# Modified Decomposition Method for Solving three- Dimensional Microscale Heat Equation

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**Abstract**

In this paper, numerical solutions of the three –dimensional microscale heat equation are obtained by using Modified decomposition method(MDM). We compare the results with solution of two methods for this equation the efficiency and power of the technique are shown for wide classes of equations of mathematical., our results show this method is powerful and efficient for solve three –dimensional microscale heat transport equation .

**Keywords**:-Modified decomposition method (MDM),Three dimensional microscale heat equation, Absolute error.

## 1.Introduction:-

The three –dimension microscl heat transport equation ,which describes the thermal behavior of thin films and other microstructures ,can be written [1]

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial t}+T_q\frac{\partial^2 T}{\partial t^2}\right)=T_q\frac{\partial^3 T}{\partial t\partial x^2}+T_q\frac{\partial^3 T}{\partial t\partial y^2}+T_t\frac{\partial^3 T}{\partial t\partial z^2}+\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+s, \qquad (1)$$

With the initial and boundary conditions :

$$T(x,y,z,0)=T_0(x,y,z), \qquad \frac{\partial T(x,y,z,0)}{dt}=T_1(x,y,z),$$
$$T(0,y,z,t)=T_2(y,z,t), \qquad T(l_x,y,z,t)=T_5(y,z,t),$$
$$T(x,0,z,t)=T_3(x,z,t), \qquad T(x,l_y,z,t)=T_6(x,z,t),$$
$$T(x,y,0,t)=T_4(x,y,t), \qquad T(x,y,l_z,t)=T_7(x,y,t),$$

Where T is the temperature, α,$T_t$,and $T_q$ are positive constants. Here α is the thermal diffusivity. $T_t$,and $T_q$ represent the time lags of the heat flux and the temperature gradient, respectively [2],s represent internal heat source.
Many application, including phonon electron iteration model[3], the single energy equation[4,5] the phonon scatting model [6],the phonon radiative transfer model[7] and the lagging behavior model [4,8,9], can by the microscale heat transport equation .

In recent years, the modified Adomain decomposition method has been applied to wide classes of stochastic and deterministic problems in many interesting mathematical and physics are [10,11]. For linear PDEs, this method is similar to the homotopy or imbedding method ,[12].this method provides analytical, verifiable, and rapidly convergent approximations which yield insight into the character and behavior of the solution just as the closed –form solution. few author deal with numerical solution of one –dimensional microscal heat transport equation .by using crank-Nicolson technique, Qui and tien [13] have solved the phonon electron interaction model. josh and Majumdar[14] have used an explicit upstream difference method to solve the phonon radiative transfer model in one –dimensional.

In this paper, we introduce modified decomposition method to the three-dimensional microscale heat transport equation. Numerical results are compare with other methods[15,16] in the aspect accuracy and error ,our results show this method is more accurate and efficient than the other methods when the approximate solution is near to the exact solution, the numerical solution is obtained in case s=0.

## 2.Modified Decomposition Method:-

The modified decomposition method [17] may be used to solve the linear problem given by Eq.(1) subject to the initial conditions .
Now let Eq.(1) can be written as:-

$$\frac{\partial^2 T}{\partial t^2} = \alpha \frac{\partial^3 T}{\partial t \partial x^2} + \alpha \frac{\partial^3 T}{\partial t \partial y^2} + \alpha \frac{T_t}{T_q} \frac{\partial^3 T}{\partial t \partial z^2} + \frac{\alpha}{T_q}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) - \frac{1}{T_q}\frac{\partial T}{\partial t} \qquad (2)$$

Defining the differential operator:-

$$L_{tt} = \frac{\partial^2}{\partial t^2} \qquad (3)$$

We rewrite Eq.(2) in the operator form :-

$$L_{tt}T = \alpha \frac{\partial^3 T}{\partial t \partial x^2} + \alpha \frac{\partial^3 T}{\partial t \partial y^2} + \alpha \frac{T_t}{T_q} \frac{\partial^3 T}{\partial t \partial z^2} + \frac{\alpha}{T_q}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) - \frac{1}{T_q}\frac{\partial T}{\partial t} \qquad (4)$$

Appling the invers operator $L_{tt}^{-1}$:-

$$L_{tt}^{-1} = \iint_0^t (.)dtdt \qquad (5)$$

To Eq.(4) and using the initial condition we obtain:-

$$T(x,y,z,t) = L_{tt}^{-1}\left[\alpha \frac{\partial^3 T}{\partial t \partial x^2} + \alpha \frac{\partial^3 T}{\partial t \partial y^2} + \alpha \frac{T_t}{T_q} \frac{\partial^3 T}{\partial t \partial z^2} + \frac{\alpha}{T_q}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)\right] + \sum_{i=0}^{1} T_i(x,y,z)\frac{t^i}{i!} - \frac{1}{T_q} L_{tt}^{-1}\left(\frac{\partial T}{\partial t}\right) \qquad (6)$$

The unknown solution T is assumed to be given by series of the form :-

$$T = \sum_{n=0}^{\infty} a_n(x,y,z)t^n \qquad (7)$$

Substitution Eq.(7) into Eq.(6) we get :-

$$\sum_{n=0}^{\infty} a_n(x,y,z)t^n = L_{tt}^{-1}[\alpha \sum_{n=1}^{\infty} n a_n^{(2)}(x)t^{n-1} + \alpha \sum_{n=1}^{\infty} n a_n^{(2)}(y)t^{n-1} +$$

$$\alpha \frac{T_T}{T_q} \sum_{n=1}^{\infty} n a_n^{(2)}(z)t^{n-1} + \frac{\alpha}{T_q}\sum_{n=0}^{\infty}(a_n^{(2)}(x)t^n + a_n^{(2)}(y)t^n + a_n^{(2)}(z)t^n)] +$$

$$\sum_{i=0}^{1} T_i(x,y,z)\frac{t^i}{i!} - \frac{1}{T_q} L_{tt}^{-1}(\sum_{n=1}^{\infty} n a_n(x,y,z)t^{n-1}) \tag{8}$$

Or

$$\sum_{n=0}^{\infty} a_n(x,y,z)t^n = L_{tt}^{-1}[\alpha \sum_{n=0}^{\infty}(n+1)a_{n+1}^{(2)}(x)t^n + \alpha \sum_{n=0}^{\infty}(n+1)a_{n+1}^{2}(y)t^n +$$

$$\alpha \frac{T_T}{T_q}\sum_{n=0}^{\infty}(n+1)a_{n+1}^{(2)}(z)t^n + \frac{\alpha}{T_q}\sum_{n=0}^{\infty}(a_n^{(2)}(x)t^n + a_n^{(2)}(y)t^n + a_n^{2}(z)t^n) \;] +$$

$$\sum_{i=0}^{1} T_i(x,y,z)\frac{t^i}{i!} -$$

$$\frac{1}{T_q} L_{tt}^{-1}(\sum_{n=1}^{\infty}(n+1)a_{n+1}(x,y,z)t^n) \tag{9}$$

We now carry out the above integration to write the follow Eq. :-

$$\sum_{n=0}^{\infty} a_n(x,y,z)t^n =$$

$$\alpha \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+2)} a_{n+1}^{(2)}(x) + \alpha \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+2)} a_{n+1}^{2}(y) + \alpha \frac{T_T}{T_q}\sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+2)} a_{n+1}^{(2)}(z) +$$

$$\frac{\alpha}{T_q}\sum_{n=0}^{\infty}(\frac{t^{n+2}}{(n+1)(n+2)} a_n^{(2)}(x) + \frac{t^{n+2}}{(n+1)(n+2)} a_n^{(2)}(y) + \frac{t^{n+2}}{(n+1)(n+2)} a_n^{(2)}(z)) + +T_0 + T_1 t -$$

$$\frac{1}{T_q}\sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+2)} a_{n+1}(x,y,z) \tag{10}$$

In the summation  on the right, n can be replaced by n-2 to write :-

$$\sum_{n=0}^{\infty} a_n(x,y,z)t^n =$$

$$\alpha \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+2)} a_{n+1}^{(2)}(x) + \alpha \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+2)} a_{n+1}^{2}(y) + \alpha \frac{T_T}{T_q}\sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+2)} a_{n+1}^{(2)}(z) +$$

$$\frac{\alpha}{T_q}\sum_{n=2}^{\infty}(\frac{t^n}{(n-1)(n)} a_{n-2}^{(2)}(x) + \frac{t^n}{(n-1)(n)} a_{n-2}^{(2)}(y) + \frac{t^n}{(n-1)(n)} a_{n-2}^{(2)}(z)) + T_0 + T_1 t -$$

$$\frac{1}{T_q}\sum_{n=2}^{\infty} \frac{t^n}{n} a_{n-1}(x,y,z) \tag{11}$$

Finally ,we can equate coefficient of like powers of t on the left side and on the right side to obtain the recurrence relations for the coefficients Thus:-

$$a_0 = T_0$$

$$a_1 = T_1$$

$$a_n = \alpha \frac{a_{n-1}^{(2)}(x)}{n} + \alpha \frac{a_{n-1}^{(2)}(y)}{n} + \alpha \frac{a_{n-1}^{(2)}(z)}{n} + \frac{\alpha}{T_q}\left[\frac{a_{n-2}^{(2)}(x)}{n(n-1)} + \frac{a_{n-2}^{(2)}(y)}{n(n-1)} + \frac{a_{n-2}^{(2)}(z)}{n(n-1)}\right] -$$

$$\frac{1}{T_q}\frac{a_{n-1}(x,y,z)}{n} \qquad \text{where n} = 2,3,4,\ldots\ldots\ldots \tag{12}$$

The final solution is now given by $T(x,y,z,t) = \sum_{n=0}^{\infty} a_n(x,y,z)t^n$

**3-Numerical results:-**

**Example 1:-**

We consider Three-dimensional model problem to test the modified decomposition method (MDM) for solving Three-dimensional microscale heat equation with initial and boundary conditions satisfying the exact solution $T(x,y,z) = e^{x+y+z+t}$ $0 \leq x,y,z,t \leq 1$

Now assume problem (1) with $T_0 = e^{x+y+z}$ ,$T_1 = e^{x+y+z}$ and take α=3/8, $T_q = 2, T_t = 1$,

We solve this problem by modified decomposition method we get

$a_0 = T_0 = e^{x+y+z}$ , $a_1 = T_1 = e^{x+y+z}$, $a_2 = T_2 = \frac{e^{x+y+z}}{2!}$, $a_3 = T_3 = \frac{e^{x+y+z}}{3!}$, $\dots\dots\dots. a_n = T_n = \frac{e^{x+y+z}}{n!}$

The absolute error = $\frac{\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\sum_{k=1}^{N_z-1}|T_{exact}-T_{approxmate}|}{(N_x-1)(N_y-1)(N_z-1)}$

Where $N_x = \frac{1}{\Delta x}$, $N_y = \frac{1}{\Delta y}$, $N_z = \frac{1}{\Delta z}, \Delta x = \Delta y = \Delta z = 0.05$

The results of absolute error for $T_{approxmate}$ solution at t=1 , n=10

| X | Y | MDM | HFDM[15] | HPM[16] |
|---|---|---|---|---|
| 0.05 | 0.05 | 1.431e-10 | 3.40e-08 | 3.599e-08 |
| 0.1 | 0.1 | 1.581e-10 | 4.42e-08 | 6.327e-08 |
| 0.15 | 0.15 | 1.748e-10 | 5.88e-08 | 7.351e-08 |
| 0.20 | 0.20 | 1.931e-10 | 7.76e-08 | 8.540e-8 |
| 0.25 | 0.25 | 2.133e-10 | 1.00e-7 | 9.922e-08 |
| 0.30 | 0.30 | 2.359e-10 | 1.26e-07 | 1.153e-07 |
| 0.35 | 0.35 | 2.607e-10 | 1.54e-07 | 1.339e-07 |
| 0.40 | 0.40 | 2.881e-10 | 1.85e-07 | 1.556e-07 |
| 0.45 | 0.45 | 3.184e-10 | 2.15e-07 | 1.808e-07 |
| 0.50 | 0.50 | 3.519e-10 | 2.46e-07 | 2.10e-07 |
| 0.55 | 0.55 | 3.889e-10 | 2.76e-07 | 2.441e-07 |
| 0.60 | 0.60 | 4.299e-10 | 3.05e-07 | 2.836e-07 |
| 0.65 | 0.65 | 4.751e-10 | 3.32e-07 | 3.294e-07 |
| 0.70 | 0.70 | 5.250e-10 | 3.57e-07 | 3.828e-07 |
| 0.75 | 0.75 | 5.802e-10 | 3.81e-07 | 4.447e-07 |
| 0.80 | 0.80 | 6.413e-10 | 4.04e-07 | 5.167e-07 |

| | | | | |
|---|---|---|---|---|
| 0.85 | 0.85 | 7.087e-10 | 4.29e-07 | 6.003e-07 |
| 0.90 | 0.90 | 7.832e-10 | 4.59e-07 | 6.974e-07 |
| 0.95 | 0.95 | 8.656e-10 | 4.99e-07 | 8.103e-07 |

Table (1) comparison of the absolute error for (MDM, HFDM, HPM)

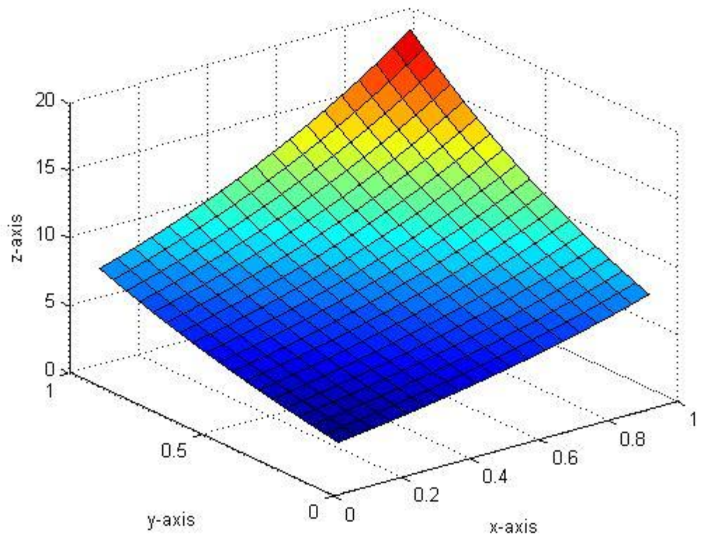


Fig1: Absolute error comparison between the (MDM,HFDM,HPM)

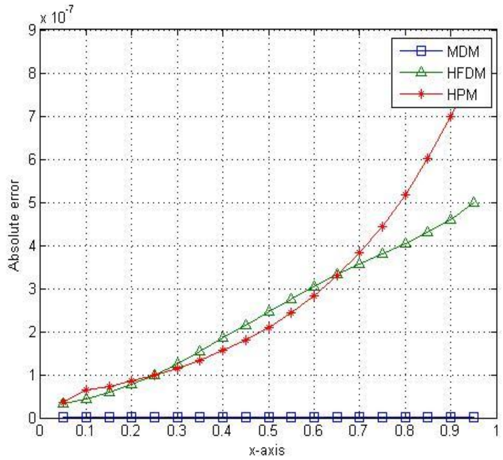


Fig2:The MDM solution when n=10

**Example 2**:-[18]

In this example, we consider the equation (1) for $\alpha = 1, T_q = \frac{1}{\pi^2} + 10^3, T_t = \frac{1}{\pi^2} - 1.99 * 10^{-5}$,
$0 \leq x, y \leq 0.1, 0 \leq z \leq 10^{-5}$, when the initial and boundary condition are

$$T(x,y,z,0) = \cos(10\pi x)\sin(10\pi y)\cos(10^5\pi z) = T_{0,}$$

$$\frac{\partial T}{\partial t}(x,y,z,0) = -\pi^2\cos(10\pi x)\sin(10\pi y)\cos(10^5\pi z) = T_1$$
$$T(0,y,z,t) = -T(0.1,y,z,t) = e^{-\pi^2 t}\cos(10^5\pi z)\sin(10\pi y),$$
$$T(x,0,z,t) = T(x,0.1,z,t) = 0,$$
$$T(x,y,0,t) = -T(x,y,10^{-5},t) = e^{-\pi^2 t}\cos(10\pi x)\sin(10\pi y),$$

The corresponding exact solution is:-

$$T(x,y,z,t) = e^{-\pi^2 t}\cos(10\pi x)\sin(10\pi y)\cos(10^5\pi z)$$

And mesh sizes $N_1 = N_2 = N_3 = 10, h_1 = h_2 = 0.01, h_3 = 0.000001$

We solve this problem by Modified decomposition method, we get

$$a_0 = T_0 = \cos(10\pi x)\sin(10\pi y)\cos(10^5\pi z)$$
$$a_1 = T_1 = -\pi^2\, cos(10\pi x)\, sin(10\pi y)\, cos(10^5\pi z)$$
$$a_2 = T_2 = \frac{\pi^4}{2} cos(10\pi x)\, sin(10\pi y)\, cos(10^5\pi z)$$
$$a_3 = T_3 = -\frac{\pi^6}{3!} cos(10\pi x)\, sin(10\pi y)\, cos(10^5\pi z)$$
$$a_n = T_n = \frac{(-\pi^2)^n}{n!} cos(10\pi x)\, sin(10\pi y)\, cos(10^5\pi z)$$

The results for approximation solution at t=1.5, n=50.

| x | Y | Z | MDM solution | Exact solution |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.01 | 0.01 | 0.000001 | 1.7261e-007 | 1.0397e-007 |
| 0.02 | 0.02 | 0.000002 | 2.3758e-007 | 1.4311e-007 |
| 0.03 | 0.03 | 0.000003 | 1.7261e-007 | 1.0397e-007 |
| 0.04 | 0.04 | 0.000004 | 5.6086e-008 | 3.3783e-008 |
| 0.05 | 0.05 | 0.000005 | 2.3155e-039 | 1.3947e-039 |
| 0.06 | 0.06 | 0.000006 | 5.6086e-008 | 3.3783e-008 |
| 0.07 | 0.07 | 0.000007 | 1.7261e-007 | 1.0397e-007 |
| 0.08 | 0.08 | 0.000008 | 2.3758e-007 | 1.4311e-007 |
| 0.09 | 0.09 | 0.000009 | 1.7261e-007 | 1.0397e-007 |
| 0.1 | 0.1 | 0.00001 | 7.5630e-023 | 4.5555e-023 |

Table (2) comparison between exact and MDM solutions

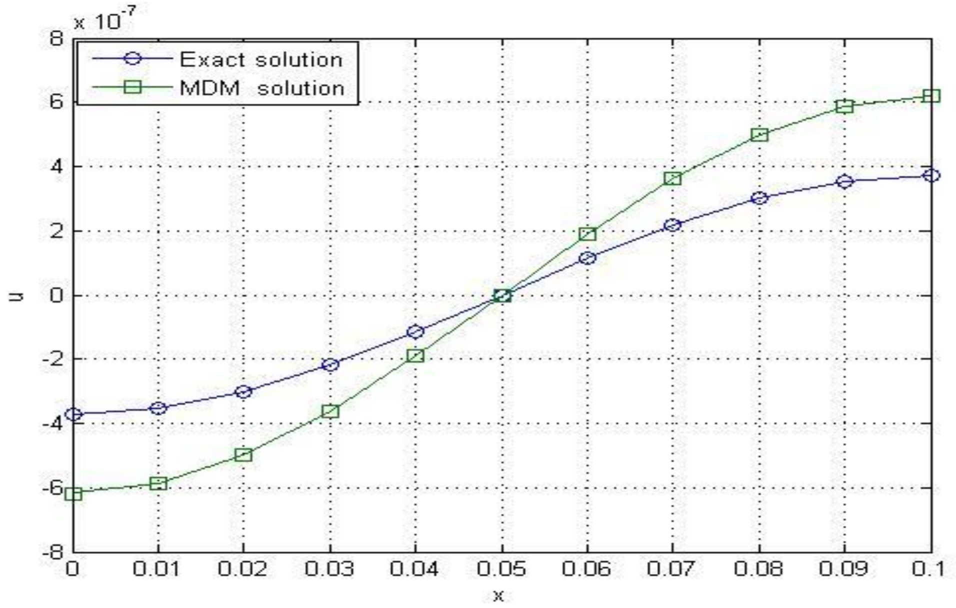


Fig (3) comparison between exact and MDM solutions for x at t=1.5,y=5/100 and z=5/100000.

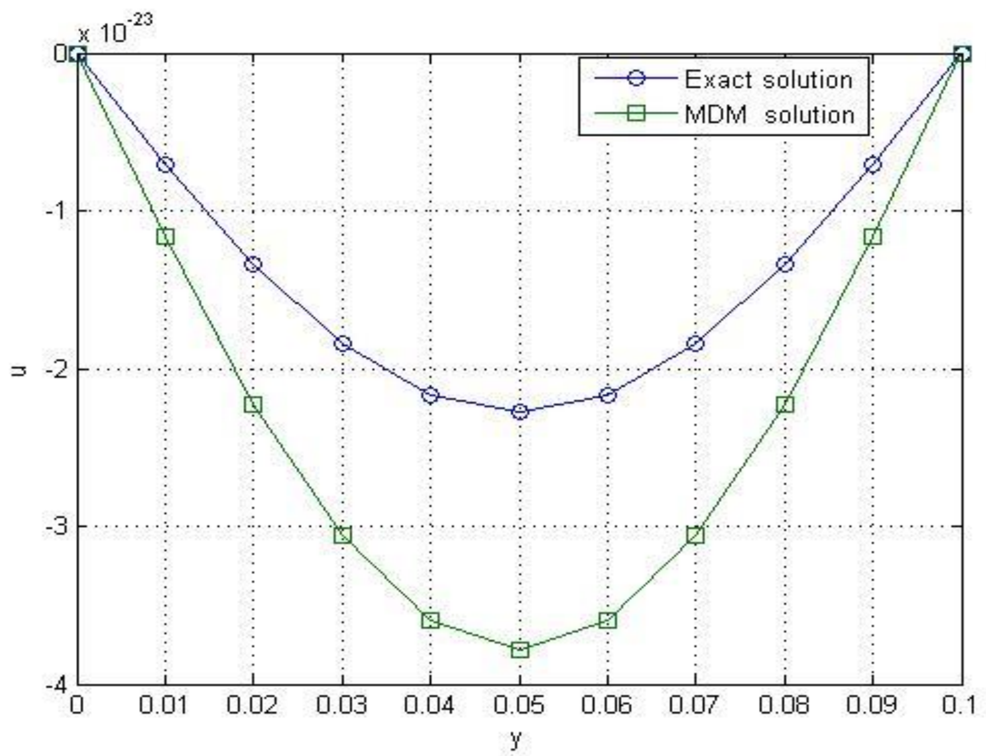


Fig (4) comparison between exact and MDM solutions for y at t=1.5,y=5/100 and z=5/100000.

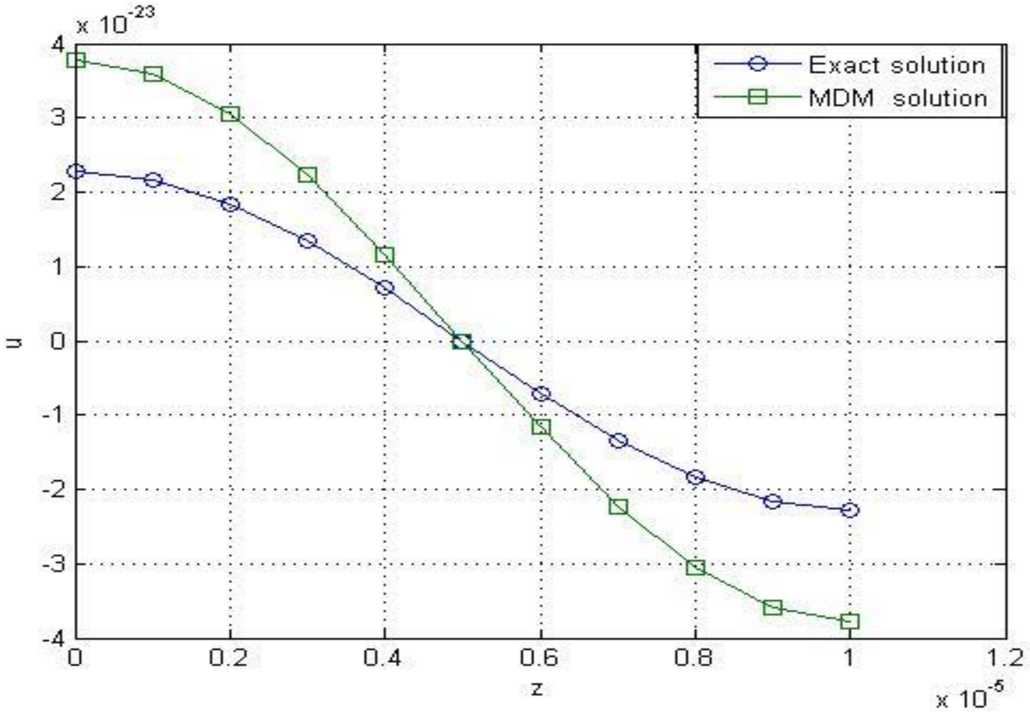


Fig (5) comparison between exact and MDM solutions for z at t=1.5,y=5/100 and z=5/100000.

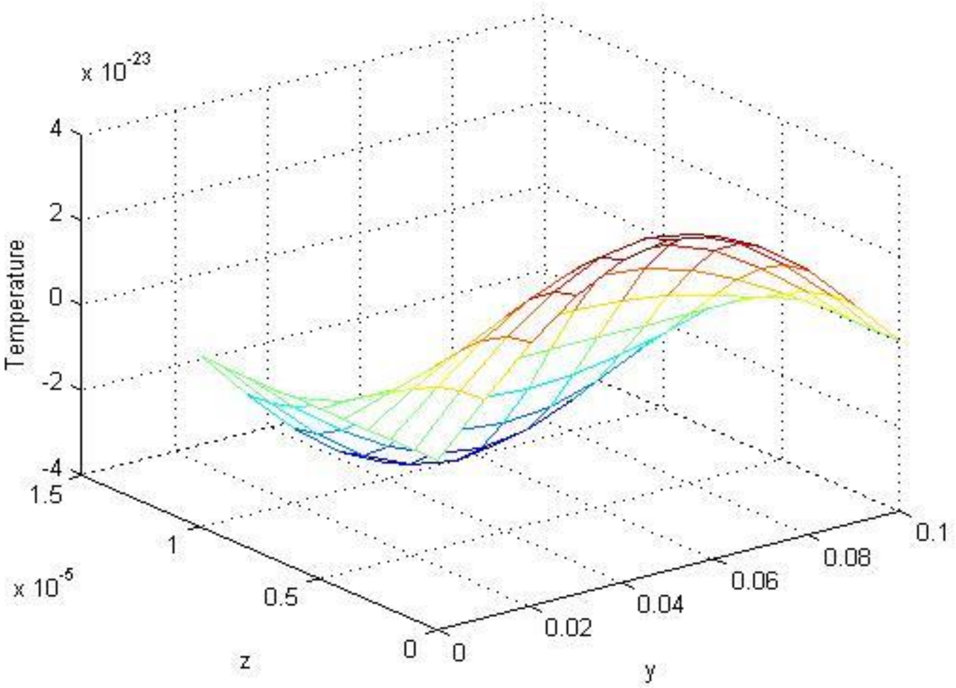


Fig (6) numerical temperature distribution at t=1.5 and x=5/100

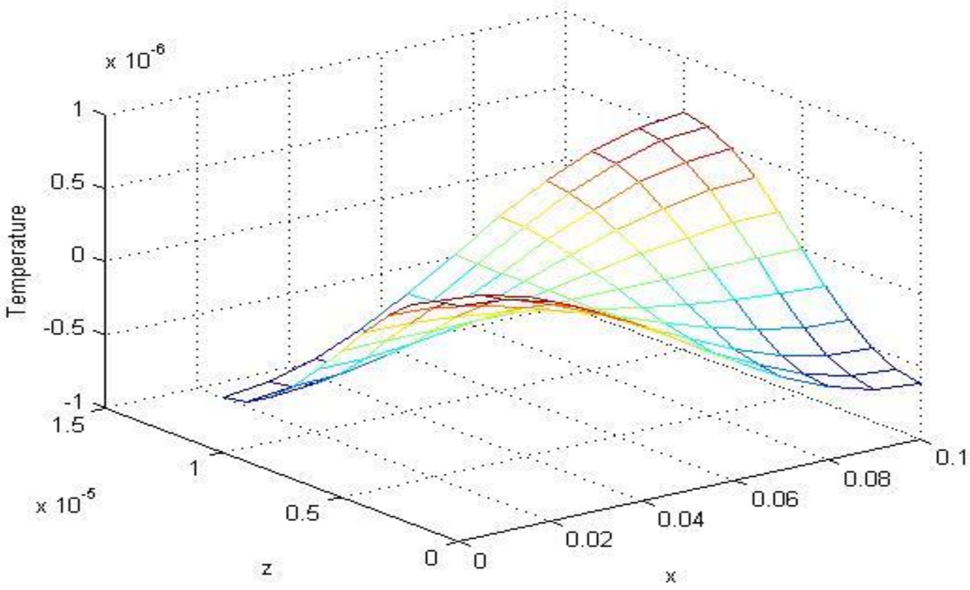


Fig (7) numerical temperature distribution at t=1.5 and y=5/100.

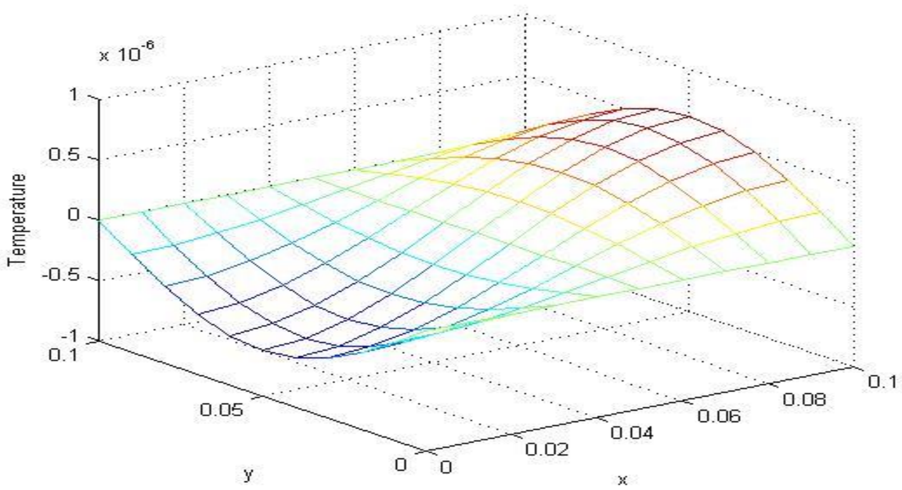


Fig (8) numerical temperature distribution at t=1.5 and z=5/100000.

**4-conclusions :-**

In this work, we calculated the approximation solution of Three-dimensional Microscale Heat equation by using modified decomposition method. The test ,the robustness ,accuracy and efficiency of this method is applied to example having analytical solution, our results exhibit good comparison with analytical solutions.

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## حل معادلة الحرارة الثلاثية الابعاد باستخدام طريقة التركيب المطورة

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**الخلاصة**

تناولنا في هذا البحث حل معادلة الحرارة الثلاثية الابعاد باستخدام طريقة التركيب المطورة حيث قمنا بمقارنة النتائج التي حصلنا عليها باستخدام هذه الطريقة مع نتائج بحثين اخرين مستخدمين طريقة الفروقات المحددة وطريقة الهموتوبي التكرارية واثبتت النتائج ان هذه الطريقة هي افضل من الطريقتين المذكورتين وبالتالي يمكن اعتبارها طريقة دقيقه وكفؤة وقريبه جدا من الحل الحقيقي.