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On Regular**-Open Separation Axioms In Ideal Topological Space

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Abstract

The concern of this study focuses on constructing a new classes of separation axioms by using the notion of Regular**− open set of ideal topological space. Using the concept of regular-open sets and Regular-local functions with *R-closure operator in an ideal topological space (X, T, I), analogous to the usual separation axioms. A few characteristics and attributes are investigated. Like separation axioms works.

Keywords: Regular-local functions, *R-closure operator, R**-open, R**-closed sets.

1- Introduction and Preliminaries.

Velicko [1] first presented the concepts of θ-open, closed, and closure subset to examine the significant class of H-closed space in terms of arbitrary filter bases. The definition of the I-open set notion by Tankovic and Hamlett [2, 3] in 1990 used Vaidyanathaswamy's local function like a starting point. In just an ideal topological space, Kuratowski [4] established the concept of the local function. A high number of mathematicians, including Hayashi [2], Three researchers Natakaniec [5], Modak, and Bandyopadhyay [6] have examined this area and demonstrated some novel findings. The idea of a regular-local function is introduced in this work, and some of its aspects are investigated.

The notation that is used is as follows. The family of open neighborhoods at point x will be T(x) if (X, T) is a topological Space. cl(A) is the set closure, and Int(A) is the set interior. Asset A is referred to as clopen if it is both open and closed. If each point in set A contains the closure of an open Neighborhood in A, then A is said to be θ-open [7]. (There is a V ∈ T(x) such that cl(v) ⊆ A). It's fine knowledge that the collection of all θ-open subsets of (X, T) are topologies on X that we will designate by Tθ. What is immediately obvious according to the definitions, $T\theta \subset T$. Then $T\theta = T$, consequently, and only when the space (X, T) is regular. If we have $cl(u) \cap A \neq \emptyset$ for any open Neighborhood u of X, then a point $x \in X$ is said to be in the θ-closure of a subset $A \subseteq X$ [8]. We'll use clθ to refer for θ-closure(A). If A = clθ then a subset $A \subseteq X$ is said to be θ-closed (A). A set need not be a θ-closed set for its θ- closure. A non - empty set collection of X subsets that fulfills the condition is an ideal I on a topological space (X, T).

i. ∅ ∈ I.
ii. A∈ I, and B ⊆ A imply B ∈ I.
iii. A ∈ I and B ∈ I imply A ∪ B ∈ I.

Some writers include a fourth requirement that X by itself is not in I. Proper ideals are ideals that have additional properties. The term ideal topological space and the symbol are used to refer to a topological space (X, T) having an ideal I on X. (X, T, I). The local function of A regarding I and T is defined as A*(l) = {x ∈ X: U ∩ A ∉ I for any U ∈ T(x)} for a subset A⊆ X [9]. To avoid any misunderstanding, we just write A*. For a topology T*(I) known as the T*-topology finer than T, cl* (A) = A ∪ A* is defined as a Kuratowski[4] closure operator. A subset A of an ideal space (X, T, I) is T*-closed [10] (resp. *-dense in itself [10]), and ∗-perfect [10] if A* ⊂ A (resp. A* ⊂ A, A = A*). There is no doubt that A is only *-perfectly formed if and only if it is internally T*-closed and *-dense.

Definition 1.1 Let (X, T) be a topological space. And A⊂ X. Then a subset A of X is said to be Regular-open If A = Int(cl(A)), [11]. Regular-closed is the term used to describe a Regular -open set's counterpart [12]. The term RO(X) (or RC(X)) stands for the collection of all Regular-open (or Regular-closed) sets in X.

Remark 1.1 We denote all Regular-open sets forms a topology T^R or TR.

Definition 1.2 Let (X, T) be a topological space. Then the Regular-interior and the Regular-closure of A in X defined as Int_R (A) = ∪ { U : U ⊆ A, U ∈ T^R } and cl_R (A) = ∩ { F : A ⊆ F, X – F ∈ T^R }.From definition, Int_R (A) is a Regular-open set and cl_R (A) is a Regular-closed set [12].

Remark 1.2

1. Every Regular-open is open.
2. Every Regular-closed is closed.

2- Regular-local functions

Definition 2.1 Assume that (X, T, and I) is an ideal topological space. We define the operator A (*R) (I, T) = {x ∈ X: for a subset A of X, where A ∩ U ∉ I for every U T∈ R (x)}. Just so there is no misunderstanding The Regular-local function of A with respect to I and T is indicated by A(*R) (I, T) and is denoted as A(*R). Also, A(*R) = A^{*R}

Remark 2.1 Assume that A ⊂ X and that (X, T, I) is an ideal topological space. Then, cl(*R) (A) = A ∪ A(*R) is a *R-closure operator.

Remark 2.2 If A ⊂ X and (X, T, I) is an ideal topological space, Then

$T^{*R} = \{ X - A : cl^{*R}(A) = A\}$.

Example 2.1 Assume that (X, T, I) is an ideal topological space and that A ⊂ X. with X = {1, 2, 3}, T = {X, {1, 2}, {2}, {1}, ∅ }, and I = { ∅, {3}}.

A	A^{*R}	cl^{*R}
∅	∅	∅
X	X	X
{1}	{1, 3}	{1, 3}
{2}	{2, 3}	{2, 3}
{3}	{3}	{3}
{1, 2}	X	X
{1, 3}	{1, 3}	{1, 3}
{2, 3}	{2, 3}	{2, 3}

Then $T^{*R} = \{ \emptyset, X, \{1, 2\}, \{2\}, \{1\} \}$.

Theorem 2.1. Given that I an ideal on X and (X, T) is a topological space.

Then B (I, T^R) = {V- i: V ∈T*R and i∈ I} is a basis for T (*R).

Lemma 2.1 [13] In the event that (X, T, I) is a perfect topological space and A ⊂X. A* = cl(A*) = cl*A if A ⊂A* (A).

Lemma 2.2. Considering that A⊂ X and (X, T, I) is an ideal topological space, In the event where A ⊂A(*R), then A(*R) = cl R(A) = cl(*R) (A).

Theorem 2.2 Suppose that (X, T, I) is an ideal topological space. Then, the under characteristics are hold.

1. If I = ∅, then cl(*R) (A) = cl R((A).

2. If I = P(X), then cl(*R) (A) = A.

3. If A ∈ I, then cl(*R) (A) = A.

Obvious proof.

Theorem 2.3 Assume that (X, T, and I) is an ideal topological space, A and B are subsets of X. As a result, the following characteristics are true for R-local functions:

1. $cl^{*R} (\emptyset) = \emptyset$.

2. If $A \subset B$, then $cl^{*R}(A) \subset cl^{*R}(B)$.

3. For an another ideal $J \supseteq I$ on X, $cl^{*R}(A, T, J) \subset cl^{*R}(A, T, I)$.

4. $cl^{*}(A) \subset cl^{*R}(A)$.

5. $cl^{*R}(A) \subset cl_R(A)$.

6. $cl^{*R}(cl^{*R}(A)) \subset cl^{*R}(A)$ if A is Regular − closed.

7. $cl^{*R}(A) \cup cl^{*R}(B) = cl^{*R}(A \cup B)$.

8. $cl^{*R}(A \cap B) \subset cl^{*R}(A) \cap cl^{*R}(B)$.

3- R**-closure and R**-interior in ideal topological spaces.

Remark 3.1. In this study, we will consider the topological space is locally indiscrete.

Definition 3.1.[14]. A space X is called locally indiscrete if every open set is closed or verse.

Example 3.1. Let X = $\{a_1, a_2, a_3\}$,

T = $\{\{a_1, a_2\}, \{a_3\}, X, \emptyset\}$. $\{a_1, a_2\}$ is open and closed $\{a_3\}$ is open and closed X and ∅ is open and closed.

Proposition 3.1. [14]. For a topological space X, If X is locally indiscrete then every dense open subset of X is regular-open.

Definition.3.2. [15] Assuming that (X, T, I) an ideal topological space, A is a subset of X, and x is arbitrary point in X.

Then if A ∩ (int (cl^* (w)) ≠∅ for all open neighborhoods w of x, then x is termed a δ-I-cluster point of A. The [A]δ − I symbol designates the family of each, δ − I cluster point of A.

If [A]δ − I = A, then a subset A is said to be, δ − I-closed. δ – I -closed set of X complement is referred to as be, δ − I-open.

Remark.3.2. [16] The family of regular open sets of (X, T) is recognized for a topology is weaker than T. This topology is designated by the symbol T_S and is known as the semi-regularization of T. In reality, Ts is identical to the family of, δ-open sets of (X, T).

Remark.3.3 We write [A]δ − l = {x ∈ X : int(cl*(w) ∩ A ≠ ∅ all w ∈T } , It is written as σcl(A) = [A]δ − l.

Lemma 3.1 [15] A and B a subset of an ideal topological space (X, T, l).Then, the subsequent characteristics are satisfied:

1. A ⊆ σcl(A).
2. If σcl(A) ⊂ σcl(B), then A ⊂ B.
3. σcl(A) = ∩ {G ⊂ X: A ⊂ G and G is, δ − l − closed}.

Definition 3.3. Let (X, T, I) bean ideal topological space, and A a subset of X. Then Rσcl*R(A) = {x ∈ X: A ∩ int(cl*R(w)) ≠ ∅, for each open neighborhood w of X}.

Definition 3.4. A subset A of an ideal topological space (X, T, I) is called R*-closed if Rσcl*R ⊂ W, whenever A ⊂ W and W is open in (X, T, I). The complement of R*-closed set in (X, T, I) is called R*-open set.

Remark 3.4. The collection of all R*-closed sets in (X, T, I) denoted by R*− C(X). The family of all R*-open sets in (X, T, I) denoted by R*− O(X).

Example 3.2. Let X = {e_1, e_2, e_3}, T = {X, ∅, {e_3}, {e_1, e_2}, I = {∅, {e_1, e_2}}.
RO(X) = {X, ∅, {e_3}, {e_1, e_2}}.

A	**Rσcl*R(A)**
X	X
∅	∅
{e_1}	{e_1, e_2}
{e_2}	{e_1, e_2}
{e_3}	{e_3}
{e_1, e_2}	{e_1, e_2}
{e_1, e_3}	X
{e_2, e_3}	X

Then R*− C(X) = {X, ∅, {e_1}, {e_2}, {e_3}, {e_1, e_2}, {e_1, e_3}, {e_2, e_3}}.
R*− O(X) = {X, ∅, {e_2, e_3}, {e_1, e_3}, {e_1, e_2}, {e_3}, {e_2}, {e1}}.

Lemma.3.2. Every Rσcl*R(A) is closed in ideal topological space.

Proof. If W ∈ T (x) and x ∈ cl(Rσcl*R(A)), then W ∩ Rσcl*R(A)/= ∅.
Since W ∈ T (y) and y ∈ Rσcl*R(A) for some y ∈ X, y ∈ W ∩ Rσcl*R(A).
We may deduce A∩ int(cl*R(w)) / = ∅. from the definition of Rσcl*R(A).
Consequently, x ∈ Rσcl*R(A). Rσcl*R(A) is closed because cl(Rσcl*R(A)) ⊂ Rσcl*R(A).

Theorem.3.1. Assume that (X, T, l) is an ideal space and that A ⊆ X. Hence, Rσcl*R(A) = Rσcl*R(B) if A ⊆ B ⊆ Rσcl*R(A).

Proof. Let A ⊆ B, Rσcl*R(A) ⊆ Rσcl*R(B), and B ⊆ Rσcl*R(A),
Rσcl*R(B) ⊆ Rσcl*R(Rσcl*R(A)) = Rσcl*R(A).
Consequently, Rσcl*R(A) = Rσcl*R(B).

Definition 3.5. Let (X, T, I) be an ideal topological space, and A a subset of X is called R**-closed if Rσcl*R ⊂ W whenever A ⊂ W, and W is Regular-open in (X, T, I). The complement of R**-closed in (X, T, I) is called R**-open set.

Remark 3.5. The collection of all R**-closed sets and R*-open set in (X, T, I) denoted by (respectively R**− C(X). and R**-open) sets

Example 3.3. Let X = {1, 2, 3}, T = {X, ∅, {1}, {2}, {1, 2}, I = {∅, {3}}.

Then RO(X) = {X, ∅, {1}, {2}}.

A	Rσcl*R(A)
X	X
∅	∅
{1}	{1, 3}
{2}	{2, 3}
{3}	{3}
{1, 2}	{1, 2, 3}
{1, 3}	{1, 3}
{2, 3}	{2, 3}

Proposition 3.2. Every R*-closed set is R**-closed set.

Proof. Let A be any R*-closed set, and W be any open set in ideal locally indiscrete topological space (X, T, I), such that A ⊆ W, and Rσcl*R(A) ⊆ W (by Def.), then every open set is closed (by Def.), therefor W is Regular-open, Hence, A is R*-closed.

Remark 3.6. The opposite of proposition (3.2) is not true. It is clear from the following example.

Example 3.4. Let X = {a_1, a_2, a_3}, T = {X, ∅, {a_1}, {a_2}, {a_1, a_2}}, I = {∅, {a_3}}.

RO(X) = {X, ∅, {a_1}, {a_2}}, if A = {a_1, , a_2}.

Rσcl*R(A) = {a_1, a_2, a_3}.

Then A is R**− closed but not R*− closed.

Remark.3.7.

1- Every Regular-open is R**-closed set.
2- Every Regular-open is R*-closed set. The opposite of Remark (3.5) is not always true,

Theorem.3.2. Rσcl*R(A)) is always R**-closed for any subset A of X if (X, T, I) is an ideal space.

Proof. Let Rσcl*R(A) ⊆ W, where W is Regular-open set.

Since Rσcl*R(Rσcl*R(A)) = Rσcl*R(A).

We have Rσcl*R(A)(Rσcl*R(A)) ⊆ W .

Whenever Rσcl*R(A) ⊆ W and W Regular-open set. Hence Rσcl*R(A) is R**-closed set.

Theorem.3.3. where A and B are R**−closed sets in a topological ideal space (X, T, and I), then A ∪ B is a R**−closed set in (X, T, I).

Proof. Suppose that A ∪ B ⊆ W, where W is any Regular-open set in (X, T, I). Then A ⊆ W and B ⊆ W. Given that A and B are R**-closed sets in (X, T, I), Rσcl*R(A) ⊆ W and Rσcl*R(B)) ⊆ W. Whenever Rσcl*R (A ∪ B) = Rσcl*R(A) ∪ Rσcl*R(B) As a result. Rσcl*R (A∪ B) ⊆ W. Thus, A ∪ B is a R**- closed sets.

Remark 3.8. the intersection of two R**-closed sets are not necessarily R**-closed set as show in the following example.

Example.3.5. If X = {1, 2, 3}, T = {X, ∅, {1}, {2}, {1, 2}}, I = {∅, {3}}.

RO(x) = {X, ∅, {1}, {2}}.

Then A = {1, 2}, B = {1, 3} are R**-closed set.

But {1, 2} ∩ {1, 3} = {1} is not R**-closed set.

Definition.3.6. The intersection of all R**-closed sets containing A is known as the R**-closure of a subset A of X, indicated by R**− cl(A). Therefore, R** − cl(A) = ∩ {F ⊆ X: A ⊆ F and F is R**-closed set}.

$$\begin{array}{ccccc} R^{**}\text{-open} & \Leftarrow & \text{Regular open} & \Rightarrow & \text{open} \\ & & \Downarrow & & \\ & & R^{*}\text{-open} & & \end{array}$$

Digrame 3.1. Relationships among the Regular-open, Regular**-open and Regular*-open.

Definition.3.7. The Regular **-interior of subset A of X denotes the union of all Regular**-open sets contained within subset A and is known as R**-int(A). Therefore,

R**-int(A) = ∪ {G ⊆ X: G ⊆ A and G is R**-open set}.

Theorem 3.3. Let A and B be any two subsets of (X, T, I) then the following properties are true.

1. A is R**-open set if and only if R**−int(A) = A.
2. R**-int(A) is the bigger R**-open set subset of X contained in A.
3. R**-int (∅) = ∅ and R**-int(X) = X.
4. R**-int(A) is a R**-open set.
5. If A ⊆ B, then R**-int(A) ⊆ R**-int(B).
6. R**-int (A ∪ B) ⊇ R**-int(A) ∪ R**-int(B).
7. R**-int (A ∩ B) = R**-int(A) ∩ R**-int(B).
8. R**-int(R**-int(A)) = R**-int(A).

Proof.

1. We know that A ⊆ R**-int(A) for any subset A of X. Let A be a R**-open set in (X, T, I).
Also A ⊆ A and A ∈ {O ⊆ X: O ⊆ A and O is R**-open set}. It means that A = ∪ {O ⊆ X: O ⊆ A and O is R**-open set} ⊆ A. Then, R**-int(A) ⊆ A. Hence, A = R**-int(A). Converse is true from the direct definition.

2. By the definition of R**-int(A) the union of all sets is open there- fore R**-int(A) is open. Also, if B is any R**-open set contained in A then R**-int(A) ⊆ B. Therefore, R**-int(A) is the biggest R**-open set in (X, T, I).

(3) and (4) it follows from.

5. We know that B ⊆ R**-int(B) for every B. If A ⊆ B, then A ⊆ R**-int(B). So, R**-int(B) is the R**-open set containing A. But R**-int(A) is biggest open set contained A. Therefore, R**-int(A) ⊆ R**-int(B).
6. We know that A ⊆ A ∪ B and B ⊆ A ∪ B, we get R**-int(A) ⊆ R**-int (A ∪ B) and R**-int(B) ⊆ R**-int (A ∪ B). Then, R**-int(A) ∪ R**-int(B) ⊆ R**-int (A ∪ B).
7. We know that A∩B ⊆ A and A ∩ B ⊆ B by using (5) we have R**-int (A ∩ B) ⊆ R**-int(A) and R**-int(A ∩ B) ⊆ R**-int(B).Then R**-int(A ∩ B) ⊆ R**-int(A) ∩ R**-int(B). On the other hand. R**-int(A) is R**-open set contained A and R**-int(B) is R**-open
set contained B. Therefore, R**-int(A) ∩ R**-int(B) is R**-open set contained A ∩ B therefore R**-int(A) ∩ R**-int(B) ⊆ R**-int (A ∩ B). Hence, R**-int (A ∩ B) = R**-int(A) ∩ R**-int(B).
8. We know that R**-int(A) is R**-open set in (X, T, I). Let R**-int(A) = O then O is R**-open set in (X, T, I). From (1) R**-int(O) = O. It means that R**-int(R**-int(A)) = R**-int(A).

4- Regular** Separation Axioms.

Definition.4.1 [17, 18]. The ideal topological space is called Regular*−$\mathbb{T}_0$ −space (briefly. R*− $\mathbb{T}_0$ space), if for each pair of distinct point x, y ∈ X there exist an R*− open set containing only one of them.

Example.4.1. If X = {a, b, c}, T = {∅, X, {b, c}, {a, c}, {a}, {c}}, I = {∅, {c} }.
T R = {X, ∅, {a}, {b, c}}.

A	Rσcl*R(A)
X	X
∅	∅
{a}	{a}
{b}	{b}
{c}	{b, c}
{a, b}	{a, b}
{a, c}	X
{b, c}	{b, c}

R*− C(X) = {X, ∅, {a}, {b}, {a, b}, {b, c}}.
R*− O(X) = {X, ∅, {a, c}, {a, c}, {c}, {a}}.
Then (X, T, I) is a R*− $\mathbb{T}_0$ space.

Definition.4.2. [17, 18]. The ideal topological space is called Regular** − $\mathbb{T}_0$space (briefly. R**− T˳ space), if for each pair of distinct point x, y ∈ X there exist an R**− open set containing only one of them.

As in the previous example then R**− C(X) = {X, ∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}. R**− O(X) = {X, ∅, {b, c}, {a, c}, {a, b}, {c}, {b}, {a}}.

Then (X, T, I) is a R**−$\mathbb{T}_0$ space.

Theorem.4.1. The ideal topological space (X, T, I) is a R**−$\mathbb{T}_0$ space if and only if for each elements x/= y there is a R**−closed set containing only one of them.

Proof. Let x and y are two distinct elements in X. Since X is a R**−$\mathbb{T}_0$-space, then there is a R**− open set U containing only one of them, then X \ U is R**− closed set containing the other one.
Conversely.
Let x and y are two distinct elements in X. And there is an R*− closed set V containing only one of them. Then, X / V is a R**− open set containing the other one.

Proposition.4.1. If (X, T, I) is a R*− $\mathbb{T}_0$space then (X, T, I) is a R**− $\mathbb{T}_0$space.

Proof. Let x and y are two distinct elements in X.
Since (X, T, I) is a R* − $\mathbb{T}_0$ space, then, there is a R**− open set U containing only one of them. Since every R*− open set is a R**− open set, then (X, T, I) is a R**−$\mathbb{T}_0$ space.

Definition.4.3. [17] The ideal topological space (X, T, I) is called Regular*−T_1 − space (briefly R* − T_1 − space) if for each elements x, y ∈ X such that x / = y, there is two R*− open sets U, V , satisfies x ∈ U, y / ∈ U and x / ∈ V, y ∈ V.

Definition.4.4. The ideal topological space (X, T, I) is called Regular**−T_1 − space (briefly R**− T_1−space) if for each elements x, y ∈ X such that x / = y, there is two such that x / = y, there is two R**− open sets U, V , satisfies x ∈ U, y / ∈ U and x / ∈ V, y ∈ V.

Definition.4.5. The ideal topological space (X, T, I) is called Regular*−T_2−space (briefly R*− T_2 − space) if for each elements x, y ∈ X such that x / = y, there is two R*− open sets U, V , satisfies x ∈ U, y / ∈ U and x / ∈ V, y ∈ V, then U ∩ V / = ∅.

Definition.4.6. The ideal topological space (X, T, I) is called Regular**–T_2 − space (briefly R**− T_2 − space) if for each elements x, y ∈ X such that x / = y, there is two R**− open sets U, V , satisfies x ∈ U, y / ∈ U and x / ∈ V, y ∈ V, then U ∩ V / = ∅.

Proposition.4.2. If (X, T, I) is R* −T_2 spaces in T, then (X, T , I) is R**− T_2 spaces in T .

Proof. Take x and y to be two separate items in X. There exist two Regular*-open sets, U1 and U_2, because (X, T, I) is a R* − T_2-space. ensure that x ∈ U_1, y ∈ U_2, and U_1 ∩ U_2 = ∅.

Since any Regular*-open set also contains a Regular**-open set. Then, U_1 and U_2 are set to Regular**-open. meet the conditions x ∈ U_1, y ∈ U_2, and U_1 ∩ U_2 = ∅.

Proposition 4.3. R*–T_0-space if R*–T_1space is inferred.

Proof. Let x and y be two separate components of X.

(X, T, I) being a R*–T_1-space Then, if x ∈ U_1, x / ∈ U_2, y ∈ U_2 and y /∈ U_1, there are R∗-open sets U_1, U_2.

After that, there is a R*-open set U that only has one of them in it.

Following this, there is a R*–T˚-space.

Remark.4.1. The following example generally shows why Proposition ( ) meant opposite interpretation is untrue.

Example.4.2. The ideal topological space (X, T, I) is a R*–T˚-space,

where X = {a, b, c}, T = {∅, X, {c}, {a}, {a, c}, {b, c}, with I = {∅, {c}}

Then,

T R = {∅, X, {a}, {b, c}}.

R*- closed (A) = {∅, X, {a}, {b}, {a, b}, {b, c}}.

R*-open(A) = {∅, X, {b, c}, {a, c}, {c}, {a}}.

The ideal Topological space (X, T, I), there are two elements b ≠ c, then,

There is no R*-open set U containing c .

But, not containing b.

Then the ideal Topological space (X, T, I) is not R*− T_1-space.

Theorem.4.2. Any ideal topological space (X, T, I) is a R**–T_1-space

if and only if there are two R**-closed sets F_1 and F_2 such that x ∈ F_1

and x / ∈ F_2, and y /∈ F_1 and y ∈ F_2 for all members x /= y.

Proof. Let x and y represent two separate X elements.

There are two R**-open sets U1, U_2 since X is a R**– T1-space.

such that x ∈ U_1, X /∈ U_2 and y ∈ U_2, y / ∈ U_1.

Then, there exist R**-closed sets X \ U_1 and X\U_2.

such that x ∈ X \ U_2–X \ U_1, y ∈ X \ U_1 – X \ U_2, where F_1= X \ U_2 and F_2 = X \ U_1.

Then, there exist two R**- closed sets F_1 and F_2 satisfy.

x ∈ (F_1 ∩ F_2^c) and y ∈ (F_2 ∩ F_1^c).

Therefor, x ∈ F_1 \ F_2) and y ∈ (F_2 \ F_1).

Conversely, let x and y be two distinct elements in X and there exist two R**-closed sets F1 and F_2 satisfy, x ∈ (F_1 ∩ F_2^c) and y ∈ (F_2 ∩ F_1^c).

Then there exists R**-open set (X \ F_1) and (X \ F_2).

When ever x ∈ X \ F_2 and X \ F_1, y ∈ X \ F_1 and X \ F_2.

Where U_1 = X \ F_2 and U_2 = X \ F_1

Proposition.4.4. If the ideal topological space (X, T, I) is R**− T_2-space, then the space is a R**− T_1-space.

Proof. Let there be two separate components in X, x and y.

Since a R**– T_2-space is (X, T, I).

Consequently, R**– T_2-open set U_1 and U_2 exist.

Then the conditions x ∈ U_1, y ∈ U_2, and U_1 ∩ U_2 = ∅.

There are R**-open sets U_1 and U_2, then.
As to have $x \in U_1$ and $y \in U_2$.
Therefore, Then, since R**-open set U_1, U_2 exists, (X, T, I) is R**− T_1-space. Remark The following example shows that the opposite interpretation in Proposition (4.4), is not often true.

$$\begin{array}{ccc} R^*\text{-}T_2\text{-space} & \Rightarrow & R^{**}\text{-}T_2\text{-space} \\ \Downarrow & & \Downarrow \\ R^*\text{-}T_1\text{-space} & \Rightarrow & R^{**}\text{-}T_1\text{-space} \\ \Downarrow & & \Downarrow \\ R^*\text{-}T_0\text{-space} & \Rightarrow & R^{**}\text{-}T_0\text{-space} \end{array}$$

Figure 4.1 Relationships among the -$\mathbb{T}_i$- R**− $\mathbb{T}_i$-space and R*− $\mathbb{T}_i$-space.

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