The Optimal Error Convergence of the Weak Galerkin Finite Element Method to the Solution of Two-Dimensional Non-Linear Convection-Diffusion Equations

[1,2]Ala' N. Abdullah and [2,3]Hashim A. Kashkool

[2] Department of Mathematics. College of Education for Pure Sciences.

University of Basrah. Basrah. Iraq.

[1] Department of Mathematics. College of Education. University of Misan,

Basrah. Iraq.

Email: [1] mr.ala_najim@uomisan.edu.iq, [1] pgs2207@uobasrah.edu.iq,

Email: [3] hkashkool@yahoo.com.

This paper presents the semi-discrete scheme of the solution to the two-dimensional nonlinear convection diffusion problem by using a weak Galerkin finite element methods. We introduce and analyze the stability, error estimate, and an optimal order of (L^2 and H^1)-norm are proved. We confirm the theoretical results with some numerical examples.

Keywords: weak Galerkin method, non-linear convection-diffusion equations, stability, optimal error.

1 Introduction

The examination of the convection-diffusion problem has piqued the interest of a growing number of mathematicians, specialists, and engineers. Due to challenges with proper resolution of the so-called boundary layers, special emphasis is dedicated to problems with convection dominating diffusion. Upwinding or artificial viscosity are commonly used in the efficient finite difference method or finite element method.

We are considered the following the Dirichlet problem for non-linear convection-diffusion equations, which seeks an unknown function $u = u(x,t)$ satisfying [12]:

$$\frac{\partial u}{\partial t} - \nabla \cdot (a(u)\, \nabla u) + \vec{b}(u) \cdot \nabla u + c\, u = f; \quad \Omega \times (0,T], \tag{1.1}$$

$$u(x,t) = g; \quad \Gamma \times (0,T], \tag{1.2}$$

$$u(x,0) = u^0(x); \quad x \in \overline{\Omega}, \tag{1.3}$$

in a domain $\Omega \subset R^2$ with boundary Γ, where $x = (x_1, x_2)$ and $f = f(x,t) \in \Omega \times (0,T]$, and the diffusion coefficient and nonlinear function are denoted by $a(u)$, and the convection coefficient and nonlinear function are denoted by $b(u)$, and $c = c(x)$ is the scalar function on Ω. Suppose that the matrix $a(u)$ satisfies the following property: There is a constant $\alpha > 0$ that ensures

the diffusion tensor $a \in R^{2\times2}$ and the convection tensor $b \in R^2$

$$\alpha\zeta^T\zeta \le \zeta^T a(u)\zeta, \forall\zeta \in R^2.$$

where ζ^T is the transpose of vector ζ. Numerical simulations and analysis for the problem $(1.1 - 1.3)$ were investigated extensively in the last decades,

Liu and Liu [13] introduced the numerical identification of diffusion parameters in non-linear convection-diffusion equations. Chan et al [14] presented a dual Petrov-Galerkin finite element method for convection-diffusion problems of a higher order. Gao and Yuan [15] adopted an upwind finite volume element method to non-linear convection-diffusion problems. Fatehi et al [16] introduced a modified finite-volume Eulerian-Lagrangian localized adjoint method extended for non-linear convection-diffusion problems. Havle et al [17] applied the analysis of the discontinuous Galerkin finite element methods to a space semidiscretization of the non-linear nonstationary convection-diffusion problems. Dolejší et al [18] applied the analysis of a discontinuous Galerkin finite element method to a non-linear convection-diffusion problem. Cockburn and Shu [19] studied the local discontinuous Galerkin methods for non-linear time-dependent convection-diffusion system. Xikui and Wenhua [20] presented the finite element procedure for the solution of a transient, multidimensional convection-diffusion problem. Salhi et al [21] presented the Galerkin characteristic finite element methods for the numerical solution of the time-dependent convection-diffusion problems.

In 2013, Wang and Ye [1] introduced the first weak Galerkin finite element method to the solution of general elliptic equations. Since then, there has been an active application of weak Galerkin methods for linear and non-linear convection-diffusion problems. Li and Wang [2] proposed a newly developed weak Galerkin method for the solution of a parabolic problem. Gao et al [3] presented numerical schemes for one class of Sobolev problems based on the weak Galerkin finite element method. Zhang and Lin[4] presented and analyzed the weak Galerkin finite element method for stationary Navier-Stokes equations. Kashkool et al ([5], [6], [7]) applied the weak Galerkin finite element method for the solution of one-dimensional and two-dimensional nonlinear convection-diffusion equations and two-dimensional linear convection-diffusion problems, respectively. Kashkool and Hussein in ([8], [9], [10]) applied the weak Galerkin method for the solution of coupled Burger's equations (one-dimensional and two-dimensional). Over-penalized weak Galerkin finite element methods for elliptic interface problems with non-homogeneous boundary conditions and discontinuous coefficients were developed by Song et al [11]. In this paper, we present a detailed study of a weak Galerkin method for solving non-linear convection-diffusion problems.

The residue of the article is organized as follows: In Section 2, we will present the variational form for non-linear convection-diffusion equations using the weak Galerkin method. In Section 3, prove the stability of the method. In Section 4, we will give a convergence for optimal rates in H^1- norm and L^2-

norm in view of nonlinear assumptions for the convection–diffusion equations. In Section 5, we have reported numerical results to validate our theoretical results.

2 Preliminary

In this article, we used notations and a standard definitions for the Sobolev space. We are concentrated on the preliminaries, including definitions of a weak function, a weak gradient, weak divergent, and approximation spaces.

The space of weak functions associated with K [1]

$$W(K) = \left\{v = \{v_0, v_b\}: v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K)\right\},$$

Assume τ_h be the shaped regular and body–fitted partition for Ω (see [11]). $P_k(T)$ denote by the set of polynomials on T with a degree greater than k, and $P_k(e)$ denotes the set of polynomials on e with a degree less than k, where $\mathcal{E}$ is represented, in particular, by a set of all edges in tau_h.

The weak gradient [1] is defined of the weak function $\omega = \{\omega^0, \omega^b\}$ on a polygon K with boundary ∂K satisfying $\omega^0 \in L^2(K), \omega^b \in H^{\frac{1}{2}}(\partial K)$, denote by $\nabla_d \omega$,in the dual space of $H(div, K)$ and weak divergent [22] denote by $\nabla_d \cdot \omega$ in the dual space of $H^1(K)$, respectively

$$(\nabla_d \omega, q) := -(\omega^0 \nabla \cdot q)_K + \langle \omega^b q \cdot n \rangle_{\partial K}, \forall q \in H(div, K)$$

$$(\nabla_d \cdot \omega, \psi) := -(\omega^0, \nabla \psi)_K + \langle \omega^b \cdot n, \psi \rangle_{\partial K}, \forall \psi \in H^1(K)$$

where n is the normal outward direction to ∂K. Basing of a weak function. Define the weak Galerkin finite element spaces by

$$V_h := \{\omega = \{\omega^0, \omega^b\}: \omega^0 \in P_k(T), \omega^b \in P_k(e), \forall \ edge \ e \forall \ \partial T, T \in \tau_h\},$$

furthermore

$$V_h^0 = \{\{\omega^0, \omega^b\} \in V_h, \omega^b|_{\partial T \cap \partial \Omega} = 0, \forall \ T \in \tau_h\}.$$

Assume that $u \in H^{k+1}(\Omega); k \geq 1$ be the exact solution of $(1.1)-(1.3)$, and the following L^2 projections are introduced

$$Q_h: L^2(T) \to P_k(T); \forall T \in \tau_h;$$

$$R_h: [L^2(T)]^2 \to [P_{k-1}(T)]^2; \ \forall T \in \tau_h.$$

3 A Weak Galerkin Finite Element Scheme

3.1 The Weak Variational Formulation

The weak variational formulation for equations $(1.1- \ 1.3)$. Find $u = \{u^0, u^b\} \in V_h$ such that

$$(u_t^0, v^0) + (a(u)\nabla_d u, \nabla_d v) - (u^0 b(u^0), \nabla_d v) - (u^0 \nabla_d b(u), v^0) + (cu^0, v^0) = (f, v^0); v \in W(K)$$

$$(u^0(x,0),v) = (u^0,v^0); v \in W(K)$$

3.2 The Semi-Discrete Weak Galerkin Method

The weak Galerkin finite element methods. Find $u_h = \{u_h^0, u_h^b\} \in S_h(j,l)$ such that

$$(u_{h,t}^0, v^0) + (a(u_h^0)\nabla_{d,r}u_h, \nabla_{d,r}v) - (b(u_h^0)u_h^0, \nabla_{d,r}v) - \left(u_h^0\nabla_{d,r} \cdot b(u_h),\ v^0\right) + (c(u_h^0), v^0) = (f, v^0); \quad v = \{v^0, v^b\} \in S_h^0(j,l),$$
(3.1)

$$(u_h^0(x,0),v) = (u_h^0, v^0); \quad v = \{v^0, v^b\} \in S_h^0(j,l).$$

3.3 Stability

A basic stability inequality for the continuous time weak Galerkin finite element method is shown by the following lemma:

Lemma 3.1 *Let u_h be the solution of the problem (3.1) satisfies the stability estimate.*

$$\| u_h^0(t) \|^2 + \beta_1 \int_0^t exp(\beta(t-s)) \| u_h(s) \|^2 \, ds \leq exp(\beta t) \| u_h^0(0) \|^2 + \frac{1}{\alpha_2} \int_0^t exp(\beta(t-s)) \| f(s) \|^2 \, ds. \quad (3.2)$$

Proof. Put $v = u_h = \{u_h^0, u_h^b\}$ the equation (3.1), we have

$$\left(u_{h,t}^0(t), u_h^0(t)\right) + \left(a(u_h^0)\nabla_{d,r} u_h(t), \nabla_{d,r} u_h(t)\right) - \left(b(u_h^0)u_h^0(t), \nabla_{d,r} u_h(t)\right) - (u_h^0(t)\nabla_{d,r} \cdot b(u_h(t)), u_h^0(t)) + (cu_h^0(t), u_h^0(t)) = (f(t), u_h^0(t)).$$

We using Young's inequality and Cauchy−Schwarz inequality the terms of the equation, we get

$$(u_{h,t}^0(t), u_h^0(t)) \leq \frac{1}{2}\frac{d}{dt} \| u_h^0(t) \|^2.$$

$$(a(u_h^0)\nabla_{d,r} u_h(t), \nabla_{d,r} u_h(t)) \leq \| a(u_h^0) \| \; \| \nabla_{d,r} u_h(t) \|^2 \leq C \;\; \| u_h(t) \|^2.$$

$$\left(b(u_h^0)u_h^0(t), \nabla_{d,r} u_h(t)\right) \leq \frac{1}{2\alpha_1} \| b(u_h^0) \| \| \nabla_{d,r} u_h(t) \|^2 + 2\alpha_1 \| u_h^0(t) \|^2 \leq C_1 \;\; \| u_h(t) \|^2 + 2\alpha_1 \| u_h^0(t) \|^2.$$

$$(u_h^0(t)\nabla_{d,r}\cdot b(u_h),u_h^0(t)) \leq\| \nabla_{d,r}\cdot b(u_h) \|\| u_h^0(t) \|^2\leq C \quad \| u_h^0(t) \|^2.$$

$$(cu_h^0(t),u_h^0(t)) \leq\| c \|_{L^\infty}\| u_h^0(t) \|^2.$$

$$(f(t),u_h^0(t)) \leq \frac{1}{2\alpha_2} \| f(t) \|^2+ 2\alpha_2 \| u_h^0(t) \|^2.$$

The above equation yields the following

$$\frac{1}{2}\frac{d}{dt} \| u_h^0(t) \|^2+ C \quad \| u_h(t) \|^2+ C_1 \quad \| u_h(t) \|^2+ 2\alpha_1 \| u_h^0(t) \|^2+ C \quad \| u_h^0(t) \|^2 +\| c \|_{L^\infty}\| u_h^0(t) \|^2 \leq \frac{1}{2\alpha_2} \| f(t) \|^2+ 2\alpha_2 \| u_h^0(t) \|^2$$

then,

$$\frac{d}{dt} \quad \| u_h^0(t) \|^2+ \beta \quad \| u_h^0(t) \|^2+ \beta_1 \quad \| u_h(t) \|^2\leq \frac{1}{\alpha_2} \quad \| f(t) \|^2,$$

where $\beta = 4(\alpha_1 - \alpha_2) + 2(C+\| c \|_{L^\infty})$ and $\beta_1 = C + C_1$.

Multiply both sides of the above inequality by the integral factor $exp(-\beta s)$ and then integrate from 0 to t, we have

$$\| u_h^0(t) \|^2+ \beta_1 \int_0^t exp(\beta(t-s)) \| u_h(s) \|^2 \, ds \leq exp(\beta t) \| u_h^0(0) \|^2+ \frac{1}{\alpha_2}\int_0^t exp(\beta(t-s)) \| f(s) \|^2 \, ds.$$

4 Optimal Order Error Estimates of the Semi−Discrete Weak Galerkin Finite Element Schemes

In this section, we are derived error estimate for a weak Galerkin finite element method (3.1). Let us begin with a derivation of the error equations for the weak Galerkin approximation u_h and the L^2 projection of the exact solution u in a weak finite element space V_h, will prove optimal order error estimate for L^2 norm and H^1 norm.

Lemma 4.1 *[23] Suppose that* $u \in L^\infty(0,T;H^{m+1}(\Omega))$ *with* $0 \le m \le j+1$ *and* $\Pi_h u \in S_h(j)$ *are the solution of the problem (1) and (37), respectively. Let* $Q_h u = \{Q_h u^0, Q_h u^b\}$ *be the* L^2 *projection of* u *in the corresponding finite element space. Then there exists a positive constant* C *independent of* h *such that*

$$\| \nabla_{d,r}(Q_h u - \Pi_h u) \|_{L^\infty(0,T;H^{m+1}(\Omega))} \le Ch^m \| u \|_{L^\infty(0,T;H^{m+1}(\Omega))},$$

$$\| Q_h u - \Pi_h u \|_{L^\infty(0,T;H^{m+1}(\Omega))} \le Ch^{m+1} \| u \|_{L^\infty(0,T;H^{m+1}(\Omega))},$$

provided that the mesh−size h is sufficiently small.

Lemma 4.2 *[23] Under the assumption of Lemma 4.1, if $u_t \in L^\infty(0,T;H^{m+1}(\Omega))$ with $0 \le m \le j+1$, then there exists a positive constant C independent of h such that*

$$\| Q_h u - \Pi_h u \|_{L^\infty(0,T;H^{m+1}(\Omega))} \le Ch^{m+1}\left(\| u \|_{L^\infty(0,T;H^{m+1}(\Omega))} + \| u_t \|_{L^\infty(0,T;H^{m+1}(\Omega))}\right),$$

provided that the mesh−size h is sufficiently small.

4.1 Optimal Order Error Estimates in the L^2 − Norm

In this subsection, we are derive the optimal order error estimate in L^2−norm for a discrete−time weak Galerkin methods. Assume $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\Pi_h u^{N+1}$ represents an elliptic projection of u onto the finite element space $S_h^0(j,l)$.

Theorem 4.1 *Assume that an exact solution of equation (3.1) is so regular that $u \in H^{k+1}(\Omega)$ and C_1 is a constant. Then*

$$\| \theta \| \le C_1 h^{m+1}\left(\| u \|^2_{L^\infty(0,T;H^{m+1}(\Omega))} + \| u_t \|^2_{L^\infty(0,T;H^{m+1}(\Omega))}\right)^{1/2}. \quad (4.1)$$

Proof. Choose $u_h = \Pi_h u$ and $u_h = Q_h u$ in equation (3.1)(see[23]), respectively.

$$((u_h^0)_t, v^0) + (a(\Pi_h u^0)\Pi_h \nabla_{d,r} u, \nabla_{d,r} v) - (b(\Pi_h u^0)\Pi_h u^0, \nabla_{d,r} v) \quad (4.2)$$

$$-(\Pi_h u^0 \nabla_{d,r} \cdot b(\Pi_h u), v^0) + (c\Pi_h u^0, v^0) = (f, v^0),$$

and,

$$((u_h^0)_t, v^0) + (a(Q_h u^0)\nabla_{d,r} Q_h u, \nabla_{d,r} v) - (b(Q_h u^0)Q_h u^0, \nabla_{d,r} v) \quad (4.3)$$

$$-(Q_h u^0 \nabla_{d,r} \cdot b(Q_h u), v^0) + (cQ_h u^0, v^0) = (f, v^0),$$

subtracting equation (4.2) from (4.3), we get

$$\big(a(\Pi_h u^0)\Pi_h \nabla_{d,r} u, \nabla_{d,r} v\big) - \big(a(Q_h u^0)\nabla_{d,r} Q_h u, \nabla_{d,r} v\big) + \big(b(Q_h u^0)Q_h u^0, \nabla_{d,r} v\big) - \big(b(\Pi_h u^0)\Pi_h u^0, \nabla_{d,r} v\big) - \big(\Pi_h u^0 \nabla_{d,r} \cdot b(\Pi_h u), v^0\big) - \big(Q_h u^0 \nabla_{d,r} \cdot b(Q_h u), v^0\big) + (c\Pi_h u^0, v^0) - (cQ_h u^0, v^0) = 0, \quad (4.4)$$

rewrite equation (??), and choose $v = \theta = \Pi_h u - Q_h u$, we have

$$\big(a(\Pi_h u^0)\nabla_{d,r}\theta, \nabla_{d,r}\theta\big) + \Big(\big(a(\Pi_h u^0) - a(Q_h u^0)\big)\nabla_{d,r} Q_h u, \nabla_{d,r}\theta\Big) - \Big(\big(b(\Pi_h u^0) - b(Q_h u^0)\big)\Pi_h u^0, \nabla_{d,r}\theta\Big) - \big(b(Q_h u^0)\theta^0, \nabla_{d,r}\theta\big) - \big(\theta^0 \nabla_{d,r} \cdot b(\Pi_h u), \theta^0\big) - (Q_h u^0 \nabla_{d,r} \cdot (b(\Pi_h u) - b(Q_h u)), \theta^0) + (c\theta^0, \theta^0) = 0. \quad (4.5)$$

Suppose the above equation as the following

$$\sum_{j=1}^{7} R^{(j)} = 0. \tag{4.6}$$

Using Cauchy–Schwarz inequality and Young's inequality in equation (4.6), we get

$$|R^{(1)}| \leq \| a(\Pi_h u^0) \| \| \nabla_{d,r}\theta \|^2 \leq C \| \theta \|^2.$$

$$\left|R^{(2)}\right| \leq \left|a(\Pi_h u^0)\nabla_{d,r}Q_h u, \nabla_{d,r}\theta\right| + \left|a(Q_h u^0)\nabla_{d,r}Q_h u, \nabla_{d,r}\theta\right| \leq \frac{\|a(\Pi_h u^0)\|^2}{4} \| \nabla_{d,r}Q_h u \|^2 + \| \nabla_{d,r}\theta \|^2 + \frac{\|a(Q_h u^0)\|^2}{4} \| \nabla_{d,r}Q_h u \|^2 + \| \nabla_{d,r}\theta \|^2 \leq 2C \| \nabla_{d,r}u_h \|^2 + (\alpha_1 + \alpha_2) \| \theta \|^2.$$

$$\left|R^{(3)}\right| \leq \left|b(\Pi_h u^0)\Pi_h u^0, \nabla_{d,r}\theta\right| + \left|b(Q_h u^0)\Pi_h u^0, \nabla_{d,r}\theta\right| \leq \frac{\|b(\Pi_h u^0)\|^2}{4} \| \Pi_h u^0 \|^2 + \| \nabla_{d,r}\theta \|^2 + \frac{\|b(Q_h u^0)\|^2}{4} \| \Pi_h u^0 \|^2 + \| \nabla_{d,r}\theta \|^2 \leq 2C \| u_h^0 \|^2 + 2C \| \theta \|^2.$$

$$|R^{(4)}| \leq \frac{\| b(Q_h u^0) \|^2}{4} \| \theta^0 \|^2 + \| \nabla_{d,r}\theta \|^2 \leq C \ \| \theta^0 \|^2 + C \| \theta \|^2.$$

$$|R^{(5)}| \leq \| \nabla_{d,r} \cdot b(\Pi_h u) \| \ \| \theta^0 \|^2 \leq C \| \theta^0 \|^2.$$

$$\left|R^{(6)}\right| \leq \left|Q_h u^0 \nabla_{d,r} \cdot b(\Pi_h u), \theta^0\right| + \left|Q_h u^0 \nabla_{d,r} \cdot b(Q_h u), \theta^0\right| \leq \frac{\| \nabla_{d,r} \cdot b(\Pi_h u) \|^2}{4} \| Q_h u^0 \|^2 + \frac{\| \nabla_{d,r} \cdot b(Q_h u) \|^2}{4} \| Q_h u^0 \|^2 + \| \theta^0 \|^2 + \| \theta^0 \|^2 \leq (\alpha_3 + \alpha_4) \| u_h^0 \|^2 + 2 \| \theta^0 \|^2.$$

$$|R^{(7)}| \leq \alpha \| \theta^0 \|^2.$$

Then equation (4.6) becomes

$$C \| \theta \|^2 + 2C \| \nabla_{d,r} u_h \|^2 + (\alpha_1 + \alpha_2) \| \theta \|^2 + 2C \| u_h^0 \|^2 + 2C \| \theta \|^2 + C \| \theta^0 \|^2 + C \| \theta \|^2 + C \| \theta^0 \|^2 + (\alpha_3 + \alpha_4) \| u_h^0 \|^2 + 2 \| \theta^0 \|^2 + \alpha \| \theta^0 \|^2 \leq 0.$$

Since $\| u_h^0 \|^2 \geq 0$ and $\| \nabla_{d,r} u_h \|^2 \geq 0$, we get

$$(4C+\alpha_1+\alpha_2) \parallel \theta \parallel^2 \leq -(2C+\alpha+2) \parallel \theta^0 \parallel^2,$$

By using Lemma 4.1 and Lemma 4.2, we get

$$(4C+\alpha_1+\alpha_2) \parallel \theta \parallel^2 \leq (-2C-\alpha-2)Ch^{2m+2}[\parallel u \parallel^2_{L^\infty(0,T;H^{m+1}(\Omega))} + \parallel u_t \parallel^2_{L^\infty(0,T;H^{m+1}(\Omega))}],$$

then

$$\parallel \theta \parallel \leq C_1 h^{m+1} \left(\parallel u \parallel^2_{L^\infty(0,T;H^{m+1}(\Omega))} + \parallel u_t \parallel^2_{L^\infty(0,T;H^{m+1}(\Omega))} \right)^{1/2}.$$

4.2 Optimal Order Error Estimates in The H^1- Norm

In this subsection, we are derive an optimal order error estimate in the H^1–norm for the discrete–time weak Galerkin methods. Assume $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\Pi_h u^{N+1}$ represents an elliptic projection of u onto the finite element space $S_h^0(j,l)$.

Theorem 4.2 *Assume Q_h and Π_h are projections, and let θ difference between $\Pi_h u$ and $Q_h u$ with C_1 is a positive constant. Then*

$$\| \nabla_{d,r}\theta \| + \| \theta \| \leq C_1 h^m \left(\| u \|^2_{L^\infty(0,T;H^{m+1}(\Omega))} + \| u_t \|^2_{L^\infty(0,T;H^{m+1}(\Omega))} \right)^{1/2}. \qquad (4.7)$$

Proof. By a similar way of Theorem (4.1), we get the following equation:

$$C \| \nabla_{d,r}\theta \|^2 + 2C \| \nabla_{d,r}u_h \|^2 + 2 \| \nabla_{d,r}\theta \|^2 + 2C \| u_h^0 \|^2 + 2 \| \nabla_{d,r}\theta \|^2 + C \| \theta^0 \|^2 + C \| \theta \|^2 + C \| \theta^0 \|^2 + (\alpha_1 + \alpha_2) \| u_h^0 \|^2 + 2 \| \theta^0 \|^2 + \alpha \| \theta^0 \|^2 \leq 0,$$

so,

$$C \| \theta \|^2 + (C+4) \| \nabla_{d,r}\theta \|^2 \leq -(2C + 2 + \alpha) \| \theta^0 \|^2,$$

By using Lemma 4.1 and Lemma 4.2, we get

$$C \| \nabla_{d,r}\theta \|^2 + (C+4) \| \theta \|^2 \leq (-2C - 2 - \alpha) \left\{ Ch^{2m+2} \left(\| u \|^2_{L^\infty(0,T;H^{m+1}(\Omega))} + \| u_t \|^2_{L^\infty(0,T;H^{m+1}(\Omega))} \right) \right\}.$$

This completes the proof

5 Numerical Experiment

We present a numerical example in order to demonstrate the effectiveness and reliability for the weak Galerkin scheme for solving non−linear convection−diffusion problems. We used MATLAB as the software development environment.

Here, we rewriter nonlineaer convection−diffusion problem $(1.1-1.3)$ [24]

$$\frac{\partial u}{\partial t} - \nabla \cdot (\varepsilon \nabla u) + u \cdot \nabla u + cu = f; \Omega \times (0,T] \quad (5.1)$$

$$u(x,t) = 0; \Gamma \times (0,T] \quad (5.2)$$

$$u(x,0) = u^0(x); x \in \overline{\Omega} \quad (5.3)$$

The uniform triangle partition is used in this example. Where $h = 1/N$; such that $(N = 2,4,8,16,32,64)$ denote the spatial mesh size. Let $\Omega = [0,1] \times [0,1]$, f and g are determined by setting an exact solution

$$u(x,y,t) = xy(1-x)(1-y)e^{-t}.$$

We take $t = 1$, $\Delta t = 0.001$ and $\varepsilon = 0.1$, we calculated the error value in a table 1 and drew a diagram of the exact solution and the approximate solution by the weak Galerkin method in fig. (1) and fig. (2) at $N = 8$ and $N = 16$, respectively.

h	$\| u - u_h \|_{H^1}$	Order	$\| u - u_h \|_{L^2}$	Order	$\| u - u_h \|_{L^\infty}$	Order

1/2	5.0074e−03	0	2.5040e−03	0	2.5069e−03	0
1/4	2.5968e−03	9.4732e−01	6.2976e−04	1.9914e+00	6.3740e−04	1.9756e+00
1/8	1.3124e−03	9.8455e−01	1.5670e−04	2.0068e+00	1.5743e−04	2.0175e+00
1/16	6.4655e−04	1.0213e+00	3.9119e−05	2.0021e+00	3.9208e−05	2.0055e+00
1/32	3.2390e−04	9.9721e−01	9.7795e−06	2.0000e+00	9.7993e−06	2.0004e+00
1/64	1.6655e−04	9.5957e−01	2.4450e−06	1.9999e+00	2.4498e−06	2.0000e+00

Table 1: Errors of weak Galerkin method with fixed $t = 1, \Delta t = 0.001$ and $\varepsilon = 0.001$.

6 Conclusion.

In this article, we introduce a weak Galerkin method, including a stabilizer of the weak functions and a semi-discrete scheme of the solution for the non−

linear convection–diffusion problems. We have analyzed the stability for semi–discrete weak Galerkin method and assured a convergence rates optimal in H^1 and L^2 norms for weak Galerkin finite element method. The convergence and flexibility of the weak Galerkin method result in good approximations of the problem (see fig. 1 and fig. 2), whereas a numerical example has been used to validate the theory.

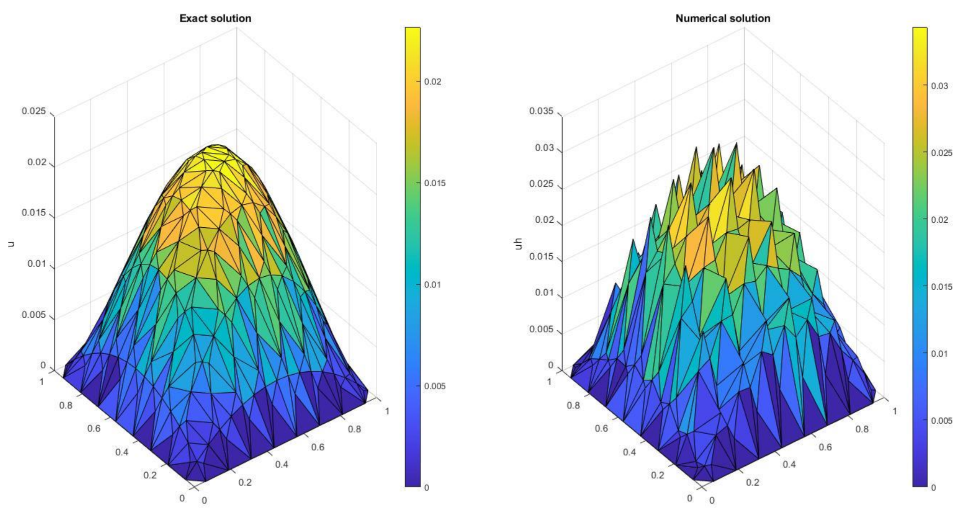


Figure a: $h = 1/8$

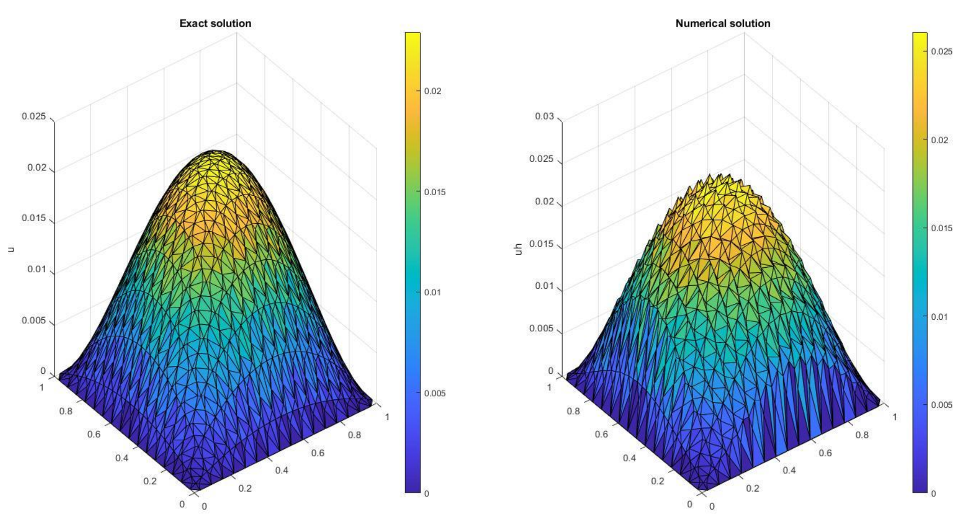
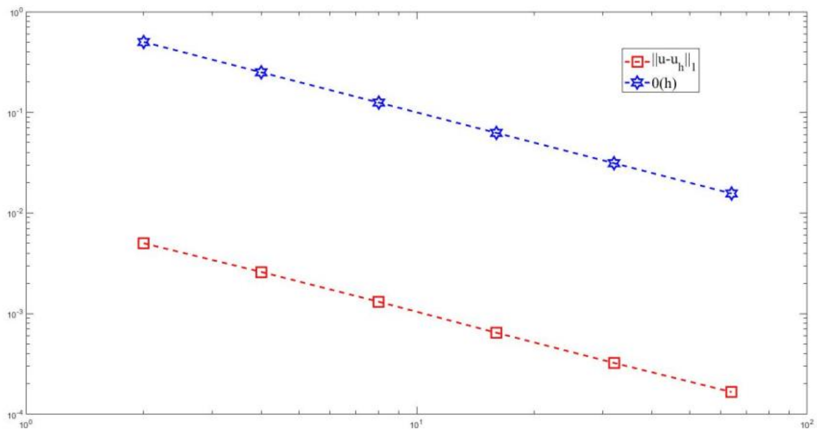
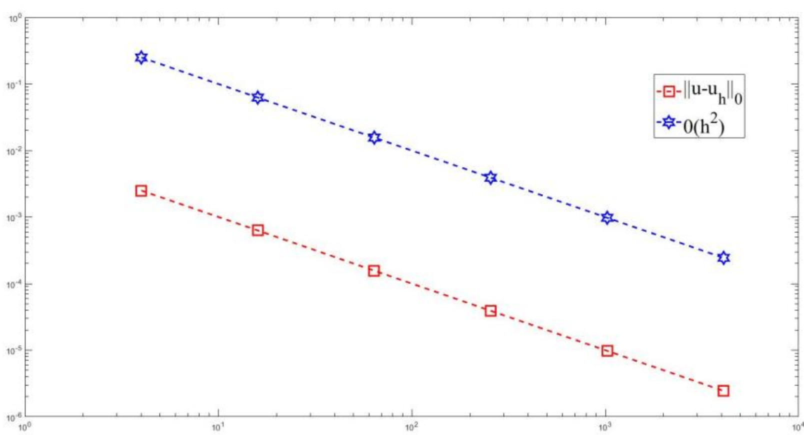


Figure b: $h = 1/16$

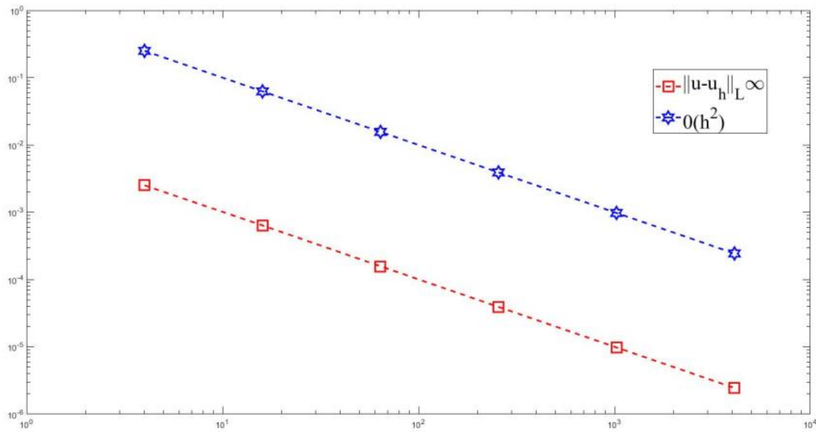
Figure 1: Weak Galerkin solutions and Exact solutions for non−linear convection−diffusion problem with $h = (1/8, 1/16), \Delta t = 0.001, t = 1$ and $\varepsilon = 0.001$.



order error for H^1 −norm



order error for L^2 −norm



order error for L^∞ −norm

Figure 2: Order error for both H^1 −norm, L^2 −norm and L^∞ −norm.

References

[1] Wang, J. and Ye, X. (2013) " A weak Galerkin finite element method for second-order elliptic problems " *Journal of Computational and Applied Mathematics*, vol. 241 ,pp.103-115.

[2] Li, Q.H. and Wang, J., (2013). Weak Galerkin finite element methods for parabolic equations. *Numerical Methods for Partial Differential Equations*, 29(6), pp.2004-2024.

[3] Gao, F., Cui, J. and Zhao, G., (2017). Weak Galerkin finite element methods for Sobolev equation. *Journal of Computational and Applied Mathematics*, 317, pp.188-202.

[4] Zhang, T. and Lin, T., (2019). An analysis of a weak Galerkin finite element method for stationary Navier–Stokes problems. *Journal of Computational and Applied Mathematics*, 362, pp.484-497.

[5] Cheichan, M. S., Kashkool, H. A., and Gao, F. (2019). "A weak Galerkin finite element method for solving nonlinear convection-diffusion problems in two dimensions". *Applied Mathematics and Computation*, 354, 149-163.

[6] Cheichan, M. S., Kashkool, H. A., and Gao, F. (2019). "A Weak Galerkin Finite Element Method for Solving Nonlinear Convection–Diffusion Problems in One Dimension". *International Journal of Applied and Computational Mathematics*, 5(2), 1-15.

[7] Cheichan, M. S., Kashkool, H. A., and Gao, F. (2020). "A weak Galerkin finite element method for solving linear convection-diffusion problems in two dimensions". *Applied of Comput. Anal. and Application*, 28(1), 425-436.

[8] Kashkool, H. A., and Hussein, A. J. (2020, May). "Error Estimate for Two-Dimensional Coupled Burgers' Equations By Weak Galerkin Finite Element Method". *In Journal of Physics: Conference Series . IOP Publishing*.(Vol. 1530, No. 1, p. 012065)

[9] Hussein, A. J., and Kashkool, H. A. (2020). "Weak Galerkin finite element method for solving one-dimensional coupled Burgers' equations". *Journal of Applied Mathematics and Computing*, 63(1), 265-293.

[10] Hussein, A. J., and Kashkool, H. A. (2020). "A weak Galerkin finite element method for two-dimensional coupled burgers' equation by using polynomials of order $(k, k-1, k-1)$". *Journal of Interdisciplinary Mathematics*, 23(4), 777-790.

[11] Song, L., Qi, W., Liu, K. and Gu, Q., (2021). A new over-penalized weak galerkin finite element method. Part II: Elliptic interface problems. *Discrete and Continuous Dynamical Systems-B*, 26(5), p.2581.

[12] Hashim, A. K. (2002). "Upwind type finite element method for nonlinear Convection-Diffusion problem and applications to numerical reservoir simulation," *school of Mathematical Sciences. Nankai Universty, China*.

[13] Liu, T. and Liu, S., (2018). Identification of diffusion parameters in a non-linear convection–diffusion equation using adaptive homotopy perturbation method. *Inverse Problems in Science and Engineering*, 26(4), pp.464-478.

[14] Chan, J., Evans, J.A. and Qiu, W., (2014). A dual Petrov-Galerkin finite element method for the convection-diffusion equation. *Computers and Mathematics with Applications*, 68(11), pp.1513-1529.

[15] Gao, F.Z. and Yuan, Y.R.,(2009). An upwind finite volume element method based on quadrilateral meshes for nonlinear convection-diffusion problems. *Numerical Methods for Partial Differential Equations: An International Journal*, 25(5), pp.1067-1085.

[16] Fatehi, R., Manzari, M.T. and Hannani, S.K.,(2009). A finite-volume ELLAM for non-linear flux convection–diffusion problems. *International Journal of Non-Linear Mechanics*, 44(2), pp.130-137.

[17] Havle, O., Dolejší, V. and Feistauer, M., (2010). Discontinuous Galerkin method for nonlinear convection-diffusion problems with mixed Dirichlet-Neumann boundary conditions. *Applications of Mathematics*, 55(5), pp.353-372.

[18] Dolejší, V., Feistauer, M. and Sobotíková, V., (2005). Analysis of the discontinuous Galerkin method for nonlinear convection–diffusion problems. *Computer methods in applied mechanics and engineering*, 194(25-26), pp.2709-2733.

[19] Cockburn, B. and Shu, C.W.,(1998). The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM Journal on Numerical Analysis*, 35(6), pp.2440-2463.

[20] Xikui, L. and Wenhua, W., (1999). Characteristic Galerkin method for convection-diffusion equations and implicit algorithm using precise integration. *Acta Mechanica Sinica*, 15(4), pp.371-382.

[21] Salhi, L., El-Amrani, M. and Seaid, M., (2021). Analysis of a Galerkin-characteristic finite element method for convection-diffusion problems in porous media. *Advances in Pure and Applied Mathematics*, 12, no special, pp. 96-122.

[22] Zhai, Q., Zhang, R., and Mu, L. (2016). "A new weak Galerkin finite element scheme for the Brinkman model". *Communications in Computational Physics*, 19(5), 1409-1434.

[23] Zhu, A., Xu, Q. and Jiang, Z., 2014, January. Characteristics weak Galerkin finite element methods for convection-dominated diffusion problems. *In Abstract and Applied Analysis (Vol. 2014). Hindawi*.

[24] Saadoon, J. J. (2020) "Discontinuous Galerkin Finite Element Method for the Fluid Flow Problems in Porous Media" *Ph. D. thesis, College of Education for Pure Sciences, University of Basrah* .