A Modification of Szãsz Operators with Two Variables

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Abstract

In the present paper, we introduce a modification of the Summation Szãsz type operators denoted by $S_{n,m}(f;x,y)$, where $f \in C_{\alpha,\gamma}([0,\infty)\times[0,\infty))$ (the space of all continuous functions on the area $([0,\infty)\times[0,\infty))$and $x,y \in [0,\infty)$ are two independent variables. First, we discuss the converges of this operator to the function$f(t,u) \in C_{\alpha,\gamma}$. Then, we establish a Voronovskaja-type asymptotic formula for the operator $S_{n,m}(f;x,y)$.

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1. Introduction

The approximation of functions by Szász-Mirakyan operators

$$L_n(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), x \in R_0 := [0,\infty), n \in N := \{1,2,\dots\}, \text{where}$$

$$q_{n,k}(x) = \frac{(nx)^k}{e^{nx}k!}, k \in N^0$$

$:= N \cup \{0\}$ has been examined in many papers and monographs e.g.([8], [2], [3]). The above operators were modified by several authors e.g. ([1], [7], [9]) which showed that new operators have similar or better approximation properties than $L_n(f;x)$.

Rempulska and Graczyk [6], introduced a modification of the Szãsz operators and studied some direct results in ordinary approximation as:

$$M_{n,r}(f,x) = \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} f\left(\frac{rk}{n}\right), \quad x \in R_0, n \in N,$$

and for every fixed $r \in N$, where $\quad A_r(t) = \sum_{k=o}^{\infty} \frac{t^{rk}}{(rk)!} \quad$ for $t \in R_0$.

In this paper, we introduced a new sequence of linear positive operators $S_{n,m}(f;x,y)$ for $f \in C_{\alpha,\gamma}([0,\infty)\times[0,\infty))$ given as follows:

$$S_{n,m}(f;x,y)=\frac{1}{G_x}\frac{1}{G_y}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}q_{n,kr}(x)\,q_{m,js}(y)\;f\left(\frac{kr}{n},\frac{js}{m}\right)$$

where $$G_x=\sum_{k=0}^{\infty}q_{n,kr}(x)\quad\text{and}\quad G_y=\sum_{j=0}^{\infty}q_{m,js}(x)$$

2. Auxiliary Results:

We give some lemmas which help us in the proofs of main theorems.

Lemma 1: [5]

For $n,r\in N$, $x\in R_0$ and

$$A_r^{(m)}(nx)=n^m\sum_{k=m}^{\infty}\frac{(nx)^{rk-m}}{(rk-m)!}\quad ;\ rk\geq m,\ \text{ we get :}$$

(1) $\displaystyle\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}rk=xA_r'(nx)$;

(2) $\displaystyle\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}(rk)^2=x^2A_r''(nx)+xA_r'(nx)$;

(3) $\displaystyle\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}(rk)^3=x^3A_r'''(nx)+3x^2A_r''(nx)+x\,A_r'(nx)$;

(4) $\displaystyle\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}(rk)^m=x^mA_r^{(m)}(nx)+\frac{m(m-1)}{2}x^{m-1}A_r^{(m-1)}(nx)+$

terms of form $Cx^lA_r^{(l)}(nx)$, where $0<l<m-1$.

Definition 1:

A function f is of smaller order than function g as $x\to\infty$, if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$, $g(x)\neq 0$,we indicate this by writing $f=o(g)$.

Lemma 2: [6]

For $n,r\in N$ and $x\in(0,\infty)$, we get:

$\dfrac{A_r^{(m)}(nx)}{n^mA_r(nx)}\to 1$, $\dfrac{A_r^{(m)}(nx)}{n^sA_r(nx)}=o(1)$ as $n\to\infty, s=r+m, r\geq 1$. And $\dfrac{A_r^{(m)}(nx)}{n^sA_r(nx)}$ $=o(n^{-r})$.

Lemma 3: [6]

For each $r, n \in N$ and for all $x \in [0, \infty)$ we get:

1) $S_{n,r}(1; x) = 1$

2) $S_{n,r}\big((t-x); x\big) = S_{n,r}(t\,; x) - xS_{n,r}(1; x) = \dfrac{xA_r'(nx)}{nA_r(nx)} - x$

3) $S_{n,r}((t-x)^2; x) = S_{n,r}(t^2\,; x) - 2xS_{n,r}(t; x) + x^2 S_{n,r}(1; x)$

$$= \frac{x^2 A_r''(nx)}{n^2 A_r(nx)} + \frac{xA_r'(nx)}{n^2 A_r(nx)} - 2x\frac{xA_r'(nx)}{nA_r(nx)} - x^2$$

Definition 2:

The norm of the space $C_{\alpha,\gamma}[0,\infty)$ which is defined as: $\|f\|_{C_{\alpha,\gamma}} = \sup_{t\in[0,\infty)} |f(t)| e^{-ht}, \quad h > 0.$

Our first theorem shows that the operators $S_{n,m}(f; x, y)$ converges to the function $f(x,y)$ as $n, m \to \infty$.

Theorem 1:

Let $n, m \in N$. If for all $x, y \in [0, \infty)$ the following conditions hold:

(1) $\lim\limits_{n,m\to\infty} \left\|S_{n,m}(1; x, y) - 1\right\|_{C_{\alpha,\gamma}} = 0;$

(2) $\lim\limits_{n,m\to\infty} \left\|S_{n,m}(t; x, y) - x\right\|_{C_{\alpha,\gamma}} = 0;$

(3) $\lim\limits_{n,m\to\infty} \left\|S_{n,m}(u; x, y) - y\right\|_{C_{\alpha,\gamma}} = 0;$

(4) $\lim\limits_{n,m\to\infty} \left\|S_{n,m}(t^2 + u^2; x, y) - (x^2 + y^2)\right\|_{C_{\alpha,\gamma}} = 0,$

then

$$\left\|S_{n,m}(f(t,u); x, y) - f(x,y)\right\|_{C_{\alpha,\gamma}} = 0 \;\; as \;\; n, m \to \infty.$$

Proof:

By using Lemmas 1 and 2, we get:

1) $$\lim_{n,m\to\infty} \left\|S_{n,m}(1; x, y) - 1\right\|_{C_{\alpha,\gamma}} = \lim_{n,m\to\infty} \left\| \frac{1}{G_y} \sum_{j=0}^{\infty} q_{m,js}(y) \frac{1}{G_x} \sum_{k=0}^{\infty} q_{n,kr}(x) - 1 \right\|_{C_{\alpha,\gamma}}$$

$$= 0\,.$$

2) $\lim\limits_{n,m\to\infty} \left\|S_{n,m}(t; x, y) - x\right\|_{C_{\alpha,\gamma}}$

$$= \lim_{n,m\to\infty} \left\| \frac{1}{G_x}\sum_{k=0}^{\infty} q_{n,kr}(x)\left(\frac{kr}{n}\right)\frac{1}{G_y}\sum_{j=0}^{\infty} q_{m,js}(y) - x \right\|_{C_{\alpha,\gamma}} = \lim_{n,m\to\infty} \left\| \frac{xA'_r(nx)}{nA_r(nx)} - x \right\|_{C_{\alpha,\gamma}}$$

$$= 0 .$$

By the same way we can prove (3), as follows:

3) $\lim\limits_{n,m\to\infty} \left\| S_{n,m}(u; x, y) - y \right\|_{C_{\alpha,\gamma}}$

$$= \lim_{n,m\to\infty} \left\| \frac{1}{G_x}\sum_{k=0}^{\infty} q_{n,kr}(x)\frac{1}{G_y}\sum_{j=0}^{\infty} q_{m,js}(y)\left(\frac{js}{m}\right) - y \right\|_{C_{\alpha,\gamma}}$$

$$= \lim_{n,m\to\infty} \left\| \frac{yA'_s(my)}{mA_s(my)} - y \right\|_{C_{\alpha,\gamma}} = 0$$

(4) $\lim\limits_{n,m\to\infty} \left\| S_{n,m}\big((t^2 + u^2); x, y\big) - (x^2 + y^2) \right\|_{C_{\alpha,\gamma}}$

$$= \lim_{n,m\to\infty} \left\| \left(\frac{1}{G_x}\sum_{k=0}^{\infty} q_{n,kr}(x)\left(\frac{kr}{n}\right)^2 \frac{1}{G_y}\sum_{j=0}^{\infty} q_{m,js}(y) + \frac{1}{G_x}\sum_{k=0}^{\infty} q_{n,kr}(x)\frac{1}{G_y}\sum_{j=0}^{\infty} q_{m,js}(y)\left(\frac{js}{m}\right)^2 \right) - (x^2 + y^2) \right\|_{C_{\alpha,\gamma}}$$

$$= \lim_{n,m\to\infty} \left\| \left(\frac{1}{G_x}\sum_{k=0}^{\infty} q_{n,kr}(x)\left(\frac{kr}{n}\right)^2 + \frac{1}{G_y}\sum_{j=0}^{\infty} q_{m,js}(y)\left(\frac{js}{m}\right)^2 \right) - (x^2 + y^2) \right\|_{C_{\alpha,\gamma}}$$

$$= \lim_{n,m\to\infty} \left\| \frac{x^2A''_r(nx)}{n^2A_r(nx)} + \frac{xA'_r(nx)}{n^2A_r(nx)} + \frac{y^2A''_s(my)}{m^2A_s(my)} + \frac{yA'_s(my)}{m^2A_s(my)} - (x^2 + y^2) \right\|_{C_{\alpha,\gamma}}$$

$$= 0 .$$

Then, by Korovkin's theorem [4] we get:

$\left\| S_{n,m}(f(t,u); x, y) - f(x,y) \right\|_{C_{\alpha,\gamma}} = 0 \qquad as \;\; n,m \to \infty.$ ■

Finally, we give a Voronovskaja-type asymptotic formula for the operators $S_{n,m}(f(t,u);x,y)$.

Theorem 2:

For $f \in C_\alpha(0,\infty) \times C_\gamma(0,\infty)$ such that, $\alpha,\gamma \in N^0$, suppose that $\frac{\partial^2 f(x,y)}{\partial^2 x}, \frac{\partial^2 f(x,y)}{\partial^2 y}$ and $\frac{\partial^2 f(x,y)}{\partial x \partial y}$ exist and are continuous at a point $(x,y) \in ((0,\infty) \times (0,\infty))$, then:

$$\lim_{n\to\infty} n\{S_{n,m}(f;x,y) - f(x,y)\} = \frac{x}{2} f''_{xx}(x,y) + \frac{y}{2} f''_{yy}(x,y)$$

Proof:

By Taylor's formula for $f \in C_\alpha(0,\infty) \times C_\gamma(0,\infty)$, about the point (x,y), we have:

$$\begin{aligned} f(t,u) &= f(x,y) + f'_x(x,y)(t-x) + f'_y(x,y)(u-y) \\ &\quad + \frac{1}{2}\{f''_{xx}(x,y)(t-x)^2 + 2f''_{xy}(t-x)(u-y) + f''_{yy}(x,y)(u-y)^2\} \\ &\quad + \varphi(t,u;x,y)\sqrt{(t-x)^4 + (u-y)^4}, \end{aligned}$$

where $\varphi(t,u) = \varphi(t,u;x,y)$ is function from the space $C_\alpha(0,\infty) \times C_\gamma(0,\infty)$, and $\varphi(t,u) \to (0,0)$ as $(t,u) \to (x,y)$ then, $\varphi(x,y) = 0$.

$$\begin{aligned} S_{n,m}\big(f(t,u);(x,y)\big) &= f(x,y) + \acute{f}_x(x,y) S_{n,n}\big((t-x);x\big) \\ &\quad + f'_y(x,y) S_{n,m}\big((u-y);y\big) + \frac{1}{2} f''_{xx}(x,y) S_{n,m}((t-x)^2;x) \\ &\quad + f''_{xy}(x,y) S_{n,m}\big((t-x);x\big) S_{n,m}\big((u-y);y\big) + \frac{1}{2} f''_{yy}(x,y) S_{n,m}((u-y)^2;y) \\ &\quad + S_{n,m}\left(\varphi(t,u)\sqrt{(t-x)^4 + (u-y)^4};x,y\right). \end{aligned}$$

Using Lemma 3, we have:

$$\begin{aligned} S_{n,m}(f;x,y) - f(x,y) &= f'_x\left(\frac{xA'_r(nx)}{nA_r(nx)} - x\right) + f'_y\left(\frac{yA'_s(nx)}{nA_s(nx)} - y\right) \\ &\quad + \frac{1}{2} f''_{xx}\left(\frac{x^2 A''_r(nx)}{n^2 A_r(nx)} + \frac{xA'_r(nx)}{n^2 A_r(nx)} - \frac{2x^2 A'_r(nx)}{nA_r(nx)} + x^2\right) \end{aligned}$$

$$\begin{aligned} &+ f''_{xy}\left(\frac{xA'_r(nx)}{nA_r(nx)} - x\right)\left(\frac{yA'_s(nx)}{nA_s(nx)} - y\right) \\ &\quad + \frac{1}{2} f''_{yy}\left(\frac{y^2 A''_s(ny)}{n^2 A_s(ny)} + \frac{yA'_r(ny)}{n^2 A_s(ny)} - \frac{2y^2 A'_r(ny)}{nA_s(ny)} + y^2\right) \end{aligned}$$

$$+S_{n,m}\left(\varphi(t,u)\sqrt{(t-x)^4+(u-y)^4};x,y\right).$$

Hence,

$$\lim_{n\to\infty} n\{S_{n,m}(f;x,y)-f(x,y)\} = \frac{x}{2}f''_{xx}(x,y)+\frac{y}{2}f''_{yy}(x,y).$$

To complete the proof, we must show that the term

$S_{n,m}\big(\varphi(t,u)\sqrt{(t-x)^4+(u-y)^4};x,y\big)\to 0 \ as \ n\to\infty.$

By using Cauchy-Schwartz inequality, we get:

$$\left|S_{n,m}\left(\varphi(t,u)\sqrt{(t-x)^4+(u-y)^4};x,y\right)\right|$$
$$\leq \left(S_{n,m}(\varphi^2(t,u);x,y)\right)^{1/2}\Big(S_{n,m}((t-x)^4;x)$$
$$+S_{n,m}((u-y)^4;y)\Big)^{1/2}.$$

Now, by the properties $\varphi(x,y)=0$, and Theorem 1, we have:

$S_{n,m}(\varphi^2(t,u);x,y)=\varphi^2(x,y)=0$.

Then from Lemma 3, we get:

$$\lim_{n\to\infty}\left(S_{n,m}((t-x)^4;x)+S_{n,m}((u-y)^4;y)\right)^{1/2}\to 0.$$

Therefore,

$$\lim_{n\to\infty} nS_{n,m}\left(\varphi(t,u)\sqrt{(t-x)^4+(u-y)^4};x,y\right)\to 0 \ .$$

Then, the proof of the Theorem 2 is over.

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الخلاصة

في هذا البحث, نقدم تحسين للمؤثر من النمط مجموع Szãsz ونرمز له $S_{n,m}(f;x,y)$, حيث أن الدالة $f \in C_{\alpha,\gamma}([0,\infty] \times [0,\infty])$ والمتغيرين $x, y \in [0,\infty)$ يكونان مستقلان. أولا, نناقش تقارب المؤثر إلى الدالة $f(t,u) \in C_{\alpha,\gamma}$ وبعد ذلك نثبت صيغة ڤرونوڤسكي للتقارب (Voronovskaja-type asymptotic formula) للمؤثر $S_{n,m}(f;x,y)$.