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# The Cartan Matrix of Weyl groups

By

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## Abstract:-

In this paper we calculated the Cartan matrices of Weyl groups of type $A_n$ , $B_n$ ,$C_n$ and $D_n$ ,and give four programs written by ( Visual Basic).

## 1. Introduction:-

*Let Φ be a root system in V .The Weyl group W(Φ) of Φ is generated by the reflections* $\omega_r$ *,*$r \in \Phi$ *.*

*Weyl groups of simple root systems have been classified (see[1]).There exist two kinds of Weyl groups ,the first kind is called the Classical types which contain the type* $A_n$ *,* $B_n$*,*$C_n$ *and* $D_n$ *.*
*The second kind is called the exceptional Weyl group which contain the type* $E_n$ *(n=6,7,8) ,*$F_n$ *and* $G_2$*.*
*Every type of Weyl group is corresponding to a root system .*

*In this paper we can obtain the Dynkin diagram and the Cartan matrices of the Weyl group of the classical types* $A_n$ *,* $B_n$*,* $C_n$ *and* $D_n$ *,and we will give example and program written by ( Visual Basic ) for all type.*

### *2. The Cartan Matrix*

*Let V be an n- dimensional real space with inner product* $\langle , \rangle$ *and GL(V) be the general linear group which is consists of all invertible endomorphism of V . H is an (n-1) – dimensional subspace of V which is called hyper plane[4],and O(V) is the group of orthogonal linear transformation of V .*
*In this section , we will give the basic definition about the reflection , a root system , cartan integers , Cartan matrix and Dyakin diagram of Weyl groups .*

***Definition (2.1) [5] :-***
*A reflection in V is a* $t \in GL(V)$*, such that* $t(v) = v-2 \langle v, h \rangle h$ *, which satisfies the following :*

1) $t_H = 1_H$ , for some hyper plan H in V ,where$H_1$ is identity element in H.
2) $t(x) = -x$ , for all $x \in H^{\perp}$, where $H^{\perp} = \{x \in V : \langle x , h\rangle = 0 \ , \ \forall \ h \in H\}$ .

Let now $r \in V$ , $r \neq 0$ ,and $Hr = \{x \in V : \langle x, r\rangle = 0\}$ be the hyper plane orthogonal to r , define $t_r \in GL(V)$ by $t_r(x) = x - [[(2\langle x , r\rangle) / \langle r , r\rangle] r]$ , for all $x \in V$ , then $t_r(r) = -r$
and $t_r(x) = x$ if $x \in H_r$ .
Thus , $t_r$ is a reflection in V with reflecting hyperplane $H_r$ .
We note that , $t_r$ is an orthogonal map, i.e $t_r \in O(V)$ .

***Definition (2.2) [5]:-***

*A subset $\Phi$ of V is called a root system in V if it satisfies the following conditions :*

i) $\Phi$ is a finite set of non zero vector and $\Phi$ generates (spans) V .
ii) if $r \in \Phi$ then , then $t_r(\Phi) = \Phi$ , i .e $t_r$ leaves $\Phi$ invariant .
iii) if $r , s \in \Phi$ , then $[2\langle r , s\rangle) / \langle r , r\rangle] \in Z$
iv) if $r , \lambda r \in \Phi$ and $\lambda \in R$ ( real number ) , then $\lambda = \pm 1$

*Also the dimension of V is called the rank of the root system .*

***Definition (2.3) [1] :-***

*Let $\Phi$ be a root system in V , the Weyl group of $\Phi$ is defind by the group generated by the reflection $t_r$ ,$r \in \Phi$ , and it is a subgroup of O(V) which is denoted by $w(\Phi)$ .*

***Definition (2.4) [1] :-***

*Let $\Phi$ be a root system in V , A subset $\pi$ of $\Phi$ is called a simple ( fundamental ) system if :*

i) $\pi$ is an R- basis for V
ii) Every root in $\Phi$ is a linear combination of root in $\pi$ with coefficient which are either all non–negative or all non– positive that is ,

if $\pi = \{r_1 ,..., r_n\}$ and $r \in \Phi$ , we have $r = \sum_{i=1}^{n} \lambda_i r_i$ , where

$\lambda_i \in R$ , and either $\lambda_i \geq 0 \ \forall i$ or $\lambda_i \leq 0 , \forall i$.

***Definition (2.5) [7] :-***

*The integers $\eta_{\alpha\beta} = [2\langle \alpha , \beta\rangle) / \langle \alpha , \alpha\rangle]$ , $\alpha , \beta \in \Phi$ are called The cartan integers of the root system $\Phi$ .*

***Founding the cartan integer (2.6):-***

*In the following work we show how we find the cartan integer and the angles between vectors* $\alpha$ *and* $\beta$ *in* $\Phi$ *.*

*Let* $\theta_{\alpha\beta}$ *be angle between* $\alpha$ *and* $\beta$ *respectively ,i.e ,* $|\alpha| = \sqrt{\langle \alpha,\alpha \rangle}$ *,* $|\beta| = \sqrt{\langle \beta,\beta \rangle}$ *, then*

$$\eta_{\alpha\beta} . \eta_{\beta\alpha} = [4\langle \alpha \ \beta \rangle^2 / \langle \alpha , \alpha \rangle . \langle \beta , \beta \rangle ] = 4 \cos^2 \theta_{\alpha\beta}$$

*Thus* $0 \leq (\eta_{\alpha\beta} . \eta_{\beta\alpha} = 4 \cos^2 \theta_{\alpha\beta}) \leq 4$ *,but* $\eta_{\alpha\beta} . \eta_{\beta\alpha} \in Z$ *, since* $\eta_{\alpha\beta} \in Z$ *,and* $\eta_{\beta\alpha} \in Z$ *,therefore* $4 \cos^2 \theta_{\alpha\beta} = 0,1,2,3$ *, or 4 .*

*The case* $4 \cos^2 \theta_{\alpha\beta} = 4$ *is not possible ,since* $\beta = \pm \alpha$ *, hence*
$4 \cos^2 \theta_{\alpha\beta} = 0,1,2$ *or 3 .*
*Now, let* $\langle \alpha , \alpha \rangle \geq \langle \beta , \beta \rangle$ *,So*

$$( |\alpha|^2 / |\beta|^2 ) = (\langle \alpha , \alpha \rangle / \langle \beta , \beta \rangle)$$
$$= (2\langle \alpha , \beta \rangle / \langle \beta , \beta \rangle) / (2\langle \alpha , \beta \rangle / \langle \alpha , \alpha \rangle)$$
$$= \eta_{\beta\alpha} / \eta_{\alpha\beta} .$$

*Thus we get the following table ,see[2]*

**Table -1-**

***Thus from above table we get***

| cases | $\eta_{\alpha\beta} . \eta_{\beta\alpha}$ | $\eta_{\alpha\beta}$ | $\eta_{\beta\alpha}$ | $\theta_{\alpha\beta}$ | $\lvert\alpha\rvert / \lvert\beta\rvert$ |
|---|---|---|---|---|---|
| 1 | *0 | 0 | 0 | $\pi/2$ | undetermined |
| 2 | 1 | 1 | 1 | $\pi/3$ | 1 |
| 3 | *1 | -1 | -1 | $2\pi/3$ | 1 |
| 4 | 2 | 1 | 2 | $\pi/4$ | 2 |
| 5 | *2 | -1 | -2 | $3\pi/4$ | 2 |
| 6 | 3 | 1 | 3 | $\pi/6$ | 3 |
| 7 | *3 | -1 | -3 | $5\pi/6$ | 3 |

i- *If* $\langle \alpha, \beta \rangle > 0$ *,then the angle between* $\alpha$ *and* $\beta$ *is an a cute angle as in the cases (2,4,6),thus* $\theta_{\alpha\beta} = \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ *respectively.*

ii- *If* $\langle \alpha, \beta \rangle < 0$ *,then the angle between* $\alpha$ *and* $\beta$ *is obtuse angle as in the cases (3,5,7),thus* $\theta_{\alpha\beta} = \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ *respectively .*

iii- *If* $\langle \alpha, \beta \rangle = 0$ *,the angle angle between* $\alpha$ *and* $\beta$ *is right angle as in the case(1),thus* $\theta_{\alpha\beta} = \frac{\pi}{2}$

***Definition (2.7):-***

*A root system Φ is irreducible if it can not be partitioned into a union of two proper subsets* $\Phi_1$*,* $\Phi_2$ *such that each root in* $\Phi_1$ *is orthogonal to each root in* $\Phi_2$ .

*i.e.* $\langle \Phi_1, \Phi_2 \rangle = 0$ *,and* $\Phi \neq \Phi_1 \cup \Phi_2$ *,* $\Phi_1 \neq \phi$ *,* $\Phi_2 \neq \phi$ *(* $\phi$ *is empty set ) .*

*Then we can find the following theorem in [4] .*

***Theorem (2.8):-***

*Φ is irreducible iff* $\pi \neq \pi_1 \cup \pi_2$ *,* $\pi_1 \neq \phi$ *and* $\pi_2 \neq \phi$ *.*

***Definition (2.9):***

*The matrix (aij) is called a cartan matrix of Φ , if all the elements of the cartan matrix are the cartan integers.*

*Now ,from the definition of (aij) we can write the entries of the cartan matrix as follows:-*

*i) aij = 2 ,for all i.*
*ii) aij = 0,-1,-2, or -3 , then :*

*If aij = 0 , then aji= 0*
*If aij =-2 ,then aji =-1*
*If aij = -3 ,then aji = -1*

***Example (2-1):-***

*The cartan matrix of $A_3$ is:*

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

***Remark (2-10):-***

*i) The cartan matrix is non singular.*
*ii) The order of the rows and columns depends on the rank of the simple root $\pi$.*

***Definition (2-11):-***

*The coxeter graph of Φ is defined to be the graph with n vertices ,i.e vertices are in 1-1 correspondence with the simple with the simple root $\pi =\{r_1,\ldots,r_n\}$ and the ith vertex is joined with jth by :*

$$\eta_{ij} = \frac{4(r_i, r_j)^2}{(r_i, r_j)(r_j, r_i)} = 4Cos^2\theta_{ij} \quad \text{edges .}$$

***Example (2-2):-***

$A_1 \times A_2$ $\quad$ $a_1$ O $\quad$ O $a_2$

*Which Ai is has one simple rootand $A_1 \times A_2$ is not connected*

$A_2$ $\quad$ $a_1$ O———O $a_2$

*Then $a_1$ connected with $a_2$*

$B_2$ $\quad$ $b_1$O———O $b_2$ and $G_2$ O———O

***Remark (2-12):-***

*Φ is irreducible if the coxeter graph is connected . if the coxeter graph is not connected ,* *i.e* $$\eta_{ij} = 0 = \frac{4\langle r_i , r_j\rangle^2}{\langle r_i , r_j\rangle \; \langle r_j , r_i\rangle} \; \textbf{then} \langle r_i , r_j\rangle = 0$$

***Definition (2-13)[4]:-***

*For every simple root $\pi$, a Dynkin diagram (D.D) is defined as follows :-*

*i) Every elements $\alpha \in \pi$ is represented by a point in the plane.*

*ii) Two distinct point $\alpha$ ,$\beta$ are connected ($\alpha$ is not orthogonal to $\beta$) by one ,two, three line segment depending on the number $\eta_{\alpha\beta}$ ,$\eta_{\beta\alpha}$ in table -1-.*

*iii) The points corresponding to $\alpha$ and $\beta$ are disconnected if $\alpha$ is orthogonal to $\beta$..*

*iv) The number of element of the system $\pi$ is the order of Dynkin system.*

***Theorem (2-14):-***

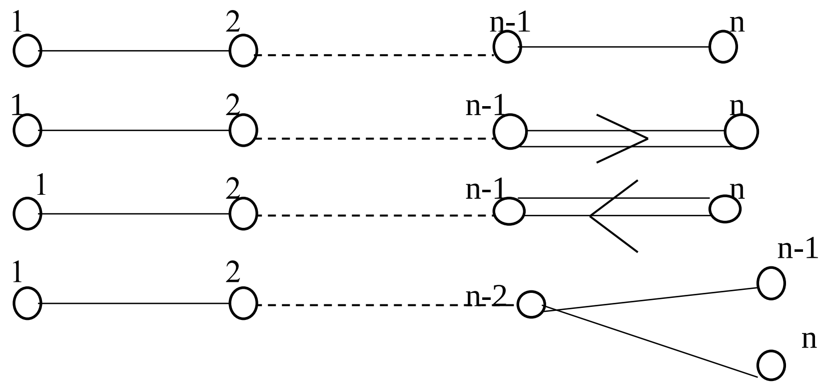
*If Φ is an irreducible root system of rank n ,its Dynkin diagram is one of the following:-*

*An ($n \geq 1$)*

*Bn ($n \geq 2$)*

*Cn ($n \geq 3$)*

*Dn ($n \geq 4$)*




***3- The Relation Between Cartan Matrix & Dynkin diagram***

*In this section we will give the special relation to obtain the cartan matrix and Dynkin diagram of classical group and we will give examples and programs written in math lap language for all type.*

*<u>Remark( 3-1):</u>*

*Let Π=[$a_1$,… ,$a_n$] be the fundamental root system of Weyl group of the classical type $A_n$ ,$B_n$ ,$C_n$ and $D_n$ .*

*In general from the next four proposition , we can obtain the Dynkin diagram and the cartan matrices of the Weyl groups of all classical types $A_n$ ,$B_n$ ,$C_n$ and $D_n$ .*

***Proposition(3-2): ( $A_n$ ($n \geq 1$))***

*i- $\langle a_i, a_j\rangle = 1/(n + 1)$ , if i= 1, …,n* ii- $\langle a_i, a_j \rangle = \begin{bmatrix} o & if\ |i-j| \succ 1 \\ -1/(2(n+1)) & if\ |i-j| = 1 \end{bmatrix}$

*Example :(3-1)*

*Let n=5 , to find the Dynkin diagram and the cartan matrix of the Weyl group of type $A_5$ :*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

$a_{11} = [\, 2\langle a_1,a_1\rangle / \langle a_1,a_1\rangle] = 2$

$a_{12} = [2\langle a_1,a_2\rangle / \langle a_1,a_1\rangle]$

$\langle a_1,a_2\rangle = [-1/2(5+1)] = -1/12$

$\langle a_1,a_1\rangle = 1/(5+1) = 1/6$

$\therefore\ a_{12} = [\, 2\,(-1/12)\, /\, (1/6)\,] = -1$ , $a_{13} = a_{14} = a_{15} = 0$

$a_{21} = [\, 2\langle a_1,a_2\rangle / \langle a_2,a_2\rangle]$

$\langle a_1,a_2\rangle = [-1/2(5+1)] = -1/12$

$\langle a_2,a_2\rangle = 1/(5+1) = 1/6$

$\therefore\ a_{21} = [\, 2\,(-1/12)\, /\, (1/6)\,] = -1$ , $a_{24} = a_{25} = a_{31} = 0$

$a_{32} = [2\,\langle a_3,a_2\rangle / \langle a_3,a_3\rangle]$

$\langle a_3,a_2\rangle = [-1/2(5+1)] = -1/12$

$\langle a_3,a_3\rangle = 1/(5+1) = 1/6$

$\therefore\ a_{32} = [\, 2\,(-1/12)\, /\, (1/6)\,] = -1$

$a_{33} = [2\,\langle a_2,a_3\rangle / \langle a_3,a_3\rangle] = 2$

$a_{34} = [2\,\langle a_3,a_4\rangle / \langle a_3,a_3\rangle]$

$\langle a_3,a_4\rangle = [-1/2(5+1)] = -1/12$

$\langle a_3,a_3\rangle = 1/(5+1) = 1/6$ , $a_{35} = 0$

$\therefore\ a_{34} = [\, 2\,(-1/12)\, /\, (1/6)\,] = -1$ , $a_{41} = a_{42} = 0$

$a_{43} = [2\,\langle a_4,a_3\rangle / \langle a_4,a_4\rangle]$

$\langle a_4,a_3\rangle = [-1/2(5+1)] = -1/12$

$a_{44} = [2\,\langle a_4,a_4\rangle / \langle a_4,a_4\rangle] = 2$

$\langle a_4,a_4\rangle = 1/(5+1) = 1/6$

$\therefore$ $a_{43}$ = [ 2 (-1 /12 ) / (1/6) ] = -1

$a_{45} = [2 \langle a_4, a_5\rangle / \langle a_4, a_4\rangle]$

$\langle a_4, a_5\rangle$ = [-1/2(5+1)] = -1/12

$\therefore$ $a_{45}$ = [ 2 (-1 /12 ) / (1/6) ] = -1

$\langle a_4, a_4\rangle$ = 1/(5+1) = 1/6 , $a_{51} = a_{52} = a_{53} = 0$

$a_{54} = [2 \langle a_5, a_4\rangle / \langle a_5, a_5\rangle]$

$\therefore$ $a_{54}$ = [ 2 (-1 /12 ) / (1/6) ] = -1

$\langle a_5, a_4\rangle$ = [-1/2(5+1)] = -1/12

$\langle a_5, a_5\rangle$= 1/(5+1) = 1/6

$a_{55} = [2 \langle a_5, a_5\rangle / \langle a_5, a_5\rangle] = 2$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$a_1$ $a_2$ $a_3$ $a_4$ $a_5$

O———O———O———O———O

*<u>Proposition ( Bn (n ≥ 2))</u> : (3-3)*

*i-* $\langle a_i, a_j\rangle = 2 / ( 2n - 1)$ *if* $i = 1,2,3,\ldots,n-1,\ j = 1,2,\ldots,n-1$

*ii-* $\langle a_n, a_n\rangle = 1 / (2n - 1 )$

*iii-* $\langle a_i, a_j\rangle = \begin{bmatrix} 0 & \textbf{\textit{if}}\ |i-j| > 1 \\ -1/(2n-1) & \textbf{\textit{if}}\ |i-j| = 1 \end{bmatrix}$

*<u>Example :(3-2)</u>*

*Let n=4 , to find the Dynkin diagram and the cartan matrix of the Weyl group of type* $B_4$ *:*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$a_{11}= [\, 2\langle a_1,a_1\rangle / \langle a_1,a_1\rangle] = 2$

$a_{12}= [2\langle a_1,a_2\rangle / \langle a_1,a_1\rangle]$
$\langle a_1,a_2\rangle = [-1/(2(4)-1)] = -1/7$
$\langle a_1,a_1\rangle = [2/(2(4)-1)] = 2/7$
$\therefore\ a_{12} = [\, 2\,(-1/7) / (2/7)\,] = -1$ , $a_{13} = a_{14} = 0$

$a_{21} = [\, 2\langle a_2,a_1\rangle / \langle a_2,a_2\rangle]$
$\langle a_2,a_1\rangle = [-1/(2(4)-1)] = -1/7$
$\langle a_2,a_2\rangle = [2/(2(4)-1)] = 2/7$
$\therefore\ a_{21} = [\, 2\,(-1/7) / (2/7)\,] = -1$ ,

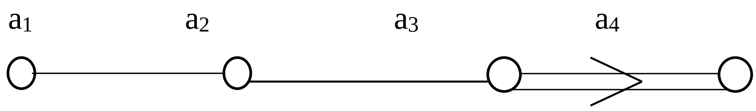
$a_{23} = [\, 2\langle a_2,a_3\rangle / \langle a_2,a_2\rangle]$
$\langle a_2,a_3\rangle = [-1/(2(4)-1)] = -1/7$
$\langle a_2,a_2\rangle = [2/(2(4)-1)] = 2/7$ , $a_{24} = a_{31} = 0$
$\therefore\ a_{23} = [\, 2\,(-1/7) / (2/7)\,] = -1$

$a_{23}= [\, 2\langle a_3,a_2\rangle / \langle a_3,a_3\rangle]$
$\langle a_3, a_2\rangle = [-1/(2(4)-1)] = -1/7$
$\langle a_3,a_3\rangle = [2/(2(4)-1)] = 2/7$
$\therefore\ a_{23} = [\, 2\,(-1/7) / (2/7)\,] = -1$

$a_{33}= [\, 2\langle a_3,a_3\rangle / \langle a_3,a_3\rangle] = 2$

$a_{34}= [\, 2\langle a_3,a_4\rangle / \langle a_3,a_3\rangle]$
$\langle a_3, a_4\rangle = [-1/(2(4)-1)] = -1/7$
$\langle a_3,a_3\rangle = [2/(2(4)-1)] = 2/7$
$\therefore\ a_{34} = [\, 2\,(-1/7) / (2/7)\,] = -1$ , $a_{41} = a_{42} = 0$
$a_{43}= [\, 2\langle a_4,a_3\rangle / \langle a_4,a_4\rangle]$
$\langle a_4,a_3\rangle = [-1/(2(4)-1)] = -1/7$
$\langle a_4,a_4\rangle = [2/(2(4)-1)] = 1/7$
$\therefore\ a_{43} = [\, 2\,(-1/7) / (1/7)\,] = -2$
$a_{44}= [\, 2\langle a_4,a_4\rangle / \langle a_4,a_4\rangle] = 2$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$




<u>*Proposition ( Cn (n ≥ 3))*</u> *: (3-4)*

*i-* $\langle a_i, a_j \rangle = 2 / (4n + 1)$ *if* $i = j = 1,2,3,\ldots,n-1$

*ii-* $\langle a_i, a_j \rangle = 4 / (4n + 1)$ *if* $i = j = n$

*iii-* $\langle a_i, a_j \rangle = -1 / (4n + 1)$ *if* $|i - j| = 1, i < n, j < n$

*iv-* $\langle a_i, a_j \rangle = -2 / (4n + 1)$ *if* $i = n, j = n-1$ *or* $i = n-1, j = n$

*v-* $\langle a_i, a_j \rangle = 0$ *in all other cases.*

<u>*Example :(3-3)*</u>

*Let n= 3, to find the Dynkin diagram and the cartan matrix of the Weyl group of type* $C_n$ *:*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{11} = [\, 2\langle a_2, a_2 \rangle / \langle a_2, a_2 \rangle ] = 2$

$a_{12} = [2\langle a_1, a_2 \rangle / \langle a_1, a_1 \rangle]$

$\langle a_1, a_2 \rangle = [-1/(4(3)+1)] = -1/13$

$\langle a_1, a_1 \rangle = [2/(4(3)+1)] = 2/13$

$\therefore\ a_{12} = [\, 2\,(-1/13) / (2/13)\,] = -1$ , $a_{13} = 0$

$a_{21} = [\, 2\langle a_1, a_2 \rangle / \langle a_2, a_2 \rangle]$

$\langle a_1, a_2 \rangle = [-1/(4(3)+1)] = -1/13$

$\langle a_2, a_2 \rangle = [2/(4(3)+1)] = 2/13$

$\therefore\ a_{21} = [\, 2\,(-1/13) / (2/13)\,] = -1$

$a_{22} = [\, 2\langle a_2, a_2 \rangle / \langle a_2, a_2 \rangle] = 2$

$a_{23} = [\, 2\langle a_2, a_3 \rangle / \langle a_2, a_2 \rangle]$

$\langle a_2, a_3 \rangle = [-2/(4(3)+1)] = -2/13$

$\langle a_2, a_2 \rangle = [2/(4(3)+1)] = 2/13$

$\therefore\ a_{23} = [\, 2\,(-2/13) / (2/13)\,] = -2$ , $a_{31} = 0$

$a_{32} = [\, 2\langle a_3, a_2 \rangle / \langle a_3, a_3 \rangle]$

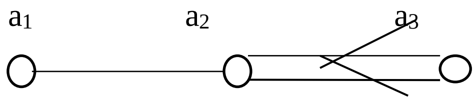
$\langle a_3, a_2\rangle = [-2/(4(3)+1)] = -2/13$

$\langle a_3, a_3\rangle = [4/(4(3)+1)] = 4/13$

$\therefore\ a_{32} = [\, 2\,(-2/13\,)\,/\,(4/13)\,] = -1$

$a_{33} = [\,2\langle a_3, a_3\rangle / \langle a_3, a_3\rangle] = 2$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$




*<u>Proposition ( Dn (n ≥4)) </u>: (3-5)*

*i- $\langle a_i, a_i\rangle = 2\,/\,(\,2n - 1)$ if $i = j$*

*ii- $\langle a_i, a_j\rangle = -1\,/\,(2n - 2)$ if $|i - j| = 1$ , $i < n$ , $j < n$*

*iii- $\langle a_i, a_j\rangle = -1\,/\,(2n - 2\,)$ if $i = n-2$ , $j = n$ , or $i = n-1$ , $j = n$*

*iv- $\langle a_i, a_j\rangle = 0$ in all other cases*

*<u>Example :(3-4)</u>*

*Let n= 5,to find the Dynkin diagram and the cartan matrix of the Weyl group of type $D_n$ :*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

$a_{11} = [\,2\langle a_1, a_1\rangle / \langle a_1, a_1\rangle] = 2$

$a_{12} = [2\langle a_1, a_2\rangle / \langle a_1, a_1\rangle]$

$\langle a_1, a_2\rangle = -1/\,(2n-2) = -1/8$

$\langle a_1, a_1\rangle = 2/\,(2n-2) = 2/8$

$\therefore\ a_{12} = [\,2\,(-1/8\,)\,/\,(2/8)\,] = -1$ , $a_{13} = a_{14} = a_{15} = 0$

$a_{21} = [\,2\langle a_2, a_1\rangle / \langle a_2, a_2\rangle]$

$\langle a_2, a_1\rangle = -1/\,(2n-2) = -1/8$

$\langle a_2, a_2\rangle = 2/\,(2n-2) = 2/8$

$\therefore\ a_{21} = [\,2\,(-1/8\,)\,/\,(2/8)\,] = -1$

$a_{22} = [\,2\langle a_2, a_2\rangle / \langle a_2, a_2\rangle] = 2$

$a_{23} = [\,2\langle a_2, a_3\rangle / \langle a_2, a_2\rangle]$

$\langle a_2, a_3\rangle = -1/\,(2n-2) = -1/8$

$\langle a_2, a_2 \rangle = 2/(2n-2) = 2/8$

$\therefore a_{23} = [2(-1/8)/(2/8)] = -1$ , $a_{24} = a_{25} = a_{31} = 0$

$a_{32} = [2\langle a_3, a_2 \rangle / \langle a_3, a_3 \rangle]$

$\langle a_3, a_2 \rangle = -1/(2n-2) = -1/8$

$\langle a_3, a_3 \rangle = 2/(2n-2) = 2/8$

$\therefore a_{32} = [2(-1/8)/(2/8)] = -1$

$a_{33} = [2\langle a_3, a_3 \rangle / \langle a_3, a_3 \rangle] = 2$

$a_{34} = [2\langle a_3, a_4 \rangle / \langle a_3, a_3 \rangle]$

$\langle a_3, a_4 \rangle = -1/(2n-2) = -1/8$

$\langle a_3, a_3 \rangle = 2/(2n-2) = 2/8$

$\therefore a_{34} = [2(-1/8)/(2/8)] = -1$

$a_{35} = [2\langle a_3, a_5 \rangle / \langle a_3, a_3 \rangle]$

$\langle a_3, a_5 \rangle = -1/(2n-2) = -1/8$

$\langle a_3, a_3 \rangle = 2/(2n-2) = 2/8$

$\therefore a_{35} = [2(-1/8)/(2/8)] = -1$ , $a_{41} = a_{42} = 0$

$a_{44} = [2\langle a_4, a_4 \rangle / \langle a_4, a_4 \rangle] = 2$

$a_{43} = [2\langle a_4, a_3 \rangle / \langle a_4, a_4 \rangle]$

$\langle a_4, a_3 \rangle = -1/(2n-2) = -1/8$

$\langle a_4, a_4 \rangle = 2/(2n-2) = 2/8$

$\therefore a_{43} = [2(-1/8)/(2/8)] = -1$

$a_{45} = [2\langle a_4, a_5 \rangle / \langle a_4, a_4 \rangle]$

$\langle a_4, a_5 \rangle = -1/(2n-2) = -1/8$

$\langle a_4, a_4 \rangle = 2/(2n-2) = 2/8$

$\therefore a_{45} = [2(-1/8)/(2/8)] = -1$ , $a_{51} = a_{52} = 0$

$a_{53} = [2\langle a_5, a_3 \rangle / \langle a_5, a_5 \rangle]$

$\langle a_5, a_3 \rangle = -1/(2n-2) = -1/8$

$\langle a_5, a_5 \rangle = 2/(2n-2) = 2/8$

$\therefore a_{53} = [2(-1/8)/(2/8)] = -1$

$a_{54} = [2\langle a_5, a_4 \rangle / \langle a_5, a_5 \rangle]$

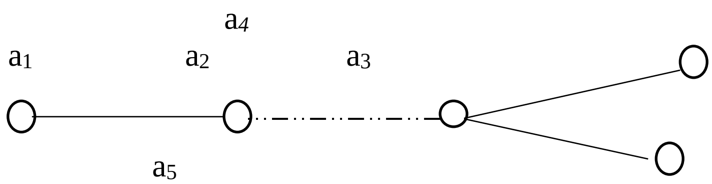
$\langle a_5, a_4 \rangle = -1/(2n-2) = -1/8$

$\langle a_5, a_5 \rangle = 2/(2n-2) = 2/8$

$\therefore a_{54} = [2(-1/8)/(2/8)] = -1$

$a_{55} = [2\langle a_5, a_5 \rangle / \langle a_5, a_5 \rangle] = 2$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$




*4 – Programs :-*

*In this section we give four programs written in ( Visual Basic ) to calculate the Cartan matrices of the Wely group of type $A_n$ ,$B_n$ , $C_n$ and $D_n$ .*

*Program $A_n$ :-(4-1)*

```
Dim  A( N, N) As integer
Dim C(N, N)  As integer
Dim  i , j  As integer
   for  I=1  To N
  for J=1  To N
      If I=J then
         C(N, N)= 1/( N +1)
     Else
        If ABS(I-J)=1 Then
       C(I,J)=((-1)*(1/(2*(N+1)))
    Else
      C(I.J)= 0
    end if
   end if
     print A(I,J)
            Next
            Next
```

*Program $B_n$ :-(4-2)*

```
Dim  A( N, N) As integer
Dim C(N, N)  As integer
Dim  i , j  As integer
      for  I=1  To N
      for J=1  To N
           If((I=J) and (I=N) and (J=N)) Then
           C(I,J)=(1/((2*N)-1))
      Else
           If  I=J Then
         C(I,J)= 2/((2*N)- 1)
     Else
```

```
            If ABS(I-J)=1 Then
            C(I,J)= -1*(1/((2*N)- 1)
      Else
            C(I.J)=0;
      end if
     end if
   end if
  for  I=1  To N
  for J=1  To N
       A(I,J)=(2*C(I,J))/C(I,J)
      Print A(I,J)
  Next
     Print
  Next
end
```

Program $C_n$ :-(4-3)

```
Dim  A( N, N) As integer
Dim C(N, N)  As integer
Dim  i , j  As integer
         for  I=1  To N
         for J=1  To N
                If ((I=J) and (I=N) and (J=N)) Then
                   C(I,J)=4/(4*(N+1))
              Else
                   If  I=J Then C(I,J)= 4/(4*(N+1))   OR
          Else
               If  ((I=(N-1))and(J=N)) Then
                  C(I,J)=-2/(4*(N+1))
                   ((I+N) and (J=(N-1)))
          Else
             If ABS(I-J)=1 Then C(I,J)= -1+(4*(N+1))
          Else
                  C(I.J)=0;
                   end if
               end if
              end if
               end if
            for  I=1  To N
           for J=1  To N
               A(I,J)=(2*C(I,J))/C(I,J)
              Print A(I,J)
           Next
```

```
          Print
         Next
       End
```

*Program* $D_n$ *:-(4-4)*

```
Dim  A( N, N) As integer
Dim C(N, N)  As integer
Dim  i , j  As integer
        for  I=1  To N
        for J=1  To N
  If ((I=J) then   C(I,J)=2/((2*N)-1))
 OR  ((ABS (I-J)=1) and (J<N) and (J<N)  then
                 C(I,J)=2/((2*N)-2))
         Else
                 If ((I=N) and (J=N-2)) OR ((I=(N-2)) and (J=N))
             Then
            C(I,J)=-1/((2*N)-2))
        Else
            C(I.J)=Ø
         end if
       end if
       for  I=1  To N
     for J=1  To N
             A(I,J)=(2*C(I,J))/C(I,J)
            Print A(I,J)
         Next
            Print
         Next
     end
```

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**الخلاصة**

في هذا البحث قمنا بحساب مصفوفات كارتن لزمر ويل من النوع $A_n$, $C_n$ , $B_n$ و $D_n$ ،كما أعطينا اربع برامج كتبت باستخدام ( فيجول بيسك ) .