AL-Qadisiya Journal for science Vol. 15 No. 2 year 2010

ON GENERALIZATION OF CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER DEFINED BY USING THE RUSCHEWEYH DERIVATIVES

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Abstract : -

In this paper , we study and introduce generalization of two classes of analytic functions of complex order defined by using the Ruscheweyh derivatives in [8] . We obtain basic properties like , coefficient inequalities and radii of close – to – convexity , starlikeness and convexity .

AMS Mathematics Subject Classification : 30 C45 .

Key words : Analytic Functions , Ruscheweyh Derivatives, Complex Order , Radius of close – to – convexity.

1. Introduction :

Let *W* denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \qquad (1)$$

which are analytic in the unit disc U = { z: | z | < 1 } , let G denote the subclass of *W* consisting of analytic and univalent functions *f (z)* in the unit disc U .

About *f(z)* belong to *W*, Salagean [10] has introduced the following operator called the Salagean operator :

$$D^{\circ} f(z) = f(z), D^1 f(z) = Df(z) = zf'(z),$$

$$D^k f(z) = D(D^{k-1} f(z))(k \in IN = \{1, 2, 3, ...\}).$$

$$Note\ that\ \ D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n , k \in IN_{\circ} = \{0\} \cup IN.$$

Now , let A denote subclass of *W* consisting of functions *f(z)* of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n , a_n \geq 0. \qquad (2)$$

Let $f(z) \in W$. Then the Ruscheweyh derivative of *f(z)* , denoted by $D^{\lambda} f$, is defined by :

$$D^{\lambda} f = f * \frac{z}{(1-z)^{\lambda+1}} = z + \sum_{n=2}^{\infty} B_n(\lambda) a_n z^n$$

$$and\ B_n(\lambda) = \frac{(\lambda+1)_{n-1}}{(n-1)!} = \frac{(\lambda+1)(\lambda+2)....(\lambda+n-1)}{(n-1)!}, (\lambda > -1),$$

where

$f * \frac{z}{(1-z)^{\lambda+1}}$ *is the Hadamard product of* $f(z)$ *and*

$\frac{z}{(1-z)^{\lambda+1}}$ ([9]).

Let $WA(\lambda, \gamma, m, b)$ be the class of functions of the form (2) , which are analytic in U and satisfy :

$$\text{Re}\{1 + \frac{1}{b}(\frac{(1-m)z(D^{\lambda} f(z))' + m(1-\gamma)z(D^{\lambda+1} f(z))' + m\gamma z(D^{\lambda+2} f(z))'}{(1-m)D^{\lambda} f(z) + m(1-\gamma)D^{\lambda+1} f(z) + m\gamma D^{\lambda+2} f(z)} - 1)\} > 0, \quad (3)$$

for some $m(0 \le m \le 1)$, $\gamma \ge 0$, b complex (Re (b) > 0) and for all $z \in U$. Also , let *WA1*(λ, γ, m, b) denote the family of functions of the form (2) , which are analytic in U and satisfy :

$$\text{Re}\{1 + \frac{1}{b}((1-m)(D^{\lambda} f(z))' + m(1-\gamma)(D^{\lambda+1} f(z))' + m\gamma(D^{\lambda+2} f(z))' - 1)\} > 0, \quad (4)$$

for some $m(0 \le m \le 1)$, $\gamma \ge 0, b$ complex (Re(b) > 0) and for all $z \in U$.

WA(0,0,0,b) and *WA*(0,0,1,b) are the classes of starlike and convex functions of complex order b. *WA1*(0,0,0,b) is the class of close – to – convex functions of complex order b.

When $\gamma = 0$, we get the classes $WA(\lambda, 0, m, b) \equiv P(\lambda, m, b)$ and $WA1(\lambda, 0, m, b) \equiv R(\lambda, m, b)$ were studied by [8] .

Aouf and Srivastava investigated the similar class of the class in [8] for some families of starlike functions with negative coefficients by using Salagean derivatives instead of Ruscheweyh derivatives in [5] and also , Kamali and Orhan studied the similar class of the class in [8] for multivalent functions with negative coefficient by using Salagean derivatives instead of Ruscheweyh derivatives in [6]. Many important properties of certain subclasses of analytic functions of complex order were studied by Atintas et al. [2 , 3 , 4] and Murugusundaramoorthy et al. [7] .

2. The Coefficient Relations for Classes *WA*(λ, γ, m, b) and *WA1*(λ, γ, m, b)

In the following theorem , we obtain the coefficient inequality of the class *WA*(λ, γ, m, b) .

<u>*Theorem 1 :*</u> Let *f(z)* ∈ *WA*(λ, γ, m, b) . Then we have

$$\sum_{n=2}^{\infty} [(1-m) + \frac{m(1-\gamma)(n-1)}{\lambda+1} + \frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}](n + |b| - 1) B_n(\lambda) a_n \le \frac{|b|^2}{\text{Re}(b)}. \quad (5)$$

Proof : From (3) , we have

$$\mathrm{Re}\{\frac{1}{b}(\frac{\sum_{n=2}^{\infty}[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}](n-1)B_n(\lambda)a_nz^n}{z-\sum_{n=2}^{\infty}[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}]B_n(\lambda)a_nz^n})\}>-1$$

If we choose z on the real axis and let $z\to 1^-$, we get

$$\frac{\sum_{n=2}^{\infty}(n-1)[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}]B_n(\lambda)a_n}{1-\sum_{n=2}^{\infty}[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}]B_n(\lambda)a_n}.\mathrm{Re}\frac{1}{b}\le 1,$$

Whence

$$\frac{\sum_{n=2}^{\infty}(n-1)[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}]B_n(\lambda)a_n}{1-\sum_{n=2}^{\infty}[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}]B_n(\lambda)a_n}.\frac{\mathrm{Re}(b)}{|b|^2}\le 1,$$

and so

$$\sum_{n=2}^{\infty}(n-1)[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}]B_n(\lambda)a_n$$

$$\le\frac{|b|^2}{\mathrm{Re}(b)}-\frac{|b|^2}{\mathrm{Re}(b)}\sum_{n=2}^{\infty}[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}]B_n(\lambda)a_n$$

$$\le\frac{|b|^2}{\mathrm{Re}(b)}-|b|\sum_{n=2}^{\infty}[(1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}]B_n(\lambda)a_n,$$

which is equivalent to (5) .

<u>*Theorem 2 :*</u> If $f(z)\in WA1(\lambda,\gamma,m,b)$, then we have

$$\sum_{n=2}^{\infty}n((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+1)(\lambda+2)})B_n(\lambda)a_n\le\frac{|b|^2}{\mathrm{Re}(b)}. \quad (6)$$

<u>*Proof :*</u> If $f(z)\in WA1(\lambda,\gamma,m,b)$, then we have from (4)

$$\mathrm{Re}\{\frac{1}{b}(-\sum_{n=2}^{\infty}n((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)})B_n(\lambda)a_nz^{n-1}\}\ge -1.$$

If we choose z on the real axis and let $z\to 1^-$, we get

$$\sum_{n=2}^{\infty}n((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)})B_n(\lambda)a_n\frac{\mathrm{Re}(b)}{|b|^2}\le 1.$$

Which is equivalent to (6).

Remark 1 : By taking $\gamma=\lambda=0$ and letting $m=\lambda$ in Theorem1 and Theorem 2 , these cases lead to results obtained by Altintas and Ozkan[1] .

3. Close – to – Convexity , Starlikeness and Convexity for the classes $WA(\lambda,\gamma,m,b)$ and $WA1(\lambda,\gamma,m,b)$

The classes $WA(\lambda,\gamma,m,b)$ and $WA1(\lambda,\gamma,m,b)$ are of special interest for it contains well – known as well as new classes of analytic univalent functions . In particular , for $\gamma=0,\lambda=0$ and $0\le m\le 1$ it provides a transition from starlike functions to convex functions . More specifically , *WA*(0,0,0,b) , *WA*(0,0,1,b) and *WA1*(0,0,0,b) are the families of functions starlike , convex and close – to – convex of complex order b , respectively , we obtain the following results :

Theorem 3 : If $f(z) \in WA(\lambda,\gamma,m,b)$, then *f*(z) is close – to – convex of complex order b in |z| < $r_1(\lambda,\gamma,m,b)$, where

$$r_1(\lambda,\gamma,m,b) = \inf_n \left\{ \frac{\left((1-m) + \frac{m(1-\gamma)(n-1)}{\lambda+1} + \frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}\right)(n+|b|-1)B_n(\lambda)\operatorname{Re}(b)}{n|b|} \right\}^{\frac{1}{n-1}},$$

$(n = 2,3,...)$.

Proof : It is sufficient to show that |*f(z)'* – 1 | < |b|. we have

$$|f'(z)-1| \le \sum_{n=2}^{\infty} na_n |z|^{n-1} \le |b| \qquad (7)$$

$$and \sum_{n=2}^{\infty}\left((1-m) + \frac{m(1-\gamma)(n-1)}{\lambda+1} + \frac{m\gamma(n-2)(n-1)}{(\lambda+1)(\lambda+2)}\right)(n+|b|-1)B_n(\lambda)a_n \le \frac{|b|^2}{\operatorname{Re}(b)} \quad (8)$$

Hence , (7) is true if

$$\frac{n|z|^{n-1}}{|b|} \le \frac{\left((1-m) + \frac{m(1-\gamma)(n-1)}{\lambda+1} + \frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}\right)(n+|b|-1)B_n(\lambda)\operatorname{Re}(b)}{|b|^2} \quad (9)$$

Solving (9) for |z| , we obtain

$$|z| = \left\{ \frac{\left((1-m) + \frac{m(1-\gamma)(n-1)}{\lambda+1} + \frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}\right)(n+|b|-1)B_n(\lambda)\operatorname{Re}(b)}{n|b|} \right\}^{\frac{1}{n-1}}$$

Theorem 4 : If $f(z) \in WA1(\lambda,\gamma,m,b)$, then *f(z)* is close – to – convex of complex order b in |z| < $r_2(\lambda,\gamma,m,b)$, where

$$r_2(\lambda,\gamma,m,b) = \inf_n \left\{ \frac{\left((1-m) + \frac{m(1-\gamma)(n-1)}{\lambda+1} + \frac{m\gamma(n-2)(n-1)}{(\lambda+1)(\lambda+2)}\right)B_n(\lambda)\operatorname{Re}(b)}{|b|} \right\}^{\frac{1}{n-1}},$$

$(n = 2,3,...)$

Proof : We must show that |*f '(z)* – 1|< |b|.Because of (7) and (6) , If

$$\frac{n|z|^{n-1}}{|b|} \le \frac{n\left((1-m) + \frac{m(1-\gamma)(n-1)}{\lambda+1} + \frac{m\gamma(n-2)(n-1)}{(\lambda+1)(\lambda+2)}\right)B_n(\lambda)\operatorname{Re}(b)}{|b|^2},$$

then $f(z) \in WA1(0,0,0,b)$.

Theorem 5 : If $f(z) \in WA(\lambda,\gamma,m,b)$, then *f(z)* is starlike of complex order b in |z| < r_3 (λ,γ,m,b) , where

$$r_3(\lambda,\gamma,m,b)=\inf_n\left\{\frac{((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)})B_n(\lambda)\operatorname{Re}(b)}{|b|}\right\}^{\frac{1}{n-1}}$$

$(n=2,3,....)$.

Proof : We must show that $\left|\frac{zf'(z)}{f(z)}-1\right|<|b|$.

$$Since\left|\frac{zf'(z)}{f(z)}-1\right|\le\frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}\le b, \qquad (10)$$

then using (5) , we see that if

$$(\frac{n+|b|-1}{|b|})|z|^{n-1}<\frac{((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}(n+|b|-1)B_n(\lambda)\operatorname{Re}(b)}{|b|^2},$$

then $f(z)\in WA(0,0,0,b)$.

Theorem 6 : If $f(z)\in WA1(\lambda,\gamma,m,b)$, then *f(z)* is starlike of complex order b in |z|< r_4 (λ,γ,m,b), *where*

$$r_4(\lambda,\gamma,m,b)=\inf_n\left\{\frac{n((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+1)(\lambda+2)})B_n(\lambda)\operatorname{Re}(b)}{|b|(n+|b|-1)}\right\}^{\frac{1}{n-1}},$$

$(n=2,3,...)$.

Proof : By using (10) and (6) , if

$$(\frac{n+|b|-1)}{|b|})|z|^{n-1}<\frac{n((1-m)+\frac{m(1-\lambda)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)})B_n(\lambda)\operatorname{Re}(b)}{|b|^2},$$

then $f(z)\in WA(0,0,0,b)$.

Theorem 7 :_If $f(z)\in WA(\lambda,\gamma,m,b)$, then *f(z)* is convex of complex order b in |z| < r_5 (λ,γ,m,b), where

$$r_5(\lambda,\gamma,m,b)=\inf_n\left\{\frac{((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)})B_n(\lambda)\operatorname{Re}(b)}{n|b|}\right\}^{\frac{1}{n-1}},$$

$(n=2,3,.....)$.

Proof : It is sufficient to show that :

$\left|\frac{zf''(z)}{f'(z)}\right| < |b|$.

From $\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}} \le b \qquad (11)$

and (5) , if

$$\frac{n(n+|b|-1)}{|b|}|z|^{n-1} < \frac{\left((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+2)(\lambda+1)}\right)(n+|b|-1)B_n(\lambda)\operatorname{Re}(b)}{|b|^2},$$

we obtain $f(z) \in WA(0,0,1,b)$.

Theorem 8 : If $f(z) \in WA1(\lambda,\gamma,m,b)$, then *f(z)* is convex of complex order b in | z | < r_6 (λ,γ,m,b) , where

$$r_6(\lambda,\gamma,m,b) = \inf_n \left\{ \frac{\left((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+1)(\lambda+2)}\right)B_n(\lambda)\operatorname{Re}(b)}{(n+|b|-1)|b|} \right\}^{\frac{1}{n-1}},$$

(n = 2 , 3 , ….) .

Proof : By using (11) and (6) , if

$$\frac{n(n+|b|-1)}{|b|}|z|^{n-1} < \frac{n\left((1-m)+\frac{m(1-\gamma)(n-1)}{\lambda+1}+\frac{m\gamma(n-2)(n-1)}{(\lambda+1)(\lambda+2)}\right)B_n(\lambda)\operatorname{Re}(b)}{|b|^2},$$

then $f(z) \in WA(0,0,1,b)$.

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حول تعميم لأصناف الدوال التحليلية من المرتبة المعقدة معرفة بواسطة استخدام مشتقات راشويا

وقاص غالب عطشان شيماء فرحان مطر
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<u>الخلاصة :</u>

في هذا البحث ، ندرس و نقدم تعميم لصنفين من الدوال التحليلية من الرتبة المعقدة معرفة بواسطة استخدام مشتقات راشويا في [8] .
حيث نحصل على نتائج متمثلة بالخصائص الأساسية مثل ، الحدود الدنيا و العليا للمعامل ، إنصاف الأقطار للتحدب المغلق ، ستارلايك و التحدب.