REDEFINE FUZZY TOPOLOGICAL VECTOR SPACE BY USING MICHÁLEK'S FUZZY TOPOLOGICAL SPACE

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ABSTRACT. In this paper, a new definition of fuzzy topological vector space was introduced. this kind of fuzzy topological vector spaces have peculiarity that differ from the other kinds. An idea of this definition was dependence on fuzzy topology that define by science of Michálek . Some theorems about this subject was proved .

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1. INTRODUCTION

The concept of fuzzy topological vector space was introduced by Katsaras in [1]. Some properties of this space was studied in those paper such as the product fuzzy topology is fuzzy topological vector space. Moreover, In [2],[3] Katsaras was develop the idea of fuzzy topological vector space . in [5], Jiři Michálek was define fuzzy topology by dependence on another idea for topological spaces and [4], F. G. Lupianez was supporter for this idea where he gave the description to the system of fuzzy neighbourhoods for any element in the system . In this paper, we gave a characterization of fuzzy topological vector space by dependence on fuzzy topology in the science of Michálek. Morover , we gave some basic theories.

2. PRELIMINARIES

Throughout this paper X is the vector space over the field $\mathbb{K}$, we wish to consider vector spaces either over $\mathbb{R}$, the field of real numbers, or over $\mathbb{C}$, the complex numbers, so for notational convenience, we use the symbol $\mathbb{K}$ to stand for either $\mathbb{R}$ or $\mathbb{C}$.

A unit interval $I=[0,1]$. A fuzzy set A in X is a function from X into unit interval I. Let $\mathcal{P}(X)$ be the system of all subsets of the set X, and let $\mathcal{F}_X$ be the system of all fuzzy sets in X. The characteristic function for a set A denoted by χ_A and defined as $\chi_A(x)=1$ if x is belong to A and $\chi_A(x)=0$ if else. A constant fuzzy set is a function $C_\alpha : X \to I$ defined by $C_\alpha(x)=\alpha$ for all $x \in X$, $\alpha \in I$.

3. MAIN RESULT

Definition 1. [5]

Let X be a non-empty set. A pair (X,u) is called fuzzy topological space supposing that u is a function from $\mathcal{P}(X)$ into $\mathcal{F}_X$ satisfying the following three axioms:

1. if $A \subset X$, then $uA(x) = 1$ for all $x \in A$,
2. if $A \subset X$ contains at most one element, then $uA(x) = \chi_A(x)$, where χ_A is the characteristic function of the set A.
3. if $A \subset X$, $B \subset X$, then $u(A \cup B)(x) = \max\{uA(x), uB(x)\}$.

Example.

Let C be a constant fuzzy set on $[0,1]$, $u:[0,1] \to \mathcal{F}_{[0,1]}$ be a function such that $uA = C(A)$ for all $A \in \mathcal{P}([0,1])$. Then u satisfying the three above axioms and we get $([0,1],u)$ is a fuzzy topological space.

From the last definition, can we gave it more generality:

Definition 2.

If $X_1, X_2, \ldots, X_n$ are non-empty sets, and let $A_j, B_j \subset X_j$ for $j = 1,\ldots,n$, then a pair $(\prod_{j=1}^{n} X_j, w)$ is Michálek's fuzzy topological space, where w is a function from $\mathcal{P}(\prod_{j=1}^{n} X_j)$ into $\mathcal{F}_{\prod_{j=1}^{n} X_j}$ satisfying the following three axioms :

1. if $\prod_{j=1}^{n} A_j \subset \prod_{j=1}^{n} X_j$, then $w\prod_{j=1}^{n} A_j(\underset{\sim}{x}) = 1$ for all $(\underset{\sim}{x}) \in \prod_{j=1}^{n} A_j$,
2. if $\prod_{j=1}^{n} A_j \subset \prod_{j=1}^{n} X_j$ contains at most one element, then $w\prod_{j=1}^{n} A_j(\underset{\sim}{x}) =$ $\chi_{\prod_{j=1}^{n} A_j}(\underset{\sim}{x})$, where $\chi_{\prod_{j=1}^{n} A_j}(\underset{\sim}{x})$ is the characteristic function of the set $\prod_{j=1}^{n} A_j$.
3. if $\prod_{j=1}^{n} A_j \subset \prod_{j=1}^{n} X_j$, $\prod_{j=1}^{n} B_j \subset \prod_{j=1}^{n} X_j$,

then $w(\prod_{j=1}^{n} A_j \cup \prod_{j=1}^{n} B_j)(\underset{\sim}{x}) =$ $\max\{w\prod_{j=1}^{n} A_j(\underset{\sim}{x}), w\prod_{j=1}^{n} B_j(\underset{\sim}{x})\}$

Theorem 1.

If $(X_1,u_1),(X_2,u_2),\ldots,(X_n,u_n)$ are Michálek's fuzzy topological space, and let $A_j, B_j \subset X_j$ for $j = 1,\ldots,n$, then a pair $(\prod_{j=1}^{n} X_j, w)$ is Michálek's fuzzy topological space, where $w = u_1 \times u_2 \times \cdots \times u_n$ is a product function for $u_1, u_2, \ldots, u_n$

from $\mathcal{P}(\prod_{j=1}^{n} X_j)$ into $\mathcal{F}_{\prod_{j=1}^{n} X_j}$ and which is defined by the form $w(A_1 \times A_2 \times \ldots \times A_n)(x_1, x_2, \ldots, x_n) = \min\{u_1 A_1(x_1), u_2 A_2(x_2), \ldots, u_n A_n(x_n)\}$

Proof.

In this theorem it's enough to prove the function w satisfies the three axioms in definition (2).

(1) it is clear.

(2) let $\prod_{j=1}^{n} A_j \subset \prod_{j=1}^{n} X_j$. If $\prod_{j=1}^{n} A_j = \phi$, then $w \prod_{j=1}^{n} A_j(\underset{\sim}{x}) = 0$. If $\prod_{j=1}^{n} A_j$ contain one element, say $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, then for $j = 1, 2, \ldots, n$, A_j contain one element x_j and $w \prod_{j=1}^{n} A_j(\underset{\sim}{x}) = 1$. Thus, $w \prod_{j=1}^{n} A_j(\underset{\sim}{x}) = \chi_{\prod_{j=1}^{n} A_j}(\underset{\sim}{x})$.

(3) $w(\prod_{j=1}^{n} A_j \cup \prod_{j=1}^{n} B_j)(\underset{\sim}{x}) = w(\prod_{j=1}^{n}(A_j \cup B_j))(\underset{\sim}{x}) = \max\{w \prod_{j=1}^{n} A_j(\underset{\sim}{x}), w \prod_{j=1}^{n} B_j(\underset{\sim}{x})\}$. ◊

Definition 3. [5]

Let $(X, u), (Y, v)$ be two Michálek's fuzzy topological spaces, and let φ be a function from X to Y. We will say that φ is *compatible* with u and v if, for all $B \subset Y$, we have that $u(\varphi^{-1}(B)) = \varphi^{-1}(vB) = vB \circ \varphi$.

Example.

Let X be a non – empty set. We define $u : \mathcal{P}(X) \to \mathcal{F}_X$, as $uA = \chi_A$. Then (X, u) is a Michálek's fuzzy topological space. Now let $\varphi : (X, u) \to (X, u)$, as $\varphi(x) = x$, then φ is *compatible* with u and u.

Theorem 2.

Let $(X, u), (X_j, u_j)_{j=1,\ldots,n}$ are Michálek's fuzzy topological spaces, and the projections $\pi_j : (\prod_{j=1}^{n} X_j, w) \to (X_j, u_j)$ are *compatible* with w and u_j , $j = 1, \ldots, n$. Then for any function $f : (X, u) \to (\prod_{j=1}^{n} X_j, w)$ is *compatible* with u and w if, and only if the functions $\pi_j \circ f : (X, u) \to (X_j, u_j)$ are *compatible* with u and u_j for $j = 1, \ldots, n$.

Proof.

Let π_j are *compatible* with w and u_j , for $j = 1, \ldots, n$ implies that $w(\pi_j^{-1}(B_j)) = \pi_j^{-1}(u_j B_j) = u_j B_j \circ \pi_j$ for $B_j \subset X_j$. If $f : (X, u) \to (\prod_{j=1}^{n} X_j, w)$ is *compatible* with u and w, then $u(f^{-1}(B)) = f^{-1}(wB)$ for $B \subset \prod_{j=1}^{n} X_j$ and with out

loss generality we can write $B = B_1 \times B_2 \times \cdots \times B_n$. Now for $j = 1,\ldots,n$, $(\pi_j \circ f)^{-1}(u_j B_j) = f^{-1}(\pi_j^{-1}(u_j B_j)) = f^{-1}(w(\pi_j^{-1}(B_j))) = u(f^{-1}(\pi_j^{-1}(B_j)))$
$= u((\pi_j \circ f)^{-1}(B_j)))$, that is mean the function $\pi_j \circ f$ is $compatible$ with u and u_j. Conversely, by the same style in above we can get the result. ◊

Theorem 3.

Let $(X_j, u_j)_{j=1,2}$, $(Y_j, v_j)_{j=1,2}$ are Michálek's fuzzy topological spaces. Suppose that for $j=1,2$, a function $f_j : (X_j, u_j) \to (Y_j, v_j)$ be $compatible$ with u_j and v_j. Then the function $f_1 \times f_2 : (X_1 \times X_2, u_1 \times u_2) \to (Y_1 \times Y_2, v_1 \times v_2)$ be $compatible$ with $u_1 \times u_2$ and $v_1 \times v_2$.

Definition 4.

Let X be a vector space over a field $\mathbb{K}$, (X,u) be fuzzy topological space (in the Michálek's sense) , then the pair (X,u) be called fuzzy topological vector space over a field $\mathbb{K}$ if,

(1) A function $\varphi : (X \times X, w) \to (X, u), (x, y) \mapsto x + y$ $compatible$ with w and u,

(2) A function $\psi : (\mathbb{K} \times X, z) \to (X, u)$, $(\alpha, x) \mapsto \alpha x$ $compatible$ with z and u.

Where w is a function from $X \times X$ into $\mathcal{F}_{X \times X}$, z is a function from $\mathbb{K} \times X$ into $\mathcal{F}_{\mathbb{K} \times X}$.

Theorem 4.

Let $(X_j, u_j), j = 1,\ldots,n$ are fuzzy topological vector spaces over $\mathbb{K}$ and (X, w) be a product fuzzy topological space. Then (X, w) is a fuzzy topological vector space.

References

[1] A. K. Katsaras and D. B. Liu, Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces, *J. Math. Anal. Appl*., **58** (1977) 135-146.

[2] A. K. Katsaras, Fuzzy Topological Vector Spaces I, *Fuzzy Sets and Systems*,**6** (1981) 85-95.

[3] A. K. Katsaras, Fuzzy Topological Vector Spaces II, *Fuzzy Sets and Systems*,**12** (1984) 143-154.

[4] F. G. Lupianez, On Michálek's Fuzzy Topological Spaces, *K*ybernetika, **37** (2001), 2, 159-163.

[5] J. Michálek, Fuzzy Topologies, *K*ybernetika,**11**(1975),5,345-354