A COMPARISON OF BAYESIAN APPROACH WITH Maximum Likelihood Method To Estimate Parameter For Rayleigh Distribution

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المستخلص

يهدف البحث لتقدير معلمة القياس لتوزيع Rayleigh باستخدام طريقة الامكان الاعظم (MLE)والطريقة البيزية باسلوبين: الاول لا يعتمد على معلومة فيشر والثاني يعتمد على معلومة فيشر وفي الاسلوبين تم اسخدام دوال خسارة(دالة الخسارة التربيعية المعدلة و دالة خسارة الانتروبي العامة) , ولايجاد افضل تقدير لمعلمة القياس تم اختيار حجوم العينات (10, 20, 30, 50, 100) وتمت المقارنة بين الطرائق باستخدام المقياس الاحصائي متوسط مربعات الخطأ (MSE) الذي سجل اقل قيمة له عند مقدر بيز باستخدام معلومة فيشر مع دالة الخسارة التربيعية المعدلة ولجميع حجوم العينات .

Abstract

The main objective of this study is to obtain and compare the performance of "Maximum likelihood estimator (MLE) and Bayesian estimators of the scale parameter θ of the Rayleigh distribution. In order to get better understanding in our Bayesian analysis we consider informative prior as well as non-informative prior using Jeffery prior information under loss functions (modified squred error loss function, Quadratic error loss function) to find the best method for estimation, which used the samples size (10, 20, 30, 50, 100). The comparison of the estimators, based on their mean squared errors (MSE's), we obtain that, MinMSE is the best estimator, while the performance of Bayes estimator under modified squared error loss function with non-informative prior with a value of the scale parameter $(\theta \geq 1.5)$ is the best estimator comparing to other for all simple size.

1. Introduction

Inferences on the Rayleigh distribution have been studied by many authors. Rasheed H. A. (2011)[7] estimated the scale parameter of the Rayleigh distribution by applying the Bayes estimators under different loss functions (using Jeffrey prior information). Dey S. (2012) [2] Bayes estimators are obtained under symmetric and asymmetric linear exponential loss functions using a non-informative prior. Oayd R. G. (2012)[6] derived the standard Bayes estimators for scale parameter and reliability function and failure rate function of Rayleigh distribution. Kazem T. H., Rashid H. A., Al Obeidi N. J.(2012)[4] used Bayes' estimators for the mean of Rayleigh distribution with three different prior prior by Jeffery's prior based on loss functions (modified squred error loss function, Quadratic error loss function). The comparison was based on the simulation method information's are presented under the squared error loss function. Al-Bderi H. J.(2013)[1] estimated the shape and scale parameters together in generalized

Rayleigh distribution by using three Nonbayesian (classical) and three Bayesian methods (Maximum Likelihood Estimator, Ordinary Least Squares Estimator, Rank Set Sampling Estimator, Standard Bayes Estimator, Lindley Approximation Estimator and Shrinkage Estimator. Globe H. I., Shafiq M. B.(2013)[3] estimated the scale parameter for Rayleigh distribution using Bayes method depending Jeffrey's information method.The aim of this study was to estimate the scale parameter of Raleigh distribution using the maximum likelihood and Bayes estimators are obtained under informative prior, as well as under non-informative Prior by Jeffery's prior basedon loss functions (modified squred error loss function, Quadratic error loss function). The comparison was based on the simulation method

2. Rayleigh distribution

Let us consider $t_1, t_2, \ldots, t_n$ to be independent and identically distributed random variable from rayleigh distribution having pdf:[1]

$$f(t|\theta) = \begin{cases} \frac{2}{\theta} t e^{-\frac{t^2}{\theta}} & \theta > 0 \\ 0 & o.w \end{cases} \quad \ldots (1)$$

Where θ is the shape parameter.The cumulative distribution function (cdf) is given by:

$$F(t|\theta) = 1 - e^{-\frac{t^2}{\theta}} \quad \ldots\ldots\ldots (2)$$

3. Maximum Likelihood Estimator

The likelihood function for Rayleigh distribution pdf is given by:

$$L(t_i;\theta) = \frac{2^n}{\theta^n} t^n e^{-\frac{\sum_{i=1}^{n} t^2}{\theta}} \quad \ldots.(3)$$

By taking the log and differentiating partially with respect to θ, We get:

$$\frac{\partial lnL(t_i;\theta)}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} t^2}{\theta^2} \quad \ldots..(4)$$

Then the MLE of θ is the solution of equation (4) after equating the first derivative to zero is given by:

$$\hat{\theta} = \frac{\sum_{i=1}^{n} t^2}{n}$$

$$\hat{\theta} = \frac{p}{n} \quad Where\ p = \sum_{i=1}^{n} t^2 \quad \ldots..(5)$$

4. Bayes Estimators

4.1. This method is based on the assumption of distribution unknown parameter to this former distribution parameter dictate as follows:

$$g_1(\theta) = \frac{1}{\theta^c}, 0 < \theta < \infty,\ c > 0 \ldots.(6)$$

So, the posterior distribution for θ using Jeffery prior is:

$$h_1(\theta|t) = \frac{L(t_i;\theta).g_1(\theta)}{\int_0^\infty L(t_i;\theta)g_1(\theta)d\theta} \quad \ldots (7)$$

Substituting (3) , (6) in (7) we get:

$$h_1(\theta|t) = \frac{p^{n+c-1} e^{\frac{-p}{\theta}}}{\theta^{n+c}\Gamma(n+c)},$$

$$where\ p = \sum t^2 \ldots\ldots(8)$$

The posterior density is recognized as the density of the Gamma distribution: $\theta \sim Gamma\ (n+c,\theta)$

i. Jeffreys prior information, under modified squred error loss function[7]

$$L(\hat{\theta},\theta)=\theta^{r}(\hat{\theta},\theta)^{2}$$

$$Risk=E(L(\hat{\theta},\theta)$$

$$=\int_{0}^{\infty}\theta^{r}(\hat{\theta},\theta)^{2}\,h_1(\theta|t)d\theta$$

By taking differentiating partially with respect to $\hat{\theta}$, and equating the first derivative to zero is given by:

$$\hat{\theta}=\frac{\int_0^\infty \theta^{r+1}\ h_1(\theta|t)d\theta}{\int_0^\infty \theta^{r}\ h_1(\theta|t)d\theta}$$

$$\hat{\theta}_1=\frac{p}{n+c-r-1} \qquad \ldots..(9)$$

ii. Jeffrey's prior information, under Quadratic error loss function[5]:

$$L(\hat{\theta},\theta)=\left(\frac{\hat{\theta}-\theta}{\theta}\right)^{2}$$

$$Risk=E(L(\hat{\theta},\theta))$$

$$=\int_{0}^{\infty}\left(\frac{\hat{\theta}-\theta}{\theta}\right)^{2}h_1(\theta|t)d\theta$$

By taking differentiating partially with respect to $\hat{\theta}$, and equating the first derivative to zero is given by:

$$\hat{\theta}=\frac{\int_0^\infty h_1(\theta|t)d\theta}{\int_0^\infty \frac{1}{\theta}h_1(\theta|t)d\theta}$$

$$\hat{\theta}_2=\frac{p}{n+c} \qquad \ldots..(10)$$

4.2. Bayes estimators for the parameter θ, was considered with non-informative prior.

$g_1(\theta)\propto\sqrt{I(\theta)}$, where $I(\theta)$ is the Fisher information . Then

$$g_1(\theta)=b\sqrt{I(\theta)} \qquad \ldots.(11),$$

where b is constant

$$I(\theta)=-nE\left(\frac{\partial^2 \ln l(t|\theta)}{\partial\theta^2}\right) \quad \ldots(12)$$

From (6),(7) we get:

$$g_1(\theta)=\frac{nb\sqrt{3}}{\theta} \qquad \ldots..(13)$$

So, the posterior distribution for θ using Jeffery prior is:

$$h_1(\theta|t)=\frac{L(\theta|t_1,t_2,\ldots,t_n)g_1(\theta)}{\int_0^\infty L(\theta|t_1,t_2,\ldots,t_n)g_1(\theta)\,d\theta}\ .(14)$$

Substituting (3) , (8) in (9) which has the following probability density function:

$$h_1(\theta|t)=\frac{p^{n+2}e^{-\frac{p}{\theta}}}{\theta^{n+1}\Gamma(n+2)} \qquad \ldots(15)$$

From the Equation (15) we note that p is distributed with Gamma distribution $\Gamma\ (n+2,\theta)$

By using modified

i- Jeffreys prior non-informative prior, under modified squred error loss function.

$$L(\hat{\theta},\theta)=\theta^{r}(\hat{\theta},\theta)^{2}$$

$$Risk=E(L(\hat{\theta},\theta))$$

$$=\int_{0}^{\infty}\theta^{r}(\hat{\theta},\theta)^{2}\,h_2(\theta|t)d\theta$$

By taking differentiating partially with respect to $\hat{\theta}$, and equating the first derivative to zero is given by:

$$\hat{\theta}_4 = \frac{p}{n-r} \quad \dots\dots(16)$$

ii- Jeffreys prior non-informative prior, under Quadratic Loss function

$$l(\hat{\theta},\theta) = \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 \quad \dots(17)$$

According to the above mentioned loss functions, we drive the corresponding Bayes' estimators for θ using Risk function $R(\hat{\theta}-\theta)$ which minimize the posterior risk

$$R(\hat{\theta}-\theta) = \int_0^\infty \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 h_1(\theta|t)d\theta$$

By taking differentiating partially with respect to $\hat{\theta}$, and equating the first derivative to zero is given by:

$$\hat{\theta}_5 = \frac{p}{n} \quad \dots(18)$$

5. Simulation Study and Results

In this paper we used the simulation in Monte Carlo to compare the different methods that are used to estimate the parameter of Rayleigh distribution, This method is summarized in the following steps:

1. Specify the default values:

- Select many different samples size (n), where n=10, 20, 30, 50, and 100.
- Different values were selected for the scale parameter θ, where $\theta = 0.5, 1.5, 2.5, 3.5$.
- Assuming the values c and r in the posterior distributon and modified squred error loss function respectively ,which are as follows: c=2, r=3.

2. Generate data:

- Random variable values are generated according to the reverse conversion method:

$$t_i = [-\theta \ln(1-u_i)]^{\frac{1}{2}} \quad \dots\dots\dots(19)$$

- Comparing all methods of estimation for the scale parameter θ by employing the mean squares error (MSE) which is defined as follows:

$$MSE(\hat{\theta}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\theta}_i-\theta)^2 \quad \dots(20)$$

Where L is the number of replications,(L=1000).

3. Results: The table(1) shows the results of the study.

6. Conclusions

The results of the simulation study for estimating the scale parameter (θ) of Rayleigh distribution, are summarized and tabulated in table (1) which contain the MSE's for estimating the scale parameter, we have observed that:

- The results showed that, the performance of Bayes estimator under modified squred error loss function with non-informative prior with a small value of $(\theta = 0.5)$ is the best estimator comparing to others.
- the performance of Bayes estimator under Quadratic error loss function with posterior distribution with a value of (θ=1.5) is the best estimator comparing to other for small simple size .

- In general, the performance of Bayes estimator under modified squred error loss function with non-informative prior with a value of the scale parameter $(\theta \geq 1.5)$ is the best estimator comparing to other for all simple size.
- The results showed that the sample size increased with decreasing values MSE and this conform to the statistical theory.

References

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- **Table 1: MSE of estimated parameter of *Rayleigh distribution* with *c =2* , *r = 3***

θ	n	M.L.E	Bayes1(M.S.L)	Bayes1(Q.L)	Bayes2(M.S.L)	Bayes2(Q.L)	Best
0.5	10	1.3004e-05	2.7777e04	2.0043e-04	6.77477e-06	1.3004e-05	Bayes2(M.S.L
	20	1.36496e-06	1.31578e-05	1.46009e-05	3.05712e-07	1.36496e-06	Bayes2(M.S.L)
	30	1.3649e-06	8.62028e-06	1.05727e-07	3.81675e-08	1.36496e-06	Bayes2(M.S.L)
	50	3.6239e-07	5.10204e-06	8.166527e-08	8.10441e-08	3.6239e-07	Bayes2(M.S.L)
	100	4.6089e-08	2.52525e-06	2.21239e-08	2.12201e-08	4.60891e-08	Bayes2(M.S.L)
1.5	10	5.71288e-03	2.50000E-04	2.21239e-04	1.38731e-03	5.71288e-03	Bayse1(Q.L)
	20	1.04161e-03	1.18421e-04	2.10672e-04	1.37333e-03	1.04161e-03	Bayse1(Q.L)
	30	2.96291e-04	7.75862e-05	2.10487e-04	3.63696e-04	2.96291e-04	Bayse1(M.S.L)
	50	2.27496e-04	4.59184e-05	1.91719e-04	2.55788e-04	2.27496e-04	Bayse1(M.S.L)
	100	1.31409e-04	2.27273e-05	1.20794e-04	1.39113e-04	1.31409e-04	Bayse1(M.S.L)
2.5	10	6.97900e-02	6.94444e-04	6.97900e-03	1.32978e-02	6.97900e-02	Bayse1(M.S.L)
	20	1.03316e-02	3.28947e-04	5.03316e-03	1.32110e-02	1.03316e-02	Bayse1(M.S.L)
	30	3.31644e-03	2.15517e-04	3.31644e-03	3.93750e-03	3.31644e-03	Bayse1(M.S.L)
	50	2.44206e-03	1.27551e-04	2.44206e-03	2.69842e-03	2.44206e-03	Bayse1(M.S.L)
	100	1.37934e-03	6.31313e-05	1.37934e-03	1.44853e-03	1.37934e-03	Bayse1(M.S.L)
3.5	10	3.17692e-02	1.36111e-03	3.65737e-02	5.37686e-02	3.17692e-02	Bayse1(M.S.L)
	20	2.38111e-02	6.44737e-04	3.19406e-02	5.33997e-02	2.38111e-02	Bayse1(M.S.L)
	30	1.46619e-02	4.22414e-04	3.17098e-02	1.72140e-02	1.46619e-02	Bayse1(M.S.L)
	50	1.06453e-02	2.50000e-04	9.30101e-03	1.16928e-02	1.06453e-02	Bayse1(M.S.L)
	100	5.96642e-03	1.23737e-04	5.57544e-03	6.24827e-03	5.96642e-03	Bayse1(M.S.L)