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The Regular Strongly Locally Connected Space

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Abstract:

In this paper ,we have dealt with the concept of strongly connected in topological spaces .Throughout its definition ,we have defined the regular strongly connected and found that the every strongly connected set is a regular strongly connected set, and the converse is not always true.We have also defined the concept of regular locally connected by means of which we have defined regular strongly locally connected where we have proved that every regular strongly locally connected space is regular locally connected ,and the converse is not always true .

Introduction :

Throughout the present paper X and Y always denote topological spaces on which no separation axioms are assumed unless explicity stated .This paper includes three sections.In the first section we have dealt with the concepts"regular connected set " and "regular connected space" .In the second section we have discussed the concepts "strongly set" and "regular strongly connected" and their relation with each other ,we have also dealt with the concepts " RT_0 -space" and " RT_1 -space" and shown that RT_0 -space is RT_1 -spaceand the opposite is not true always.Finally ,in the third section ,we have dealt with "regular strongly locally connected" and some theorems related to it .

1. Prilimeries

Definition 1.1 [2]

A subset A of a topological space X is called a regular open iff $A = \overset{0}{\overline{A}}$,and its complement is called a regular closed .

Remarks 1.2

1) A is regular closed iff $A = \overline{A^0}$.

2) If A is any subset of a topological space ,then ($\overset{0}{\overline{A}}$) is regular open .

3) The intersection,but not necessarily the union of two regular open sets is regular open .(Thus the same proposition with " union " and " intersection " in changed holds for regular closed sets .

Definition 1.3

A topological space X is said to be regular disconnected iff it is the union of two non-empty disjoint regular open sets, otherwise is said to be regular connected .

Lemma 1.4

A subset A in a topological space which is both open and closed is regular open and regular closed in the same time .

Proof :

Let $A \subseteq X$ such that A is both open and closed .

$A = \overline{A}$ because A is closed, and since A is open ,then $A = \overset{0}{A} = \overset{0}{\overline{A}}$,then A is regular open .

Similarly

Since $A = \overset{0}{A}$,because A is open ,and since A is closed ,then $A = \overline{A} = \overline{A^0}$,then A is regular closed .

Theorem 1.5

The following statements are equivalent.

1) X is regular disconnected .

2) There are two non-empty regular closed A, B such that $A \cap B = \phi$,and $A \cup B = X$.

3) There exist regular open and regular closed A in the same time such that $A \neq \phi, A \neq X$.

4) There exist $A \subseteq X$ such that $A \neq \phi, A \neq X$ and $b(A) = \phi$.

Proof :-

1 $\Rightarrow$ 2

Since X is regular disconnected ,then there are two non-empty regular open sets A, B such that $A \cap B = \phi$,and $A \cup B = X$.

Then $A^c \cup B^c = X$,and $A^c \cap B^c = \phi$.

Since A and B are regular open,then A^c and B^c are regular closed .

2 $\Rightarrow$ 3

Since there are two non-empty regular closed A and B such that $A \cap B = \phi$,and $A \cup B = X$.

Since $A \cap B = \phi$,then $A \subseteq B^c$,and since $A \cup B = X$,then $B^c \subseteq A$.

Then $A = B^c$.

Since B^c is regular open,then A is regular open and regular closed in the same time.

If $A = X$,since $A \cap B = \phi$,then $B = \phi$,this contradicting.

Then $A \neq \phi, A \neq X$.

3 $\Rightarrow$ 4

Let A be regular open and regular closed in the same time such that $A \neq \phi, A \neq X$.

Since A is regular closed ,then A is closed,so $b(A) \cap A = \phi$.

Since A is regular open,then A^c is closed ,so $b(A^c) \cap A^c = \phi$.

Then $b(A) = b(A^c)$,thus $b(A) = \phi$.

4 $\Longrightarrow$ 1

There exists $A \subseteq X$,such that $A \neq \phi, A \neq X$,and $b(A) = \phi$.

Since $b(A) = \phi$,then A is open and closed in the same time,so A is regular open and regular closed in the same time by (1.4) .

Since $A \neq X$,then $A^c \neq \phi,$ and since $A \cap A^c = \phi$,and $A \cup A^c = X$ .Then X is regular disconnected .

Theorem 1.6

A topological space X is regular connected iff it is connected .

Proof :-

Suppose X is disconnected ,then there are open sets A and B such that $A \neq \phi, B \neq \phi, A \cap B = \phi$,and $A \cup B = X$.

Then $A^c \cap B^c = \phi$,thus $B^c \subseteq A$.

Since $A \cap B = \phi$,then $A \subseteq B^c$.

Then $A = B^c$,that is A is both open and closed .

Then A is regular open from (1.4)

Similarly ,we can prove that B is regular open .

Then X is regular disconnected .

The converse it is clear ,since every regular open set is open .

Now ,we introduce the following definition

A subset A of a topological space (X,T) is said to be regular connected (briefly $R.C$) if the subspace (A,T_A) is regular connected ,otherwise A is regular disconnected .Then from (1.6),we have the following results.

Proposition 1.7

A subset $A \subseteq X$ is regular connected iff it is connected .

Proposition 1.8

If $\{A_i / i \in I\}$ is a collection of regular connected sets from a topological space (X,T) such that $\bigcap_{i \in I} \mathrm{A_i} \neq \phi, \bigcup_{i \in I} A_i = X$ .Then X is regular connected .

Proposition 1.9

If (X,T) be a topological space such that $\forall a,b \in X, a \neq b$,and a,b belong to a same regular connected set ,then X is regular connected .

Proposition 1.10

X is regular connected iff there is No continuous $f : X \to 2$ is surjective .

Proposition 1.11

Let $A \subseteq X$ be regular connected ,then any set B satisfying $A \subseteq B \subseteq \overline{A}$ is also regular connected.In Particular the closure of a regular connected set is regular connected.

2. Regular Strongly Connected Sets in Topology

A subset A of a topological space X is said to be strongly connected (briefly S.C) iff for each open sets B and C such that $A \subseteq B \bigcup C$,then $A \subseteq B$ or $A \subseteq C$.

Theorem 2.1 [1]

If A is S.C set ,then A is connected.

Definition 2.2 [1]

A subset A of a topological space (X,T) is said to be regular strongly connected (briefly R.S.C) iff for each regular open sets B and C such that $A \subseteq B \bigcup C$,then $A \subseteq B$ or $A \subseteq C$.

Theorem 2.3

Every S.C set is R.S.C

Proof:-

Let $A \subseteq X$ such that A is S.C set and let B, C be two regular open sets such that $A \subseteq B \bigcup C$.

Since B and C are regular open ,then B and C are open sets,and since A is S.C ,then $A \subseteq B$ or $A \subseteq C$,So A is R.S.C.

The converse of theorem 4 is not true ingeneral if we take the topology $T = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$ on the set $X = \{1,2,3\}$.Then $\{2,3\}$ is regular strongly connected but it is not S.C .

Remark 2.4

It is easy to see that if A is R.S.C ,then A is R.C

Theorem 2.5

If A is R.S.C ,then A is connected .

The proof is clear from (2.4) and (1.7)

The converse of the above theorem is not true in general,see the following example.

Example 2.6

The open interval (0,1) is connected in the usual topological space (R,T) ,but $(0,1) = (0,\frac{1}{2}) \bigcup (\frac{1}{4},1)$,and $(0,1) \not\subset (0,\frac{1}{2})$,and $(0,1) \not\subset (\frac{1}{4},1)$ .Then A is not R.S.C.

Definition 2.7

A topological space (X,T) is R.S.C iff the only non-empty subset of X which is both regular open and regular closed in X is X itself .

Theorem 2.8

A topological space (X,T) is R.S.C iff is connected space

Proof :-

Suppose X is disconnected ,then there exists $A \subseteq X$ such that $A \neq \phi, A \neq X$,and A is both open and closed in the same time.Then by (1.4), A is both regular open and regular closed in the same time,which contradicts being X is R.S.C

The converse proof is trivial .

Definition 2.9

A subset A of a topological space X is said to be regular weakly disconnected (briefly R.w.d) iff it is not R.S.C

Remark 2.10

Not every subset of R.S.C set is R.S.C ,because the usual topological space R is R.S.c ,but the open interval $(0,1)$ is not R.S.C (see example 1).

Definition 2.11

Let f be a mapping from topological space (X,T) into a topological space (Y,T'),then f is said to be regular continuous iff the inverse image of any regular open (regular closed) in Y is regular open (regular closed) in X .

Theorem 2.12

If $f:(X,T) \to (Y,T')$ is regular continuous and if A is R.S.C in X ,then $f[A]$ is R.S.C in Y .

Proof :-

Let $f[A]$ be a R.w.d in Y ,then there exist two regular open sets B, C in Y such that $f[A] \subseteq B \bigcup C$, $f[A] \not\subset B$ and $f[A] \not\subset C$.

Thus $A \subseteq f^{-1}(f[A]) \subseteq f^{-1}(B \bigcup C) = f^{-1}(B) \bigcup f^{-1}(C)$.

Then $A \subseteq f^{-1}(B) \bigcup f^{-1}(C)$, $A \not\subset f^{-1}(B)$ and $A \not\subset f^{-1}(C)$.

Since f is regular continuous and since B, C are regular open ,then $f^{-1}(B), f^{-1}(C)$ are regular open in X .Then A is R.w.d .

Definition 2.13

A topological space (X,T) is said to be RT_0 iff $\forall x, y \in X$ such that $x \neq y$,there exists a regular open set in X which continuous one of them and not the other.

Example 2.14

The topology $T = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ on the set $X = \{a,b,c\}$ is RT_0 -Space .

Definition 2.15

A topological space (X,T) is said to be RT_1 iff $\forall x, y \in X$ such that $x \neq y$,there exists A, B be two regular open sets such that $x \in A, y \notin A, x \notin B, y \in B$.

Theorem 2.16

RT_1-Space is RT_0 -Space .

Proof :-

Let (X,T) be RT_1 -Space and let $x, y \in X$,such that $x \neq y$.

Since X is RT_1-Space ,then there are regular open sets A,B such that $x \in A, y \notin A, x \notin B, y \in B$.Then there exists regular open set in X which contains one of them and not the other .

Remark 2.17

The convers of the above theorem is not true in general (see example 2.14) .

Definition 2.18

A topological space (X,T) is regular totally weakly disconnected (R.t.w.d) iff singelation sets are the only R.S.C sets .

Theorem 2.19

A topological space (X,T) is RT_1 iff it is (R.t.w.d) .

Proof :-

Singelation sets are clearly R.S.C

Now

Suppose A is a subset of X with two or more points .

Let $x \neq y$ in A,then $\{x\}$ and $\{y\}$ are non-empty disjoint regular closed subsets of A,then A is R.w.d .Then X is (R.t.w.d).

Conversly

Let $x, y \in X$,such that $x \neq y$ ..

Then $A = \{x, y\}$ is not R.S.C.Thus there are two regular open sets B,C such that $A \subseteq B \bigcup C$, $A \not\subset B$ and $A \not\subset C$.

Since $A \subseteq B \bigcup C$,then $x \in B \bigcup C$.

If $x \in B$,then $y \notin B$ and $y \in C, x \notin C$ (because $A \not\subset B$,and $A \not\subset C$).

Then X is RT_1-Space .

3. Regular Strongly Locally Connectivity

Definition 3.1

A topological space (X,T) is said to be regular locally connected iff $\forall a \in X$ and $A \in T$ such that $a \in A$,there exists a regular connected open set B such that $a \in B \subseteq A$.

Example 3.2

An indiscrete topological space is regular locally connected .

Theorem 3.3

Every regular locally connected space is locally connected .

Proof :-

Let (X,T) be regular locally connected topological space,and let $a \in X, A \in T$ such that $a \in A$.

Since (X,T) is regular locally connected ,then there exists regular open connected set B such that $a \in B \subseteq A$.

Then X is locally connected (because every regular open connected set is open connected).

Remark 3.4

The convers of the above theorem is not true in general if we take the topology $T = \{\phi, X, \{1\}, \{1,3\}\}$ on the set $X = \{1,2,3\}$.

Then X is locally connected ,but not regular locally connected ,because $1 \in X$ and $\{1\} \in T$ such that $1 \in X, \{1\} \in T$ such that $1 \in \{1\}$,but there is not regular open connected set B such that $1 \in B \subseteq \{1\}$.

Definition 3.5

If $x \in X$,the largest regular connected subset C_x of X containing x is called the regular component of x .It exists being just the union of all regular connected subsets of X containing x.

Proposition 3.6

The regular component of a regular connected space are regular open (regular closed).

Proof :-

Let (X,T) be regular locally connected space.Then X is locally connected space by theorem (3.3)

Then the proof is complet by theorem (1.6) .

Theorem 3.7

Let (X,T) be a topological space,then the following statements are equivalent .

1) X is regular locally connected

2) If C is a regular component of a regular subspace Y in X ,then $b(C) \subseteq b(Y)$.

3) Every regular component of a regular open subspace in X be **regular open .**

Proof :-

1 $\Longrightarrow$ 2

Let $Y \subseteq X$,and C be regular component of a subspace Y in X .

Let $a \in b(C)$,then $a \in \overline{C} = \mathrm{b(C)} \bigcup \mathrm{C}$.

Since $C \subseteq Y$,then $\overline{C} \subseteq \overline{Y}$ and $a \in \overline{Y}$,thus $a \in b(Y) \bigcup Y^0$.

Suppose $a \notin b(Y)$,then $a \in Y^0$.

Since $Y^0 \in T$ and X is regular locally connected ,then there is a regular open connected set A in X such that $a \in A \subseteq Y^0$.

Since $a \in b(C)$,then $A \bigcap C \neq \phi$.

Since A and C are connected (because every regular connected set is connected). Then $A \bigcup C$ is connected ,but C is regular component ,then $C = A \bigcup C$,thus $A \subseteq C$.

So $A \in C^0$,this contradicts,then $a \in b(Y)$ and $b(C) \subseteq b(Y)$.

2 $\Longrightarrow$ 3

Let Y be regular open subspace in X ,and let C be regular component of Y .
Then we have $b(C) \subseteq b(Y)$ from 2.

Since $C \subseteq Y$,then $b(C) \bigcap C \subseteq b(Y) \bigcap Y \subseteq b(Y) \bigcap Y^0 = \phi$.

Then $b(C) \bigcap C = \phi$ .Thus $C = C^0$,hence C is regular open .

3 $\Longrightarrow$ 1

Let $x \in X$,and $A \in T$ such that $x \in A$.

Let C_x be a regular component of x in a general subspace (A, T_A).

From (3) C_x is regular open connected set and $x \in C_x \subseteq A$.

Then X is regular locally connected.

Corollary 3.8

Every regular component in a general locally connected space is regular open .

Definition 3.9

A topological space (X, T) is said to be regular strongly locally connected (briefly as R.S.L.C) iff $\forall a \in X$ and $\forall A \in T$ such that $a \in A$,there exists a regular strongly connected open set B such that $a \in B \subseteq A$.

Theorem 3.10

If X is R.S.L.C space ,then X is R.L.C

Proof :-

Let $a \in X$ and $A \in T$ such that $a \in A$.

Since X is R.S.L.C,then there is a regular strongly connected open set B such that $a \in B \subseteq A$.

Since every R.S.C is R.C ,then (X, T) is R.L.C .

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الخلاصة :

في بحثنا هذا تناولنا مفهوم الأتصال المنتظم في الفضاءات التبولوجية ومن خلاله قمنا بتعريف الأتصال الاقوى المنتظم ووجدنا ان كل مجموعة متصلة بقوة تكون منتظمة متصلة بقوة والعكس غير صحيح دائما .
ايضا تناولنا مفهوم الاتصال المحلي المنتظم ومن خلاله عرفنا مفهوم الاتصال المحلي الاقوى المنتظم وبرهنا ان كل فضاء يحقق مفهوم الاتصال المحلي الاقوى المنتظم فانه يحقق مفهوم الاتصال المحلي المنتظم والعكس غير صحيح دائما .