

We introduce the following definitions.

Definition 3.1.

Let (X, τ) be a topological space, a family ϐ of subset of X is said to be a b*-covering (resp.b-covering) of X if ϐ covers X and $ϐ \subset \tau_b$ (resp. $ϐ \subset$ BO(X)) .

Definition 3.2.

A space X is said to be b-compact (resp. b*-compact), if and only if every b-covering (resp.b*-covering) of X has a finite subcover.

Remark 3.3.

By Remark 2.2, it follows that b-compactness is stronger than b*-compactness and this is stronger than compactness. The converse is not true, as the following example shows.

Example 3.4.

Let $X = \{x\} \cup \{x_\alpha : \alpha \in \Delta\}$ where the indexed set Δ is uncountable. Let $\tau = \{X, \phi, \{x\}\}$ be the topology on X. It is clear that X is compact space, but it is not b*-compact and hence it is not b-compact, since $\{\{x, x_\alpha\}: \alpha \in \Delta\}$ is a b*-covering of X but it has no finite subcover. Hence X is not b-compact.

The next result gives several characterizations of b*-compact spaces. We notice that property (5) holds for b*-compact spaces, but in general it fails for compact spaces.

Theorem 3.5.

Let (X,τ) be a topological space. Then the following are equivalent:

1. X is b*-compact.
2. Every b*-covering of X has a finite subcover.
3. Every filter over X possesses at least one b*-cluster point.
4. Any ultrafilter over X is b*-converges.
5. Each family of b*-closed sets in X whose intersection is empty contains a finite subfamily whose intersection is empty.

Proof.

(1 ⇔ 2): Follows directly from Definition 3.2.

(3⇒4): Let (3) holds and ų be any ultrafilter over X. Since ų is a filter, hence by (3) there is a b*-cluster point x of ų. Therefore by Theorem 2.15, there is a filter finer than ų which b*-converges to x, this is ų itself, since it is an ultrafilter.

(4⇒3): Let (4) holds and £ be a filter on X. There is an ultrafilter ų finer than £, which is ų is b*-convergent to a point x . Therefore x is b*-cluster point of £.

(3⇒5): Let (3) holds and ξ be a family of b*-closed sets in X whose intersection is empty. If the intersection of any finite subfamily of ξ is not empty, then ξ will generate a filter on X. Hence by (3), ξ will have a b*-cluster point. Since each element of ξ is b*-closed, so by Remark 2.8 this b*-cluster point belongs to each element of ξ. Thus the intersection of ξ is not empty which contradicts the assumption. Therefore (5) holds.

(5⇒3): Suppose that (5) holds, and £ is a filter on X which has no b*-cluster point. Then the family {b*cl (F): F∈ £} is a family of b*-closed sets in X whose intersection is empty, but there exists no finite subfamily of sets of £ which does not intersect, and this contradicts (5).

(2 ⇔ 5): This follows from Demogan's formulae.

Definition 3.6.

A point x of a space X is said to be a b-limit (resp. b*-limit) point of a set A if every b- open (resp. b*-open) set containing *x* contains a point of A distinct from x.

It is clear that, if *x* is a b-limit point, then it is a b*- limit point, by Remark 2.2.

Theorem 3.7.

If X is b-compact and A is an infinite subset of X, then A has at least one b-limit point in X.

Proof:

Suppose that A has no b-limit point in X. Then for each x ∈ X, there is a b-open set V_x containing x such that $V_x \cap A = \{x\}$ (or $= \phi$), then $V = \{V_x : x \in X\}$ is a b-covering of X. Since X is b-compact, there is a finite subcover of V, say $V^* = \{V_{x_1}, V_{x_2}, \ldots, V_{x_n}\}$, n ∈ N. Since each V_{x_i}, i=1,2,…,n contains at most one element of A, so A is a finite set. This is a contradiction.

Similarly we can prove the following theorem.

Theorem 3.8.

If X is b*-compact and A is an infinite subset of X, then A has at least one b*-limit point in X.

Lemma 3.9.[5]

Let (X, τ) a topological space, Y be a subspace of X, and $A \subset Y$ then,

(1) $cl_Y(A) = cl(A) \bigcap Y$.

(2) $\text{int}(A) = \text{int}_Y(y) \bigcap \text{int}(A)$.

Where int_Y (resp. cl_Y) denotes the interior (resp. closure) operator relative to Y.

Lemma 3.10.[3]

Let A be an open set in X, then for every subset B of X we have

$$A \cap cl(B) \subset cl(A \cap B)$$

Now we introduce the following result.

Theorem 3.11.

If Y is an open subspace of a space X and A$\in$ BO(Y), then A $\in$ BO (X).

Proof:

Since A $\in$BO(Y), Then $A \subset \text{int}_Y(cl_Y(A)) \bigcup cl_Y(\text{int}_Y(A))$,. Since $A \subset Y$ and $\text{int}(Y) = Y$.

Therefore $A = A \bigcap Y = A \bigcap \text{int}(Y) \subset [\text{int}_Y(cl_Y(A)) \bigcap \text{int}(Y)] \bigcup [cl_Y(\text{int}_Y(A) \bigcap \text{int}(Y)]$. Now

$\text{int}_Y(cl_Y(A)) \bigcap \text{int}(Y) = \text{int}(cl_Y(A))$, (by Lemma 3.9)

$= \text{int}(cl(A) \bigcap Y)$, (by Lemma 3.9)

$\subseteq \text{int}(cl(A))$

and $cl_Y(\text{int}_Y(A)) \bigcap \text{int}(Y)$

$= [cl(\text{int}_Y(A)) \bigcap Y] \bigcap \text{int}(Y)$

$= cl(\text{int}_y(A) \bigcap \text{int}_Y(Y)] \bigcap Y$

$= cl(\text{int}_Y(A)) \bigcap Y$

$\subset cl(\text{int}_Y(A) \bigcap Y)$, (by Lemma 3.10)

$= cl(\text{int}_Y(A) \bigcap \text{int}(Y))$

$= cl(\text{int}(A))$, (by Lemma 3.9)

Therefore, $A \subset \text{int}(cl(A)) \bigcup cl(\text{int}(A))$ .Thus $A \in BO(X)$.

Theorem 3.12.

If Y is an open, b-closed (resp.b*-closed) subspace of a b-compact (resp.b*-compact) space X, then Y is b-compact (resp.b*-compact).

Proof:

For the case where Y is open, b- closed and X is b- compact.
Let $\{A_\alpha: \alpha \in \Delta\}$ be a b-covering of Y relative to Y. Since Y is open in X, so by Theorem 3.11, $A_\alpha \in$ BO(X) for each $\alpha \in \Delta$. Since Y is a b-closed set, hence X – Y $\in$ BO(X). Therefore the family $\{A_{\alpha}: \alpha \in \Delta\} \cup \{ X - Y \}$ is a b-covering of X. Since X is b-compact, there is a finite subcovering say $\{A_{\alpha_1}, A_{\alpha_2},.., A_{\alpha_n}\} \cup \{X - Y\}$, but $Y \subseteq X$ and {X – Y} covers no part of Y, so $\{A_{\alpha_1}, A_{\alpha_2},\ldots, A_{\alpha_n}\}$ covers Y. Hence Y is b-compact.

The second case holds, because τ_b is a topology.

Similarly, we can prove the following result.

Theorem 3.13.

Every b-closed (resp.b* closed) subset of a b-compact (resp. b*-compact) space is compact.

Now we introduce the following definition.

Definition 3.14.

A function $f : X \to Y$ is said to be b-irresolute (resp. b-continuous) if the inverse image of every b- open (resp. open) set of Y is a b-open set in X.

Remark 3.15.

Continuous function and b-irresolute functions are independent as the following examples show:

Example 3.16.

Let $X = \{1, 2, 3\}$, $Y = \{a, b\}$, let $\tau = \{X, \phi, \{1\}, \{1, 3\}\}$, and $\sigma = \{Y, \phi, \{a\}\}$ be a topologies on X and Y respectively. Then $BO(X)=\{X, \phi, \{1\}, \{1, 3\}, \{1, 2\}\}$ and $BO(Y)= \{Y, \phi, \{a\}\}$. Define f: $(X, \tau) \to (Y, \sigma)$ as follows $f(1) = f(2) = a$, $f(3) = b$ Then f is b-irresolute but not continuous.

Example 3.17.

Let $X= Y = \{a, b, c\}$, $\tau = \{X, \phi, \{c\}\}$, $\sigma = \{Y, \phi, \{b, c\}\}$,
$f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = a$, and $f(c) = c$,
$BO(X) = \{X, \phi, \{c\}, \{a, c\}, \{b, c\}\}$,
$BO(Y) = \{X, \phi, \{b, c\}, \{a, b\}, \{a, c\}, \{c\}, \{b\}\}$. Then f is continuous, but not b-irresolute since $f^{-1}(\{a, b\}) = \{a, b\}$ which is not b-open in X.

Remark 3.18.

Every continuous function is b-continuous but not conversely, as the following example shows.

Example 3.19.

Let $X=Y= \{a, b, c\}$, let $\tau=\{X, \phi\}$ and $\sigma=\{Y, \phi, \{a\}\}$. Then $BO(X) = P(X)$, where $P(X)$ is the power set of X. Define $f: (X,\tau) \to (Y, \sigma)$ to be the identity function, then f is b-continuous, but not continuous

It is easy to prove the following result.

Theorem 3.20.

Every b- irresolute is b-continuous, but not conversely as it is shown by the following example.

Example 3.21.

Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{Y, \phi, \{b, c\}\}$. Then $BO(X)=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$, and

$BO(Y)=\{Y, \phi,\{b,c\},\{b\},\{c\},\{a,c\},\{a,b\}\}$

Let $f: (X,\tau) \to (Y, \sigma)$ be the identity map. Then f is b-continuous function but not b- irresolute, since $f^{-1}(\{c\}) = \{c\}$ which is not b-open in X.

It is easy to prove the following:

Theorem 3.22.

Let $f: (X, \tau) \to (Y, \sigma)$ be an open and continuous function, then f is b- irresolute.

Proof:

Let $A \in BO(Y)$, then $A \subset int(cl(A)) \cup cl(int(A))$. Hence $f^{-1}(A) \subset f^{-1}(int(cl(A) \cup cl(int(A))$. Since f is continuous, so $f^{-1}(int(cl(A)) \subseteq int(f^{-1}(cl(A)))$. Since f is open, so f^{-1} is continuous and $f^{-1}(cl(A)) \subseteq cl(f^{-1}(A))$. Therefore $f^{-1}(cl(int(A))) \subseteq cl(f^{-1}(int(A))) \subseteq cl(int(f^{-1}(A)))$, and $f^{-1}(int(cl(A)) \subseteq int(cl(f^{-1}(A)))$. Hence $f^{-1}(A) \subseteq int(cl(f^{-1}(A))) \cup cl(int(f^{-1}(A)))$. Hence $f^{-1}(A)$ is b-open set, thus f is b-irresolute.

Theorem 3.23.

Let $f: (X,\tau) \to (Y, \sigma)$ be a function. Then the following are equivalent

1. f is b- irresolute.
2. For $x \in X$ and any b-open set V of Y containing f(x), there exists $U \in BO(X)$, such that $x \in U$ and $f(U) \subset V$.
3. The inverse image of every b-closed set of Y is a b-closed set of X.

Proof: Obvious.

Theorem 3. 24.

Let $f: X \to Y$ be a b-irresolute function of X onto Y. If X is b-compact, then Y is b-compact.

Proof: Obvious.

Corollary 3.25.

Let $f: X \to Y$ be an open and continuous function of X onto Y. If X is b-compact, then Y is b-compact.

Proof: This follows from Theorems 3.22 and 3.24.

Corollary 3.26.

b-compactness is a topological property.

Theorem 3.27.

Let $\{X_\alpha : \alpha\in\Delta\}$ be any family of spaces if the product space $\prod X_\alpha$ is b-compact, Then X_α is b-compact for each $\alpha\in\Delta$.

Proof :

Let the product space $\prod X_\alpha$ be a b-compact, and $p_\alpha : \prod X_\alpha \rightarrow X_\alpha$ be the projection function of $\prod X_\alpha$ onto X_α which is continuous and open . Then by Corollary 3.25, each X_α is b-compact.

Note:

In Definition 3.14, if b-open set is replaced by b*-open set, then we have the definition of the b*-irresolute and b*-continuous functions, and the results of the first definition are true for the new one.

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