SOLVING THE NON-HOMOGENOUS SECOND ORDER (LPDEs) USING (L.T)

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ABSTRACT

Our aim in this paper is to used Laplace transformations (L.T) for solving the non-homogenous second order linear partial differential equations (LPDEs) with constant coefficients without using any initial conditions(I.C) and boundary conditions(B.C).

1.INTRODUCTION

The operator D ,denotes partial derivative of some (PDEs) with respect to independent variables and it is important to obtain solutions for (LPDEs).

The Laplace transformation is used to solve linear partial differential equations with real constant coefficients with initial and boundary conditions by taking (L.T) of both sides of linear partial differential equations and by substituting (L.T) for partial derivatives and writing initial and boundary conditions. After this we getting linear ordinary differential equations its order is the same order of linear partial differential equations, and by taking inverse (L.T) of both sides, after writing (L.T) for general solution, we obtaining the solution of linear partial differential equations.

In this paper we used (L.T) for solving the second order linear partial differential equation with non-homogenous and with constant coefficients with out using any initial and boundary conditions.

2.BASIC DEFINITIONS, THEOREMS AND PROBERTES

Definition 2.1,[3]:- A partial differential equation is an equation that contains partial derivatives. In contrast to ordinary differential equation, where unknown function depends only on one variable in partial differential equations; the unknown function depends on several variables.

*Definition 2.2,[2],[4]:-*A second order linear partial differential equation in the independent variable x and y is an equation of the form

$$A_1(x,t)Z_{xx} + A_2(x,t)Z_{xt} + A_3(x,t)Z_{tt} + A_4(x,t)Z_x + A_5(x,t)Z_t + A_6 Z = F(x,t) \quad \text{... (1}$$

where $A_i(x,t), i=1,2,...,6$ are functions of x and t or constants. If $F(x,t)=0$ then the equation (1) is said to be homogenous and non-homogenous if $F(x,t)\neq 0$. If

$A_i(x,t), i=1,...,6$ are constant functions then the equation (1) is called with constant coefficient, and if at lest one of $A_i(x,t)$, $i=1,2,...,6$ are non-constants then the equation (1) is called with variable coefficients. And if $A_4 = A_5 = A_6 = 0$ then the equation(1) is said to be linear partial differential equation with homogenous terms.

Definition 2.3,[2],[5]:- To obtains the (L.T) of a function $Z(x,t)$,we multiply the function $Z(x,t)$ by e^{-st} and integrate the product $e^{-st} Z(x,t)$ with respect to the (t) between $t=0$ and $t=\infty$,i.e. $L(Z(x,t)) = \int_0^\infty e^{-st} Z(x,t)dt$

the constant parameter (s) ,is assumed to be positive and larger enough to make the product $e^{-st} Z(x,t)$ converges to zero as. $t \to \infty$

Note that(1),is a definite integral with limits to be substituted for (t),

so that the resulting expression will not contain (t) but will be expressed in terms of (s) only ,i.e. $L(Z(x,t)) = \int_0^\infty e^{-st} Z(x,t)dt = v(x,s)$

Definition 2.4,[2]:- Let $v(x,s)$ represents the(LT) of the function $Z(x,t)$,then $Z(x,t)$ is said to be the inverse (L.T) and is denoted by $Z(x,t) = L^{-1}(v(x,s))$ when we know the (L.T) containing fraction functions of (s) then the inverse (L.T) can be obtained directly or written it in partial fractions.

Theorem 2.5,[1]:-Let $Z(x,t)$ be a continuous function such that $t>0$ then

1) $L(Z_t) = sv(x,s) - Z(x,0)$

2) $L(Z_{tt}) = s^2 v(x,s) - s\,Z(x,0) - Z_t(x,0)$

3) $L(Z_x) = \dfrac{d}{dx} v(x,s)$

4) $L(Z_{xx}) = \dfrac{d^2}{dx^2} v(x,s)$

Property 2.6:- :-Let $Z(x,t)$ be a continuous function such that $t>0$ then

$$L(Z_{xt}) = s\,v'(x,t) - Z_x(x,0)$$

Proof:-

by using the definition (2.3),we get

$$L(Z_{xt}) = \int_0^\infty e^{-st} Z_{xt} dt$$

$$= e^{-st} Z_x(x,t) \Big|_0^\infty + s\int_0^\infty e^{-st} Z_x dt$$

,hence $= -Z_x(x,0) + s\, v'$

$L(Z_{xt}) = s\, v'\,(x,t) - Z_x(x,0)$

3.NEW MAIN RESULT FOR SOLVING THE SECOND ORDER (LPDEs)

The non-homogenous second order linear partial differential equation with constant coefficients its general form:

$$A_1(x,t)Z_{xx} + A_2(x,t)Z_{xt} + A_3(x,t)Z_{tt} + A_4(x,t)Z_x + A_t(x,t)Z_t + A_6 Z = F(x,t) \quad \ldots (2$$

for solving the equation (2) by using (L.T),taking the (L.T)of both sides of the equation (2) without using any (I.C) and (B.C),i.e. $Z\,(x,0), Z_t(x,0), Z_x(x,0)$ are known and the (L.T) of $F(x,t)$ is known we get:

$$v(x,s) = \frac{D_1(s)}{(A_3 s^2 + A_5 s + A_6)} + \frac{D_2(x,s)}{(A_3 s^2 + A_5 s + A_6) \bullet K(s)}$$

Let $(A_3 s^2 + A_5 s + A_6) = H(s)$,then the above equation becomes

$$v(x,s) = \frac{D_1(s)}{H(s)} + \frac{D_2(x,s)}{H(s) \bullet k(s)} \quad \ldots (3$$

Where $D_1(s)$ is a polynomial of (s) it's degree smaller than the degree of $H(s)$,and $K(s)$
is a polynomial of (s) represents denominator of (L.T) of the function $F(x,t)$,and $D_2(x,s)$ represent numerator of (L.T) of the function $F(x,t)$.

Now, since $L(Z(x,t)) = v(x,s)$,then the equation(3) becomes:-

$$L(Z(x,t)) = \frac{D_1(s)}{H(s)} + \frac{D_2(x,s)}{H(s) \bullet k(s)}$$

and by taking $(L^{-1})$of both sides of the above equation, then we get:-

$$Z(x,t)=L^{-1}\left(\frac{D_1(s)}{H(s)}\right)+L^{-1}\left(\frac{D_2(x,s)}{H(s)\bullet K(s)}\right) \qquad \text{... (4}$$

Since, $D_2(x,s)$ represent numerator of (L.T) of the function $F(x,t)$ hence $D_2(x,s)=F_1(x)F_2(s)$,then the above equation becomes:-

$$Z(x,t)=L^{-1}\left(\frac{D_1(s)}{H(s)}\right)+F_1(x)L^{-1}\left(\frac{F_2(s)}{H(s)\bullet K(s)}\right)$$

$F_2(s)$is a polynomial of (s)it's degree smaller than the degree of $(H(s)\bullet K(s))$.
Then the complete solution is given by

$$Z(x,t)=A_1g_1(t)+A_2g_2(t)+\dots+A_ng_n(t)+F_1(x)\left[B_1h_1(t)+B_2h_2(t)+\dots+B_mh_m(t)\right] \qquad \text{... (5}$$

Where $A_1,A_2,\dots,A_n$ and $B_1,B_2,\dots,B_m$ are constants, $g_1,g_2,\dots,g_n$ and $h_1,h_2,\dots,h_m$ are functions of (t).

The number of the constants $A_i, i=1,\dots,n$ and the number of the functions $g_i, i=1,\dots,n$ are equal to the degree of $H(s)$which is supposed to be (n),and the number of the constants $B_i, i=1,\dots,m$,and the number of the functions $h_i, i=1,\dots,m$ are equal to the degree of $(H(s)\bullet K(s))$ which is supposed to be (m).

We can eliminate some of these constants $A_1,A_2,\dots,A_n$ and $B_1,B_2,\dots,B_m$ by obtaining whose values by finding Z_{xx},Z_{xt},Z_{tt},Z_x and Z_y for the equation(4),and substituting this values in the equation (2),so we get some of these constants, and by this method we get the solution of the equation (2) without using any initial and boundary conditions for the second order linear partial differential equation by using (L.T).

4.Examples:

Example 4.1:-to solve the equation $Z_{tt}-Z_{xx}-2Z_t=e^x\sin t$

$L(e^x\sin t)=e^xL(\sin t)=\dfrac{e^x}{s^2+1}$,hence

$$Z(x,t)=L^{-1}\left(\frac{D_1(s)}{s^2-2s}\right)+e^xL^{-1}\left(\frac{F_2(s)}{(s^2-2s)(s^2+1)}\right)$$

$$=L^{-1}\left(\frac{A}{s}\right)+L^{-1}\left(\frac{B}{s-2}\right)+e^x\left[L^{-1}\left(\frac{c}{s}\right)+L^{-1}\left(\frac{D}{s-2}\right)+L^{-1}\left(\frac{Es+F}{s^2+1}\right)\right]$$

hence

$$Z(x,t) = A + Be^{2t} + ce^{x} + De^{x}e^{2t} + Ee^{x}\cos t + Fe^{x}\sin t$$

we can get some eliminate of the constant by finding

$$,Z_{tt} = 4Be^{2t} + 4De^{x}e^{2t} - Ee^{x}\cos t - Fe^{x}\sin t$$

$$Z_{t} = 2Be^{2t} + 2De^{x}e^{2t} - Ee^{x}\sin t + Fe^{x}\cos t$$

and $Z_{xx} = ce^{x} + De^{x} + Ee^{x}\cos t + Fe^{x}\sin t$

and by substituted Z_t, Z_{tt}, Z_{xx} in the original equation then we get

$$-2Ee^{x}\cos t - 2Fe^{x}\sin t - ce^{x} - De^{x}e^{2t} - +2Ee^{x}\sin t - 2Fe^{x}\cos t = e^{x}\sin t$$

hence $\quad -2E - 2F = 0, -2F + 2E = 1, c = 0, D = 0$

by solving these equations we get $c = 0, D = 0, F = \frac{-1}{4}, E = \frac{1}{4}$

Then the complete solution is given by

$$Z(x,t) = A + Be^{2t} + \frac{1}{4}e^{x}\cos t - \frac{1}{4}e^{x}\sin t$$

where A and B are arbitrary constants.

<u>*Example 4.2*</u>:-to solve the equation $Z_{tt} + Z_{x} - Z = \cos x e^{t}$

$$L(\cos x e^{t}) = \cos x L(e^{t}) = \frac{\cos x}{s-1} \text{ ,hence}$$

$$Z(x,t) = L^{-1}\left(\frac{D_1(s)}{s^2-1}\right) + \cos x L^{-1}\left(\frac{F_2(s)}{(s^2-1)(s-1)}\right)$$

$$= L^{-1}\left(\frac{A}{s-1}\right) + L^{-1}\left(\frac{B}{s+1}\right) + \cos x\left[L^{-1}\left(\frac{c}{s-1}\right) + L^{-1}\left(\frac{D}{(s-1)^2}\right) + L^{-1}\left(\frac{E}{s+1}\right)\right]$$

hence

$$Z(x,t) = Ae^{t} + Be^{-t} + c\cos x e^{t} + D\cos x e^{t} t + E\cos x e^{-t}$$

we can get some eliminate of the constant by finding

$$,Z_{tt} = Ae^{t} + Be^{-t} + c\cos x e^{t} + 2D\cos x e^{t} + D\cos x e^{t} t$$

$$Z_{x} = -c\sin x e^{t} - D\sin x e^{t} t - E\sin x e^{-t}$$

and by substituted $Z_{tt}, Z_{x} Z$ in the original equation then we get

$$2D\cos x e^{t} - c\sin x e^{t} - D\sin x e^{t} t - E\sin x e^{-t} = \cos x e^{t}$$

hence $\quad 2D = 1, -c = 0, -E = 0, -D\sin x e^{t} t = 0$

by solving these equations we get $D=\frac{1}{2}, c=0, E=0, \frac{-1}{2}\sin x e^{t} t=0$

Then the complete solution is given by

$$Z(x,t)=Ae^{t}+Be^{-t}+\frac{1}{2}\cos x e^{t} t$$

where A and B are arbitrary constants.

Example 4.3-to solve the equation $Z_{xt}+Z_{xx}-Z_{x}+Z_{t}=t\cos t$

$L(t\cos t)=\frac{s}{(s^2+1)^2}$,hence

$$Z(x,t)=L^{-1}\left(\frac{D_1(s)}{s}\right)+L^{-1}\left(\frac{F_2(s)}{s\cdot(s^2+1)^2}\right)$$

$$=L^{-1}\left(\frac{A}{s}\right)+L^{-1}\left(\frac{B}{s}\right)+\left[L^{-1}\left(\frac{cs+D}{s^2+1}\right)+L^{-1}\left(\frac{Es+F}{(S^2+1)^2}\right)\right]$$

hence

$$Z(x,t)=A+B+c\cos t+D\sin t+Et\cos t+Ft\sin t$$

we can get some eliminate of the constant by finding

$Z_x=0, Z_{xx}=0, Z_{xt}=0, Z_t=-c\sin t+D\cos t-Et\sin t+E\cos t+Ft\cos t+F\sin t$

and by substituted Z_x, Z_{xx}, Z_{xt}, Z_t in the original equation then we get

$-c\sin t+D\cos t-Et\sin t+E\cos t+Ft\cos t+F\sin t=t\cos t$ hence

$-c+F=0, D+E=0, -E=0, F=1$

by solving these equations we get $F=1, c=1, E=0, D=0$

Then the complete solution is given by

$$Z(x,t)=A+\cos t+t\sin t$$

where A is an arbitrary constant.

Example 4.4:-to solve the equation $Z_{xx}+Z_{xt}+Z_{tt}+Z_{x}-2Z_{t}+Z=xt$

$L(xt)=xL(t)=\frac{x}{s^2}$,hence

$$Z(x,t)=L^{-1}\left(\frac{D_1(s)}{s^2-2s+1}\right)+xL^{-1}\left(\frac{F_2(s)}{(s^2-2s+1)s^2}\right)$$

$$= L^{-1}\left(\frac{A}{s-1}\right) + L^{-1}\left(\frac{B}{(s-1)^2}\right) + x\left[L^{-1}\left(\frac{c}{s}\right) + L^{-1}\left(\frac{D}{s^2}\right) + L^{-1}\left(\frac{E}{s-1}\right) + L^{-1}\left(\frac{F}{(s-1)^2}\right)\right]$$

hence

$$Z(x,t) = Ae^{t} + Bte^{t} + cx + Dxt + Exe^{t} + Fxte^{t}$$

we can get some eliminate of the constant by finding

$$Z_{xx} = 0$$

$$Z_{t} = Ae^{t} + Bte^{t} + Be^{t} + Dx + Exe^{t} + Fxte^{t} + Fxe^{t} \text{، } Z_{x} = c + Dt + Ee^{t} + Fte^{t} \text{،}$$

$$Z_{tt} = Ae^{t} + Bte^{t} + 2Be^{t} + Exe^{t} + Fxte^{t} + 2Fxe^{t} \; Z_{xt} = De^{x} + Ee^{t} + Fe^{t} + Fte^{t} \text{،}$$

and by substituted $Z_t, Z_{tt}, Z_x, Z_{xt}, Z_{xx}, Z$ in the original equation then we get

$$D + Fe^{t} - 2Dx + c + Dt + 2Ee^{t} + 2Fte^{t} + cx + Dxt = xt$$

hence $\quad 2E + F = 0, 2F = 0, D + c = 0, (-2D + c)x = 0, Dt = 0, D = 1$

by solving these equations we get $D = 1, t = 0, -3x = 0, c = -1, F = 0, E = 0$

Then the complete solution is given by

$$Z(x,t) = Ae^{t} + Bte^{t} - (1-t)x$$

where A and B are arbitrary constants.

5.Conclusion:

We can find the complete solution of the non-homogenous second order (LPDEs) by Laplace transformation without using any initial and boundary conditions. If we use initial and boundary conditions then we get easy solutions, and this method is illustrated in the under graduated text book. Thus the complete solution of the non-homogenous second order (LPDEs) contains arbitrary constants.

There is not any relation between the arbitrary constants and the initial and boundary conditions because there are no initial and boundary conditions

There for we can solve the non-homogenous second order (LPDEs) in this paper by (L.T) without using initial and boundary conditions, such the solution of non-homogenous second order (LPDEs) in the dependent variable u and an independent variables x and t can be found by the proposition:

$$Z(x,t) = L^{-1}\left(\frac{D_1(s)}{H(s)}\right) + F_1(x) L^{-1}\left(\frac{F_2(s)}{H(s) \bullet K(s)}\right),$$

hence the complete solution is given by

$$Z(x,t) = A_1 g_1(t) + A_2 g_2(t) + \dots + A_n g_n(t) + F_1(x)\left[B_1 h_1(t) + B_2 h_2(t) + \dots + B_m h_m(t)\right]$$

where $A_1, A_2, A_n, B_1, ..., B_n$ are arbitrary real constants have not relation with initial and boundary conditions.

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<u>الخلاصة العربية</u>:-

هدفنا في هذا البحث استخدام تحويلات لابلاس لحل المعادلة التفاضلية الجزئية الخطية من الرتبة الثانية الغير متجانسة وذات المعاملات الثابتة بدون استخدام أي شروط ابتدائية وحدودية.