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Solving high order of non-linear Volterra-fredholm Integro-differential equation by using bou-baker Polynomials approximation Method

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Abstract

In this paper, the Bou-baker polynomial method is used to evaluate an approximate so1ution initia1 value problem of high-order of non1inear Vo1terra-Fredholm Integro-Differential equation of the second kind. Three different examp1es and their graphics are displayed.

Keywords: Bou-baker, polynomial, Vo1terra - Fredholm.

1. Introduction

Many biological, social, technical and physical issues were characterized using integral and Integro-differential equations. Analytically, nonlinear integral and Integro-differential equations are employed, where precise solutions are difficult to acquire. Numerous numerical techniques have been investigated, including the differential transform (Behiry and Mohamed 2012) as well as a mechanization algorithm (Wang 2006).

Numerous authors have provided techniques for solving a nonlinear Integro-differential equation, for example, Deepa et al. (2000), Taylor polynomial solution (Maleknejad and Mahmoudi 2003), Numerical Solution by Approximate Methods in some mathematical models (Nasser and Hamid 2009), Chebyshev Polynomial (Behrooz and Mohammad 2013), Homotopy Perturbation Method, Rus et at. (2006). The nonlinear Volterra-Fredholm Differential equation of the second kind.

$$y(x) = f(x) + \lambda_1 \int_a^x k_1 (x,y)[y(t)]^r dt + \lambda_2 \int_a^b k_2 (x,t)[y(t)]^s dt \qquad (1)$$

where $k_1(x,t)$, $k_2(x,t)$ and $f(x)$ represent known functions, $\lambda_1, \lambda_2, a, b$ represent constant values, r, s are integers while $y(x)$ denotes an unknown function to be determined.

Therefore, the high-order non-linear Volterra-Fredholm Integro-differential equation of the second kind is given by (Behiry and Mohamed 2012; Wang, 2006):

$$\sum_{i=0}^{m} \mu_i(x)\left[y^{(i)}(x)\right] = f(x) + \int_a^x k(x,t)\,[y(t)]^r dt + \int_a^b k(x,t)\,[y(t)]^s dt \qquad (2)$$

having the following initial conditions given by $y(a)^i = y_i$, $i = 0,1,2,\dots\dots, m-1$,

In this research, we used the Bou-baker polynomials technique to propose the approximation method for solving the high-order nonlinear Volterra Fredholm Integro-differential equation of the second kind.

2. Bou-baker Polynomials Method

The Bou-baker polynomials of n degree are expressed as (Handan and Ayşegül 2006), (Biazar and Eslami 2010).

$$B_n(t) = \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} c_{n-p}^{p}\right] (-1)^p x^{n-2p}, \quad (3)$$

Where $\xi(n) = \left[\frac{n}{2}\right] = \frac{2n+((-1)^n-1)}{4}$.

Here, $\xi(n) = \left[\frac{n}{2}\right]$ resembles the floor function.

Moreover, the standard Bou-baker polynomials are expressed as follows:

$B_0(x) = 1$

$B_1(x) = x$

$B_2(x) = x^2 + 2$

$B_3(x) = x^3 + x$

$\vdots$

$B_m(x) = xB_{m-1}(x) - B_{m-2}(x)$ for $m > 2$

3. Bou-baker Polynomial's approximation Method

This section discusses Bou-baker polynomials approximation solution of the following form:

$y(x) = \sum_{n=0}^{N} c_n B_n(x), -\infty < x \leq b \leq \infty.$ (4)

Here, $B_n(x)$ $n = 0,1,2,\dots$ denotes the Bou-baker polynomials, a_n , $0 \leq n \leq N$ represents the unknown Bou-baker coefficients, while N represents some positive integers provided that $N \geq m$. We employ the collocation points described as following to obtain a numerical solution of eq. (4).

$$x_i = a + \frac{b-a}{N}\, i,$$

$$i = 0,1,2,\dots\dots, N. \qquad (5)$$

Substituting eq. (4) into eq. (2) gives

$$\sum_{i=0}^{m} \mu_i\,(x)\left[\sum_{n=0}^{N} c_n B_n(x)\right]^i = f(x) + \int_a^x k(x,t)\left[\sum_{n=0}^{N} c_n B_n(t)\right]^r dt + \int_a^b k(x,t)\left[\sum_{n=0}^{N} c_n B_n(t)\right]^s dt \qquad (6)$$

Eq. (6) can be written in a simpler form such that

$$\sum_{i=0}^{m} \mu_i(x)\,[c_0B_0(x) + c_1B_1(x) + c_2B_2(x) + c_3B_3(x) + \cdots]^i = f(x)$$

$$+ \int_a^x k(x,t)\,\big[[c_0B_0(t) + c_1B_1(t) + c_2B_2(t) + c_3B_3(t) + \cdots]\big]^r dt \qquad (7)$$

$$+ \int_a^b k(x,t)\,\big[[c_0B_0(t) + c_1B_1(t) + c_2B_2(t) + c_3B_3(t) + \cdots]\big]^s dt$$

$$\sum_{i=0}^{m} \mu_i\,(x)[c_0 + c_1 * x + c_2(x^2+2) + c_3(x^3+x) + \cdots]^i = = f(x)$$

$$+ \int_a^x k(x,t)\,\big[[c_0 + c_1 * x + c_2(x^2+2) + c_3(x^3+x) + \cdots]\big]^r dt \qquad (8)$$

$$\int_a^b k(x,t)\,\big[[c_0 + c_1 * x + c_2(x^2+2) + c_3(x^3+x) + \cdots]\big]^s dt$$

The right-hand side of eq (8) is integrated and simplified, resulting in the collocation points of eq. (5). The initial condition and collocation points resulted in a (N+1) linear algebraic equation with (N+1) unknown constants. The unknown constants are then inserted in eq. (4) once this is solved. I to obtain the numerical solution to eq. (2) with the help of the MATLAB program.

4. Examples and Results

The following examples of nonlinear high order Fredholm Integro-differential equations will be presented in this section. Let's have a look at the Fredholm Integro-differential equations once again. These examples were chosen from (Behiry and Mohamed 2012; Wang 2006).

Examp1e 1:

$$y^{(3)}(x) + y(x) = -\frac{x^5}{5} + \frac{2x^3}{3} + \frac{5x^2}{6} - \frac{113x}{105} - 1 + \int_0^x y^2\,(t)dt + \int_0^1 xt(x+t)y^2\,(t)dt$$

$0 \leq x \leq 1$

with respect to initial conditions

$y(0) = -1, y'(0) = 0$ and $y''(0) = 2$

having $y(x) = -1 + x^2$ as the exact solution.

Figure 1 and Table 1 compares between approximate and exact solutions for several values of N in Example 1.

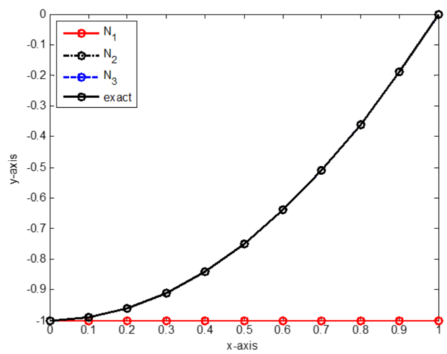

Figure 1: Comparison of the solutions of example 1.

Example 2:

$$x^4 y^{(6)}(x) + y^{(3)}(x) + y'(x) = -x^4\cos(x) + \frac{1}{2}\sin(2x) + 3x + 0.4 -0.1\, e^{\{[\cos(1)+\sin(1)]x[\cos^2(1)+3e]\}} -2\int_0^x [1 + y^2(t)]\,dt + \int_0^1 e^t y^3(t)dt\,, \quad 0 \leq x \leq 1$$

having initial condition

$y(0) = 1, y'(0) = 0\,, y''(0) = -1$

$y'''(0) = 0\,, y^{(4)}(0) = 1$ *and* $y^{(5)}(0) = 0$

Here, $y(x) = \cos(x)$ is the exact solution.

The numerical results of this problem is shown in Table 2 and Fig (2).

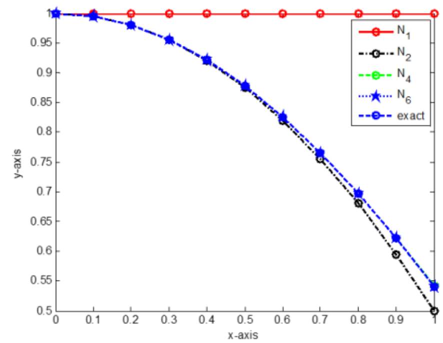

Figure 2: Comparison of the solutions of example 2.

Example 3:

$$y^{(8)}(x) - \pi^8 y(x) = \frac{x}{2} - \int_0^x y^2(t)dt + \frac{\sin 2\pi x}{2\pi}\int_0^1 [\cos(\pi t) - y(t)]\,dt\,, \qquad 0 \leq x \leq 1$$

With initial condition

$y(0) = 0, y'(0) = \pi, y''(0) = 0,$

$y'''(0) = -\pi^3, y^{(4)}(0) = 0,$

$y^{(5)}(0) = -\pi^5, y^{(6)}(0) = 0$ and $y^{(7)}(0) = -\pi^7$

Here, $y(x) = \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7$ is the exact solution.

Numerical results of this problem are shown in Table 3 and Fig (3).

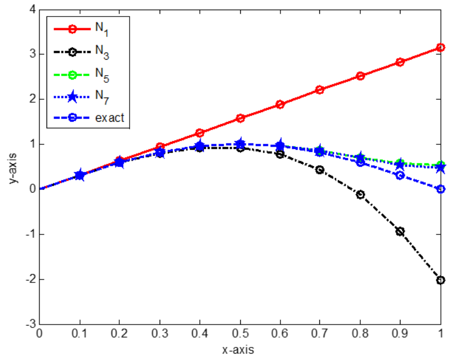

Figure 3: Comparison of the solutions of example 3.

5. Conclusion

Most nonlinear Volterra Fredholm Integro-Differential equations are difficult to solve analytically, necessitating the use of approximate solutions in many situations. For this reason, we provide the solution of high-order nonlinear Volterra Fredholm Integro-Differential equations. Our technique uses Bou-baker polynomials to convert a high-order non-linear Volterra Fredholm Integro-Differential equation to a collection of linear algebraic equations that MATLAB Program can easily solve. The final outcome demonstrates that the approach employed can effectively handle such problems, as shown in the tables.

1. Using Bou-baker polynomials basis function to approximate when the nth degree of Bou-baker po1ynomia1s is increases then the error is decreases.

We can see also from Fig (1), Fig (2) and Fig (3), tab (1), tab (2) and tab (3) that the approximation is good. when compare approximation with the exact so1ution.

6. Appendix

Table 1: Comparison between the approximate and the exact solutions for several values of N in Example 1.

X	Exact solution	Differentia1 transform method	Bou baker po1ynomia1s method			1.S. E
			N=1	N=2	N=5	
0	-1.0000	-1.00000	-1.00000	-1.00000	1.00000	0.0000
0.1	-0.9900	-0.9900	-1.00000	-0.9900	-0.9900	0.0000
0.2	-0.9600	-0.9600	-1.00000	-0.9600	-0.9600	0.0000
0.3	-0.9100	-0.9100	-1.00000	-0.9100	-0.9100	0.0000
0.4	-0.8400	-0.8400	-1.00000	-0.8400	-0.8400	0.0000
0.5	-0.7500	-0.7500	-1.00000	-0.7500	-0.7500	0.0000
0.6	-0.6400	-0.6400	-1.00000	-0.6400	-0.6400	0.0000
0.7	-0.5100	-0.5100	-1.00000	-0.5100	-0.5100	0.0000
0.8	-0.36000	-0.36000	-1.00000	-0.36000	0.36000	0.0000
0.9	-0.19000	-0.19000	-1.00000	-0.19000	0.19000	0.0000
1	0	0	-1.00000	0	0	0

Table 2: Numerical comparison of results in Example 2.

X	Exact solution	Differential transform method	Bou-baker polynomia1s method				L.S. E
			N=1	N=2	N=4	N=6	
0	1.0000	1.0000	1.00000	1.0000	1.0000	1.0000	0.0000
0.1	0.9950	0.9950	1.00000	0.9950	0.9950	0.9950	0.0000
0.2	0.9801	0.9801	1.00000	0.9800	0.9801	0.9801	0.0000
0.3	0.9553	0.9553	1.00000	0.9550	0.9553	0.9553	0.0000
0.4	0.9211	0.9211	1.00000	0.9200	0.9211	0.9211	0.0000
0.5	0.8776	0.8776	1.00000	0.8750	0.8776	0.8776	0.0000
0.6	0.8253	0.8253	1.00000	0.8200	0.8254	0.8253	0.0000
0.7	0.7648	0.7648	1.00000	0.7550	0.7650	0.7648	0.0000
0.8	0.6967	0.6967	1.00000	0.6800	0.6971	0.6967	0.0000
0.9	0.6216	0.6216	1.00000	0.5950	0.6223	0.6216	0.0000
1	0.5403	0.5403	1.00000	0.5000	0.5417	0.5403	0.0000

Table 3: Numerical comparison of resu1ts in Example3.

X	Exact solution	Differential transform method	Bou-baker polynomia1s method				L.S. E
			N=1	N=3	N=5	N=7	
0	0	0	0	0	0	0	0
0.1	0.3090	0.3090	0.3142	0.3090	0.3090	0.3090	0.0000
0.2	0.5878	0.5878	0.6283	0.5870	0.5878	0.5878	0.0000
0.3	0.8090	0.8091	0.9425	0.8029	0.8091	0.8091	0.0000
0.4	0.9511	0.9519	1.2566	0.9259	0.9520	0.9519	0.0000
0.5	1.0000	1.0041	1.5708	0.9248	1.0045	1.0041	0.0000
0.6	0.9511	0.9653	1.8850	0.7687	0.9670	0.9653	0.0000
0.7	0.8090	0.8502	2.1991	0.4266	0.8552	0.8502	0.0000
0.8	0.5878	0.6903	2.5133	-0.1326	0.7030	0.6903	0.0000
0.9	0.3090	0.5370	2.8274	-0.9398	0.5660	0.5370	0.0000
1	0.0000	0.4633	3.1416	-2.0261	0.5240	0.4633	0.0000

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