

Approximate Solution of Linear Volterra Integro-Differential Equation by Touchard Polynomials Method

الحل التقريبي لمعادلة فولتيرا التفاضلية التكاملية الخطية باستخدام طريقة متعددات حدود توجارد.

الخلاصة: الفكرة لهذا البحث هي لإيجاد الحل العددي التقريبي لمعادلة فولتيرا التفاضلية التكاملية الخطية من الرتبة الاولى والدرجة الثانية باستخدام درجات مختلفة من متعددات حدود توجارد. بسبب بساطة هذه الطريقة سوف تكون مستخدمة لتعطي نتائج عالية الدقة في اقل كلفة و اقل وقت كما ويمكن بسهولة تفاضلها ومكاملتها وتقاربها من الحل الدقيق . الخوارزمية والمثال المعطى هي لتوضيح الحل بهذه الطريقة ومقارنته مع الحل الدقيق.

Abstract: The idea of this research is to find approximate solution of Linear Volterra Integro-Differential Equation (LVIDE) of the first order and second kind by using different degrees of the Touchard polynomials is presented. Because the simplicity definition applying this method will give high resolution results in low cost and short time, also can be easily differentiated, integrated and converge to an exact solution.

The algorithm and example given are to illustrate the solution in this way and compare it with the exact solution.

Keywords: Volterra Integro-Differential Equation & Touchard Polynomials Method

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1. Integration

Integro-differential “equation is a hybrid of integral and differential equations which have found extensive applications in science and engineering since it was established by Volterra^[1, 2]. A special class of these equations are the volterra type have been used to model heat and mass diffusion processes, biological species coexisting together increasing and decreasing rate of growth, electromagnetic theory and ocean circulations”, in many physical applications such as glass forming, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert^[3, 4]. In the solution survey of linear Volterra integro-differential equations, several methods of that solutions have been developed in recent years, such as in^[5] Galerkin weighted residual method is proposed with touchard polynomials as trial functions by Aqsa Nazir, Muhammad et al. super implicit multistep collocation^[6] methods to find “numerical solution of Volterra integro-differential equations” by Somayyeh Fazeli and Gholamreza Hojjati, in^[7] A. H. Khater et al., uses Legendre Polynomials to obtained “Numerical

Solutions of integral and integro-differential Equations”, M. El-shahed^[8] and Behrouz Raftari^[9] used homotopy perturbation method to developed a numerical solution Volterra’s integro-differential equation, S. “G. Venkatesh, S. K. Ayyaswamy, and S.Raja Ba(achandar)^[10] applied Legendre for a class of higher-order Volterra integro-differential equation”, “Siraj-ul-Islam, Imran Aziz and Muhammad Fayyaz^[11] presented A new approach for numerical solution of integro-differential equations via Haar wavelets”, Maleknejad, K. and F. Mirzaee used rationalized Haar functions^[12] method for solution integro-differential equations, and Heleema S. Ali^[13] used the Bernstein Polynomials method for solving “Volterra integro- differential equation”.

In this research we solve the Linear “Volterra integro-differential Equation” of second kind by using very well-known Touchard Polynomials. For this, we give a short introduction about definition integro-differential equation, Touchard polynomials, and then we apply this technique on the integro-differential equation. To verify our formulation by one example, we obtain approximate solutions by using different degrees of the Touchard polynomials.

2. Integral Equations

The general form of integral equation has the form^[14]

$$“u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x, t) u(t) dt \dots (1)”,$$

where “ $\alpha(x)$ and $\beta(x)$ ” are the limits of integration that may be both variables, constants respectively, or mixed, and they may be in one dimension or more. “ $K(x, t)$ is a known function of two variables x and t , called the kernel or the nucleus of the integral equation”. “The unknown function $u(x)$ appears under the integral sign and out of the integral sign in most other cases. It is important to point out that the Kernel $K(x, t)$ and the function $f(x)$ ” are given in advance, and λ is known constant parameter. when the finite interval $[\alpha(x), \beta(x)] \subseteq R$

3. Volterra integral equations

The standard form of Volterra integral equation is given as^[4, 14, and 15].

$$“u(x)\varphi(x) = f(x) + \lambda \int_a^x k(x, t) u(t) dt \dots (2)”$$

$$[a, x] \subseteq R$$

“where the upper and lower limits of integration are variable and constant respectively”, “the unknown function $u(x)$

appears linearly or nonlinearly under the integral sign”.

There are three kinds of the Volterra integral equations:

1. “Volterra integral equation of the first kind, when the function $\varphi(x) = 0$, then equation (2) becomes:

$$f(x) + \lambda \int_a^x k(x, t) u(t) dt = 0 \dots (3),$$

$$[a, x] \subseteq R”$$

2. “Volterra integral equation of the second kind, when the function $\varphi(x) = 1$, then equation (2) becomes:

$$u(x) = f(x) + \lambda \int_a^x k(x, t) u(t) dt \dots (4),$$

$$[a, x] \subseteq R”$$

3. “Volterra integral equation” of the third kind, when $\varphi(x)$ neither 0 nor 1.

3.1. Volterra Integro-differential Equations^[3, 4]

We get “Volterra integro-differential equations when we convert initial value problems to integral equations. In this type of equations, the unknown function $u(x)$ appears as the combination of the ordinary derivative and under the integral sign”.

The linear “Volterra integro-differential equation” of the second kinds given as^[3, 4]

$$u^{(n)}(x) = f(x) + \lambda \int_0^x k(x, t) u(t) dt \dots (5),$$

$$x \in [a, b]$$

“where $u^{(n)}(x)$ indicates the n th derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}(x)$ at the left side, then it is necessary to define initial condition $u(0)$, $u'(0)$, ..., $u^{(n-1)}(0)$ for determine the particular solution and the exact solution of the linear Volterra integro-differential equation”.

4. Touchard Polynomials

The Touchard polynomials, studied in (1939) by J. Touchard, consist of (a polynomial sequence of binomial type). Touchard polynomials [5, 16, 17, and 18] are given as

$$T_n(x) = \sum_{k=0}^n S(n, k) x^k = \sum_{k=0}^n \binom{n}{k} x^k \dots (6)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, n is the degree of polynomials, (k) is the index of polynomials and (x) is the variable.

First six Touchard polynomials are given as

$$T_0(x) = 1,$$

$$T_1(x) = 1 + x,$$

$$T_2(x) = 1 + 2x + x^2,$$

$$T_3(x) = 1 + 3x + 3x^2 + x^3,$$

$$T_4(x) = 1 + 4x + 6x^2 + 4x^3 + x^4$$

$$T_5(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

The derivatives the Touchard polynomials are

$$\begin{aligned} \frac{d}{dx} T_n(x) &= \frac{d}{dx} \sum_{k=0}^n \binom{n}{k} x^k \\ &= \sum_{k=1}^n \binom{n}{k-1} x^{k-1} \dots (7) \end{aligned}$$

5. A Matrix Representation for Finding Approximate Solution

Given Touchard polynomials, we can write the approximate solution as follows

$$T(x) = c_0 T_0(x) + c_1 T_1(x) + \dots + c_n T_n(x), \dots (8)$$

where c_i ($i = 0, 1, 2, \dots, n$) are unknown coefficients to be determined. It is easy to write equation (8) as a dot product of two vectors

$$T(x) = [T_0(x) \ T_1(x) \ \dots \ T_n(x)] \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}, \dots (9)$$

we can convert equation (9) to the form

$$= [1 \ t \ t^2 \ \dots \ t^n] \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots & b_{1,n} \\ 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}, \dots \quad (10)$$

where the $b_{i,j}$ are the coefficients of the power basis that are used to determine the Touchard polynomials. We note that the matrix in this case is upper triangular.

The matrix derivatives of Touchard polynomials

$$u'(x) = [0 \ 1 \ 2x \ \dots \ nx^{n-1}] \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots & b_{1,n} \\ 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}, \dots \quad (11)$$

Now in linear case ($n=1$), the matrix representation is

$$u(t) = [1 \ t] \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}.$$

In the quadratic case ($n=2$), the matrix representation is

$$u(t) = [1 \ t \ t^2] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.$$

In the cubic case ($n=3$), the matrix representation is

$$u(t) = [1 \ t \ t^2 \ t^3] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

In the fourfold case ($n=4$), the matrix representation is

$$u(t) = [1 \ t \ t^2 \ t^3 \ t^4] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

And in the fivefold case ($n=5$), the matrix representation is

$$u(t) = [1 \ t \ t^2 \ t^3 \ t^4 \ t^5] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

6. Solution of the linear Volterra

Integro-differential Equation by Touchard Polynomials.

In this section, we will use Touchard polynomials method to estimate the approximate solution of the linear “Volterra integro-differential equation of second kind”, will be introduced.

Let us consider the linear Volterra integro-differential equation of the first order and second kind in equation (5)

$$u'(x) = f(x) + \lambda \int_0^x k(x,t) u(t) dt \dots \quad (12),$$

and by using equation (9), we have

$$u(x) = T(x) = [T_0 \ T_1 \ \dots \ T_n] \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}, \dots \quad (13)$$

also by using equation (7)

$$u'(x) = \sum_{k=1}^n \binom{n}{k-1} x^{k-1}$$

Now applying the Touchard polynomials method for equation (12), by using equations (10) and (11) we get the following formula

$$\begin{aligned} & [0 \ 1 \ 2x \ \dots \ nx^{n-1}] \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots & b_{1,n} \\ 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \\ & = \lambda \int_0^x k(x,t) [1 \ t \ t^2 \ \dots \ t^n] \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots & b_{1,n} \\ 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} dt \dots \quad (14) \end{aligned}$$

Now we find all integrations in the equation (14), in order to determine the unknown values $(c_0, c_1, \dots, c_n)$, we need (n) equations.

We choose (x_i) where $(i = 1, 2, \dots, n)$ in $[a, b]$, which gives (n) equations. We solve this equations by “Gauss elimination” to determine the values $(c_0, c_1, \dots, c_n)$.

The following algorithm summarizes the steps for finding the approximate solution for the “second kind of linear Volterra integro_differential equation”.

7. Algorithm

Step (1):

We choose (n) , the degree of Touchard Polynomials,

$$T_n(x) = \sum_{k=0}^n \binom{n}{k} x^k$$

Step (2):

Substitute the Touchard Polynomials in the linear “Volterra integro-differential equation of second kind”,

$$\begin{aligned} & u'(x) \\ & = [0 \ 1 \ 2x \ \dots \ nx^{n-1}] \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots & b_{1,n} \\ 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \\ & = \lambda \int_0^x k(x,t) [1 \ t \ t^2 \ \dots \ t^n] \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots & b_{1,n} \\ 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} dt \end{aligned}$$

Step (3):

Compute the following

$$[0 \ 1 \ 2x \ \dots \ nx^{n-1}] \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots & b_{1,n} \\ 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Compute the following integration

$$\int_0^x k(x,t) [1 \ t \ t^2 \ \dots \ t^n] \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots & b_{1,n} \\ 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} dt$$

Step (4):

Compute c_i ($i = 0, 1, 2, \dots, n$), where

x_i ($i = 0, 1, 2, \dots, n$) $\in (a, b)$.

8. Numerical Examples

Example1:

Consider the following linear “volterra integro-differential equation of second kind “on $[0, 1]$ given in ^[3],

$$u'(x) = \frac{1}{4} \int_0^x u(t) dt,$$

where

$$f(x) = 2 - \frac{x^2}{4} \text{ and } \lambda = \frac{1}{4}$$

with initial condition $u(0) = 0$ and the exact solution $u(x) = 2x$.

now we can applying the algorithm described above in this example, when choosing the Touchard Polynomials of degrees 2, 3, and 4.

1. When Choosing the degree of Touchard polynomials $n=2$, we have

$$[0 \ 1 \ 2x] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \\ = 2 - \frac{x^2}{4} + \frac{1}{4} \int_0^x [1 \ t \ t^2] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} dt,$$

which can be rewritten as

$$c_1 + c_2(2 + 2x) = 2 - \frac{x^2}{4} \\ + \frac{1}{4} \int_0^x (c_0 + c_1(1 + t) + c_2(1 + 2t + t^2)) dt,$$

after computing the integrals, then in order to determine c_0 , c_1 and c_2 , we need three

equations. By Choosing x_i ($i = 1, 2, 3$) in $[0, 1]$, with substitution in the initial condition will give three equations, solve these equations by “Gauss elimination” to find the values of c_0 , c_1 and c_2 . To get

$$c_0 = -2$$

$$c_1 = 2$$

$$c_2 = 0$$

which make the solution of linear “Volterra integro- differential equation of the second kind” is

$$u(x) = c_0 + c_1(1 + x) + c_2(1 + 2x + x^2)$$

$$u(x) = -2 + 2(1 + x)$$

2. When Choosing the degree of Touchard polynomials $n=3$, we have

$$[0 \ 1 \ 2x \ 3x^2] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ = 2 - \frac{x^2}{4} + \frac{1}{4} \int_0^x [1 \ t \ t^2 \ t^3] \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} dt,$$

which can be rewritten as

$$c_1 + c_2(2 + 2x) + c_3(3 + 6x + 3x^2) \\ = 2 \\ - \frac{x^2}{4} + \frac{1}{4} \int_0^x (c_0 + c_1(1 + t) \\ + c_2(1 + 2t + t^2) + c_3(1 + 3t + 3t^2 + t^3)) dt,$$

after computing the integral, and by choosing x_i ($i = 0, 1, 2, 3$) in the interval $[0, 1]$, with substitution in the initial condition which gives four equations. Then solve these

equations by “Gauss elimination” to find the values c_0, c_1, c_2 and c_3 . To get

$$c_0 = -2.0000$$

$$c_1 = 2.0000$$

$$c_2 = -2.4575e-012$$

$$c_3 = -5.5201e-014$$

which make the solution of linear “Volterra integro- differential equation of the second kind” is

$$u(x) = c_0 + c_1(1+x) + c_2(1+2x+x^2) + c_3(1+3x+3x^2+3x^3)$$

$$u(x) = -2 + 2(1+x) + (-2.4575e-012)(1+2x+x^2) + (-5.5201e-014)(1+3x+3x^2+x^3)$$

3. In the similar way when choose the degree of Touchard polynomials as $n=4$,

we get

$$c_0 = -2,$$

$$c_1 = 2,$$

$$c_2 = 0,$$

$$c_3 = 0,$$

$$c_4 = 0,$$

which makes the approximate solution is

$$u(x) = (-2) + (2)(1+x)$$

Approximated solutions (A. S.) of some values of (x) got by using “Touchard polynomials method” and the exact solution $u(x) = 2x$ of example, depending on “Least Square Error” (L. S. E.),

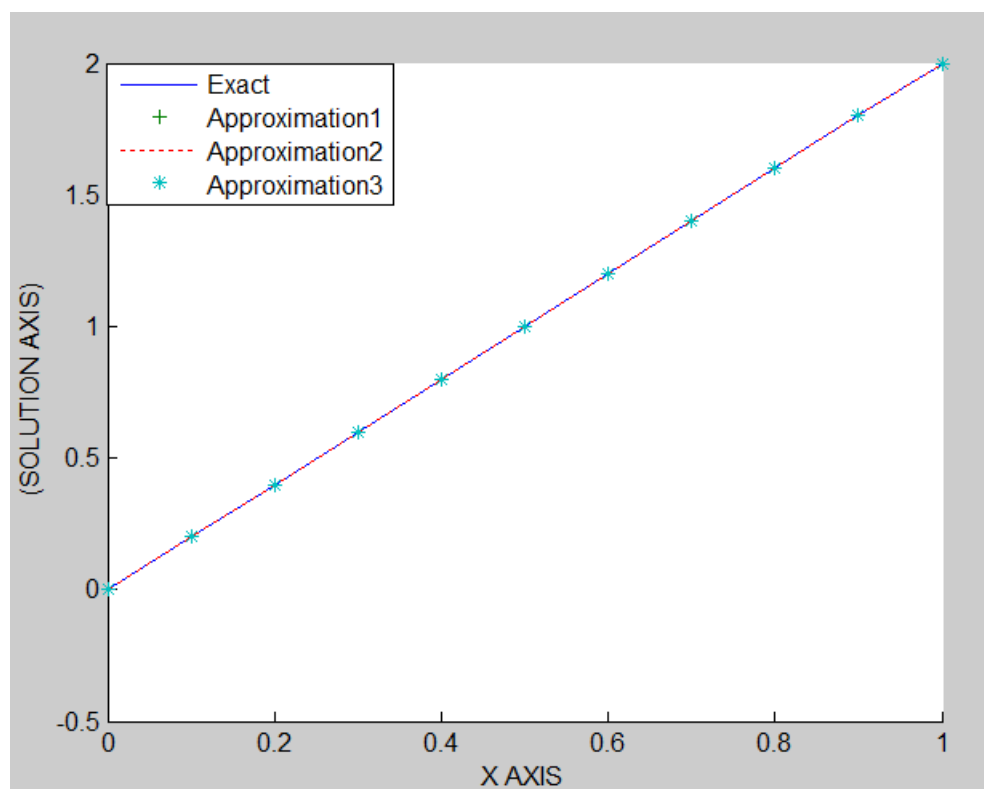
Error = $\sum_{i=1}^m (u(x)_{Exact} - u(x)_{Approximation})^2$ are presented in Table (1) and Figure (1).

Table 1: Exact and Approximate Solutions of (LVIDE) of Example1.

x	Exact Solution = 2 x	App. S. (n=2)	App. S. (n=3)	App. S. (n=4)
0.0	0 .0000	0	-0.0000	0.0000
0.1	0.2000	0.2000	0.2000	0.2000
0.2	0.4000	0.4000	0.4000	0.4000
0.3	0.6000	0.6000	0.6000	0.6000
0.4	0.8 000	0.8000	0.8000	0.8000
0.5	1.0000	1.0000	1.0000	1.0000
0.6	1.2000	1.2000	1.2000	1.2000
0.7	1.4000	1.4000	1.4000	1.4000
0.8	1.6000	1.6000	1.6000	1.6000
0.9	1.8000	1.8000	1.8000	1.8000
1.0	2.0000	2.0000	2.0000	2.0000
L.S.E.= $\sum_{i=0}^{10} (u(x)_{Exact} - u(x)_{APP.})^2$		1.8797e-031	4.6097e-022	1.8797e-031

Where App. S. : represents the Approximate Solution

Figure 1: Exact and Approximate Solutions of (LVIDE) of Example 1



Example2:

Consider the following linear “volterra integro-differential equation of second kind “on $[0, 1]$ given in ^[4],

$$u'(x) = 1 + \int_0^x u(t) dt,$$

where

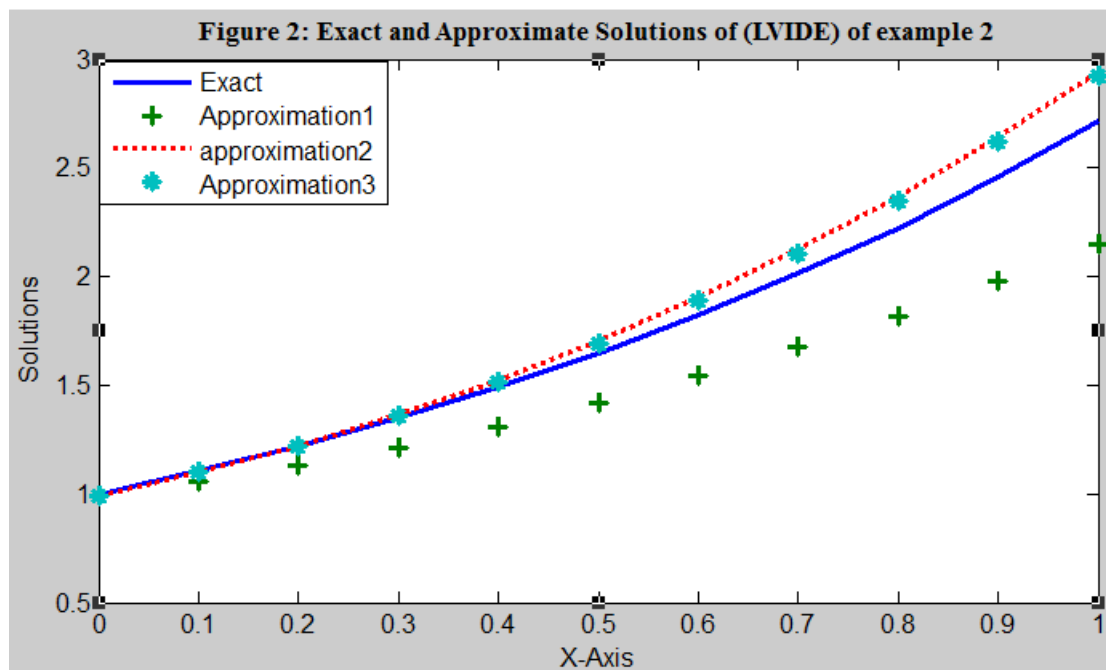
$$f(x) = 1 \text{ and } \lambda = 1$$

with initial condition $u(0) = 1$ and the exact solution $u(x) = e^x$.

When “Touchard polynomials algorithm” is applied, Table(2) and Figure(2), show the comparison between the approximate solutions using Touchard polynomials method and exact solution $u(x) = e^x$ depending on “Least Square Error (L. S. E.). “

Table 2: Exact and Approximate Solutions of (LVIDE) of Example2.

x	Exact Solution= e^x	App. S. (n=2)	App. S. (n=3)	App. S. (n=4)
0.0	1.0000	0.9914	0.9913	0.9924
0.1	1.1052	1.0536	1.0990	1.0995
0.2	1.2214	1.1276	1.2226	1.2217
0.3	1.3499	1.2134	1.3635	1.3601
0.4	1.4918	1.3111	1.5229	1.5161
0.5	1.6487	1.4206	1.7022	1.6913
0.6	1.8221	1.5419	1.9028	1.8874
0.7	2.0138	1.6750	2.1259	2.1063
0.8	2.2255	1.8200	2.3728	2.3500
0.9	2.4596	1.9768	2.6450	2.6209
1.0	2.7183	2.1454	2.9436	2.9212
L. S. E. = $\sum_{i=0}^{10} (u(x)_{\text{Exact}} - u(x)_{\text{App.}})^2$		1.0339	0.1300	0.0981



9. Conclusion

In general, the integral equations are difficult to be solved analytically, therefore in many equations we need to get the approximate solutions, and for this case the “Touchard Polynomials method” for the solution linear “Volterra integro- differential equation” is implemented.

The solutions are based on Touchard Polynomials method which reduces a Volterra integro- differential equation to a set of “algebraic equations” that can be easily solved by using “MATLAB” program. From the figure, it is obvious that the obtained results show that the convergence to the exact solution is very fast in “Touchard Polynomials method” and, double precision in the example can be achieved with low error in order (n).

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